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STATISTICAL THEORY OF ELASTIC PROTON-PROTON SCATTERING AT LARGE ANGLES

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ABSTRACT

It is assumed that the collision of two nucleons of high energy sometimes forms a "compound system" in thermal equilibrium. The secondary processes following the collision are then the decay products of this compound system. It is conjectured that the elastic scattering at large angles is primarily due to this mechanism. Its cross-section can then be calculated by simple thermodynamical methods. The pion cloud in the compound system is approximated by a massless boson gas. The Stefan-Boltzmann equation leads to an exponential energy dependence $\exp(-E/T)$ of the cross-section with $T \sim E^{\frac{1}{4}}$ where E is the kinetic energy of the nucleons. The order of magnitude of the cross-section can also be estimated. These results agree with the experimental observations and with previously published computer calculations of the relevant phase space by Fast and Hagedorn and by Jones.

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1. Recently, there has been done very extensive measurements of the elastic pp cross-section at high energies - from 10 - 31 GeV/c in the laboratory system - and high momentum transfers, i.e., large scattering angles ¹⁾. The results of these experiments seem to indicate the statistical character of the scattering at large angles as pointed out by several authors ^{2),3),4),5),6)}.

The elastic scattering at large angles is definitely greater than the diffraction scattering. The cross-section around 90° in the centre of mass system shows a characteristic exponential decrease with energy, which is an indication of a statistical effect (Boltzmann factor).

A statistical theory makes the following assumption : during the collision, the total energy available is concentrated in a certain volume in which a state of thermal equilibrium is produced. The volume expands and, when it reaches a maximum value V , the thermal equilibrium ceases to be maintained ; then the products of the collision leave the centre of collision roughly with the energies and the properties which they had at thermal equilibrium in the volume V .

To draw quantitative conclusions in regard to the elastic scattering from such a statistical model, one should compute the number of all possible states within a given volume V (with the same total charge, strangeness and baryon number as those of the initial two protons) at a given energy and compare it with the number of states of two protons only at the same energy. The probability of an elastic scattering at large angles (outside the diffraction scattering) should then be proportional to the ratio of the number of two proton states and the total number of states. This amounts to the calculation of many phase-space integrals. It is known that such numerical calculations are extremely difficult and need many hours of computer time. This work was performed at CERN ⁷⁾. On this basis, Hagedorn and Jones ^{4),5),6)} were able to find the energy dependence of the "statistical part" of the elastic pp cross-section at high energies. Their predictions fit the existing data on the pp elastic scattering at large angles ¹⁾ with good accuracy.

In the present paper, we would like to consider the same problem using a thermodynamical approach. We consider the mixture of boson and fermion gases in the volume V and calculate the number of possible states by the usual methods used in thermodynamics. In our opinion this approach has two main advantages :

- i) it offers a simple picture of the high energy "statistical" collisions,
- ii) it permits to simplify considerably the calculations - they can be performed analytically and are indeed very simple. They are, however, less accurate than those reported in Ref. ⁷⁾.

It seems remarkable that our simple theory fits the existing data with at least the same accuracy as the theories based on the exact statistical calculations.

2. MAIN ASSUMPTIONS

Let us consider the collision of two high energy protons. We will distinguish between "central" and "peripheral" collisions. The first type refers to those collisions, where, in fact, a thermal equilibrium occurs within a volume V as indicated above. In the other collisions, the peripheral ones, the situation is such that the energy is not distributed over all possible degrees of freedom. We introduce a factor y giving the ratio of central collisions to all others :

$$y = \frac{\sigma_{\text{central}}}{\sigma_{\text{total}}} \quad (2.1)$$

We expect y to be roughly of the order of magnitude 0.1 to 0.01. This division is analogous to the division of nuclear reactions into "compound nucleus formation" and "direct reactions". In fact, the statistical interpretation of the elastic pp scattering corresponds to what one calls compound-elastic scattering in nuclear physics. We expect that the peripheral collisions will lead predominantly to scattering into smaller angles. It is plausible, therefore, that the elastic scattering at 90° should come predominantly from central collisions and should be predictable on the basis of a statistical theory.

The cross-section for the statistical contribution to the elastic scattering can be written as follows :

$$\sigma_{\text{el}} = y \sigma_{\text{tot}} \frac{\omega_{p+p}(W)}{\omega_{\text{all}}(W)} \quad (2.2)$$

where y is the factor defined in (2.1) ; σ_{tot} is the total cross-section, which, in our energy region, is about 40 mb ; the third factor in (2.2) is the branching ratio into two protons of total energy W . It is the ratio of the density of states for two protons to the density of all states at the energy W within the volume V . Here W is the total energy of the system.

Obviously, the states in question consist of a mixture of baryons and mesons. We will make the following simplifying assumptions :

- i) we neglect the antibaryons since they are rare at the energies considered ;
- ii) we consider only π mesons. It is true that K mesons and other bosons do occur at our energies. A calculation in Appendix 2 shows that the contribution of K mesons is not very important ⁸⁾ ;
- iii) we assume that the pions have no rest mass. It will be shown in Appendix 1 that this causes only small errors and allows us to treat the pion-system in the same way as one treats a photon gas. We then can use the Stefan-Boltzmann law with its characteristic energy-temperature relation ($E \sim T^4$) which simplifies and clarifies our calculations ;
- iv) we assume that the volume V is independent of energy. This assumption is in contrast with other work in this field ⁹⁾. We base it on the following idea : when a central collision occurs, the shape of the volume in which there is thermal equilibrium, most probably is a flat disc perpendicular to the incident direction, because of the relativistic contraction of the meson clouds around each proton. This volume will not remain in that shape, however. It will quickly expand in all directions and will cease to be in equilibrium when its dimensions are of the order of a certain interaction length r in all directions. This is why we approximate the maximum volume V of thermal equilibrium by a sphere of radius r , independent of energy ;
- v) we are using some simplifying assumptions in order to take care of the conservation laws of momentum and charge.

3. NUMBER OF STATES OF THE BARYON-PION SYSTEM

It is useful to consider our system in the volume V as consisting of two interacting subsystems : 1) the pion gas, 2) two baryons. The density of states $\omega_{\text{all}}(W)$ of the total system can be expressed in terms of the densities of the subsystems ω_1 and ω_2 :

$$\omega_{\text{all}}(W) = \int_{W_0}^W \omega_1(W-E) \omega_2(E) dE \quad (3.1)$$

where $W_0 = 2M$ is the smallest energy of subsystem 2).

Let us first compute the density $\omega_2(E)$ of the baryon system. We will need it at the energy W as seen in (2.2) and also at lower energies when the system is in equilibrium with the pion gas. For the energy W the system of two baryons is highly relativistic since the values of W which we are considering lie between 4.7 and 8 GeV in the c.m. system. In thermal equilibrium with pion gas, however, the average energies of the two baryons will be of the order of only one or two pion-masses more than the rest mass. Furthermore, we neglect here the multiplicity of states coming from charge or strangeness, and we consider the system at fixed total momentum because of reasons which will be clear later on. Hence we get

$$\omega_2(E) = \frac{V}{2\pi^2} \frac{1}{8} \sqrt{E^2 - 4M^2} E \quad (3.2)$$

$$\omega_2(E) = \frac{V}{8\pi^2} \frac{1}{2} M^2 \sqrt{E - 2M} \equiv A \sqrt{E - 2M} \quad E - 2M \ll 2M \quad (3.3)$$

We now come to the density of states of the pion gas. This expression can be found by pure thermodynamical considerations. We write the density in the form

$$\omega_1(E_1) = e^{S_1(E_1)} \quad (3.4)$$

where $S_1(E_1)$ is the entropy of the system. We then use the well-known relation between entropy and temperature :

$$\frac{\partial S}{\partial E} = \frac{1}{T(E)} \quad \text{or} \quad S = \int \frac{dE}{T(E)} \quad (3.5)$$

We know the energy-temperature dependence for a Bose gas in the volume V ; it is the Stefan-Boltzmann law

$$E = \frac{\pi^2}{30} V n T^4 \quad (3.6)$$

where n is the number of degrees of freedom of the boson. In our case $n = 3$ corresponding to the three isotopic spin states. Introducing $V = 4\pi r^3/3$ we get

$$E = \kappa n r^3 T^4 \quad \kappa = \frac{2\pi^3}{45} \approx 1.4 \quad (3.7)$$

and finally from (3.5)

$$S_1(E) = \frac{4}{3} (\kappa n r^3)^{1/4} E^{3/4} + \text{const} \quad (3.8)$$

or

$$\omega_1(E) = C \exp \left\{ \frac{4}{3} (\kappa n r^3)^{1/4} E^{3/4} \right\} \quad (3.9)$$

Let us now determine the constant C in the following way : we first introduce the total number of states Z up to the energy E

$$Z(E) = \int_0^E \omega_1(E') dE' = \omega(E) T(E) \quad (3.10)$$

If the energy E is equal to $E_{\min}$, the lowest possible energy a pion can have in the volume V , the number $Z(E_{\min})$ should be n . This allows us to determine the constant in (3.9). The calculated values of C are given in the following table :

r (in pion masses)	0.8	0.9	1.0	1.1	(3.11)
C	1/15	1/13	1/12	1/11	

Let us now return to the expression (3.1) for the density of states of the total system. We now calculate the integral (3.1) by inserting the expressions (3.9) and (3.3) and we get

$$\omega_{\text{all}}(W) = \int_0^{W-2M} C \exp\left\{(\kappa n^3)^{1/4} (W-2M-\epsilon)^{3/4}\right\} A \sqrt{\epsilon} d\epsilon \quad (3.12)$$

In order to simplify this integration and to see the essential connections, we expand the exponent around $(W - 2M)$:

$$\begin{aligned} S_1(W-2M-\epsilon) &= \log C + \frac{4}{3} (\kappa n^3)^{1/4} (W-2M-\epsilon)^{3/4} = \\ &= \log C + \frac{4}{3} (\kappa n^3)^{1/4} (W-2M)^{3/4} - \frac{\epsilon}{T} \end{aligned} \quad (3.13)$$

Here T is the temperature of the pion gas at the energy $(W - 2M)$:

8.

$$T = \left(\frac{W - 2M}{k n r^3} \right)^{1/4} \quad (3.14)$$

and we get for (3.12)

$$\omega_{all}(W) = \frac{\sqrt{\pi}}{2} T \omega_1(W - 2M) \omega_2(E = 2M + T) \quad (3.15)$$

It is seen from this that the number of states of our total system is dominated by the density $\omega_1(W - 2M)$ of the states of the pion system at the maximum energy available. Obviously, the state density of the proton system appears at the energy corresponding to the temperature of the pion system.

We now can insert (3.15) into our expression for the statistical contribution to the elastic cross-section (2.2)

$$\sigma_{el} = \frac{2}{\sqrt{\pi}} \gamma \sigma_{tot} \frac{1}{T \omega_1(W - 2M)} \frac{\omega_{p+p}(W)}{\omega_{p+p}(E = 2M + T)} \quad (3.16)$$

One sees immediately that this cross-section will have an exponential dependence on the energy $(W - 2M)$. A look at (3.9) shows that the leading factor will be $\exp \left\{ - \text{const}(W - 2M)^{3/4} \right\}$. The other state densities depend on the energy only in the form of low powers of W or T .

4. CORRECTION DUE TO THE CONSERVATION LAWS

The number of states of the pion system we obtained in Section 3 should be corrected in order to take into account the conservation laws of momentum and charge. The formula (3.10) gives the sum over all possible states of a free pion system. In the case we consider now, the pion system is not free - it interacts with the proton system. The main feature of this interaction is described by the conservation laws of charge and momentum. Their effect will reduce the possible number of states of the pion system : only such states can occur for which the momentum of the pions and momentum of the baryons add up to the total momentum and for which the total charge equals 0.+1 or +2.

Let us first estimate the correction due to the momentum conservation. Looking at the actual collision in the c.m. system, the total momentum P will always be zero, or the value of the momentum $|p_N|$ of the baryons will be equal to the one $|p_\pi|$ of the pions. This is why we have kept the momentum p_N fixed, when we have determined the number of baryon states.

Clearly, as a consequence of conservation we obtain the equality

$$\overline{p_N^2} = \overline{p_\pi^2} \quad (4.1)$$

If we count the states of the pion and baryon system independently of each other then equality need not be fulfilled. For the pion system we have

$$\overline{p_\pi^2} = \frac{V}{2\pi^2} \int \frac{p^4 dp}{e^{p/T} - 1} = a n r^3 T^5 \quad (4.2)$$

where

$$a = \frac{2 \times 2.4 \times 1.04}{3\pi} \sim 5.3 \quad (4.3)$$

The average of the square of the total momentum of the baryon system at the temperature T is

$$\overline{p_N^2} = \overline{p_1^2} + \overline{p_2^2} = 6MT \quad (4.4)$$

where p_1 and p_2 are momenta of the baryons. In the energy region of interest, one can check that the numbers given by Eqs. (4.2) and (4.4) obey an inequality

$$\overline{p_\pi^2} \gg \overline{p_N^2} \quad (4.5)$$

instead of the equality (4.1). It is clear that many states of the pion system (for which the total momentum is too big) cannot occur. A rough estimate gives the following result for the corrected number of states

$$\omega_{\text{corrected for } p} = \Lambda_p \omega_1 \quad (4.6)$$

where

$$\Lambda_p = \left(\frac{\overline{p_N^2}}{\overline{p_\pi^2}} \right)^{3/2} \quad (4.7)$$

We estimate the correction due to the charge conservation in a similar way. We calculate the average total charge squared of the pion system :

$$\overline{Q_\pi^2} = \frac{2}{3} N_\pi = \frac{2 \times 2.4}{3\pi} n \tau^3 T^3 \quad (4.8)$$

Taking into account that the charge of the baryon system can be 0, +1 or +2 (we are dealing only with nucleons) it is very easy to calculate the corresponding quantity for the baryon system. It appears that the average total charge squared of pions and baryons calculated in this way are of the same order. Therefore, it seems reasonable to use only the obvious correction due to the fact that the total charge of the mesons cannot be negative

$$\omega_{1 \text{ corrected for } e} = \Lambda_e \omega_1 \quad (4.9)$$

where

$$\Lambda_e = \frac{1}{2} \quad (4.10)$$

Combining the formulae (4.6) and (4.9) we obtain the following expression

$$\omega_{1 \text{ corrected}} = \Lambda_p \Lambda_e \omega_1 \quad (4.11)$$

where Λ_p and Λ_e are given by (4.7) and (4.10), respectively.

5. COMPARISON WITH EXPERIMENT

Taking into account the formulae (3.16), (3.9), (3.2), (3.3) and (4.11) we find for the calculated cross-section

$$\sigma_{el} = \frac{2}{\sqrt{\pi}} \gamma \sigma_{tot} \frac{1}{CT\Lambda_e\Lambda_p \exp\left\{\frac{4}{3}(\alpha r^3 n)^{1/4}(W-2M)^{3/4}\right\}} \frac{W\sqrt{W^2-4M^2}}{4M^{3/2} T^{1/2}} \quad (5.1)$$

where T , Λ_e , Λ_p are given by the Eqs. (3.14), (4.7), (4.10). The constant C is given in the table (3.11).

We would like to compare this cross-section with the measured one at 90° in the c.m. system. We do not expect spherical symmetry of the statistical part of the elastic cross-section because of the high angular momenta involved. The average angular momentum $\bar{l}$ is of the order of the ratio r/λ , which, in our case (20 GeV), gives a value $\bar{l} \sim 20$. In this case, one rather expects an angular distribution $\frac{d\sigma}{d\Omega} \sim \frac{1}{\sin\theta}$ which leads to

$$\left(\frac{d\sigma}{d\Omega}\right)_{90^\circ} = \frac{2}{2\pi^2} \sigma_{tot} \quad (5.2)$$

Hence we get for the experimental cross-section at 90° , as measured by Cocconi et al.

$$\left(\frac{d\sigma}{d\Omega}\right)_{exp} = \frac{2}{2\pi^2} \left(\sigma_{tot}\right)_{elastic\ theor.} \quad (5.3)$$

The factor 2 appears because of the identity of the particles.

Figure 1 gives the logarithmic plot of both calculated and measured quantities. The curve was calculated for $r = 0.9/m_\pi$ and $y = 1/20$. One can see that the obtained curve fits the existing data quite well. The comparison with the curves for other values of the interaction radius shows that the results do not depend very much on r : one can obtain a good fit to the data with values of r between $0.8/m_\pi$ and $1.0/m_\pi$. To achieve this one should change also the y (by a factor of order 5 in the whole region).

We therefore conclude that the value and the energy dependence of the elastic pp cross-section at large angles can be represented by a simple statistical theory. The resulting value r of the interaction radius has a plausible magnitude and the resulting ratio y of central collisions is also within the range which should be expected. The main merit of this consideration lies in a simple explanation of the strong exponential dependence. This dependence appears here as nothing else than the Boltzmann factor $e^{-E/T}$, where E is the kinetic energy available to the protons and T is proportional to $E^{1/4}$ according to the Stefan-Boltzmann law. The theory does not pretend to be able to determine the absolute value of the cross-section ; it only can give an indication of its order of magnitude by choosing reasonable values of r and y .

In principle, Eq. (3.16) could be used also to calculate the statistical contribution to any inelastic channel. One only replaces $\omega_{pp}(W)$ with the state density at W which represents the channel in question.

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A P P E N D I X 1

We will show that neglecting the pion mass does not introduce any large error. The expression for the energy-temperature relation of a pion gas with finite rest mass is given by

$$E = n \frac{V}{2\pi^2} \int \frac{\epsilon p^3 dp}{e^{\epsilon/T} - 1} = n \frac{V}{2\pi^2} T^4 \phi(\delta) \quad (\text{A1.1})$$

with

$$\epsilon = \sqrt{p^2 + m_\pi^2} \quad \delta = \frac{m_\pi}{T} \quad (\text{A1.2})$$

and

$$\phi(\delta) = \int_0^\infty \frac{\sqrt{x^2 + \delta^2} x^2 dx}{e^{\sqrt{x^2 + \delta^2}} - 1} \quad (\text{A1.3})$$

The function $\phi(\delta)/\phi(0)$ is shown in Fig. 2. One can see that the change due to the mass is very small (less than 10%) as long as $\delta \leq 1$. This means that a substitution of $\phi(0)$ for $\phi(\delta)$ is a good approximation for $T \geq m_\pi$.

APPENDIX 2

We will try to estimate the number of states of the boson system admitting the possibility of the production of strange particles. We confine ourselves to pions and kaons.

To evaluate the number of states of the boson system in this more general case we need the energy-temperature dependence. This can be obtained by means of the formula

$$E = \frac{V}{2\pi^2} n_\pi \int_0^\infty \frac{\epsilon_\pi p^2 dp}{e^{\epsilon_\pi/T} - 1} + \frac{V}{2\pi^2} n_\kappa \int_0^\infty \frac{\epsilon_\kappa p^2 dp}{e^{\epsilon_\kappa/T} - 1} \quad (\text{A2.1})$$

where

$$\epsilon_{\pi, \kappa} = \sqrt{m_{\pi, \kappa}^2 + p^2} \quad (\text{A2.2})$$

This formula can be written as

$$E = \kappa +^3 N T^4 \quad (\text{A2.3})$$

where

$$N = \frac{n_\pi \phi\left(\frac{m_\pi}{T}\right) + n_\kappa \phi\left(\frac{m_\kappa}{T}\right)}{\phi(0)} \quad (\text{A2.4})$$

Figure 3 gives the plot of N as a function of temperature. One can see that this function does not change very much in the region of interest ($1.5 m_\pi \leq T \leq 2 m_\pi$).

Therefore, in a not too bad approximation we can put

$$N = n_{\text{eff}} = 5.5 \quad (\text{A2.5})$$

After this approximation the theory proceeds along exactly the same lines as in the case without strange particles. The only difference is that we have to make the correction due to strangeness conservation. This can be done in a similar way as in the case of charge conservation and gives another factor

$$\Lambda_s = \frac{1}{2} \quad (\text{A2.6})$$

We end up with the following formula for the cross-section

$$\left(\frac{d\sigma}{d\Omega} \right)_{90^\circ} = \frac{2}{\pi^2 \sqrt{\pi}} \gamma \sigma_{\text{tot}} \frac{1}{c T \Lambda_e \Lambda_p \Lambda_s \exp \left\{ \frac{4}{3} (\kappa + r^3 n_{\text{eff}})^{1/4} (W - 2M)^{3/4} \right\}} \frac{W \sqrt{W^2 - 4M^2}}{4 M^{3/2} T^{1/2}} \quad (\text{A2.7})$$

All quantities in this formula are given by the same expressions as the quantities in Eq. (5.1) except that we have to substitute $n \rightarrow n_{\text{eff}}$.

The plot of the function (A2.7) is given in Fig. 2 for $r = 0.8/m_\pi$ and $\gamma = 1/20$.

A P P E N D I X 3

COMPARISON WITH THE PHASE-SPACE CALCULATIONS OF FAST AND HAGEDORN ⁷⁾

These calculations are direct computations of the phase space of the end products of the pp reaction without any of our approximations and with exact consideration of all conservation laws. The main difference, however, lies in the fact that Fast and Hagedorn do not assume an interaction volume which is independent of energy. Their interaction volume changes with energy according to the expression ⁹⁾

$$V = V_0 \frac{2M}{W} \quad (\text{A3.1})$$

It is interesting to see what result arises from our method of calculation, if the above relation (A3.1) is used. The calculation in this case is not more complicated than with a constant volume. We do not like to go through all its details, so we will point out only the differences between the two calculations.

The main difference is the energy-temperature relation for the boson system. We now get

$$E = \frac{2M}{W} \propto n_{\text{eff}} r^3 T^4 \quad (\text{A3.2})$$

Since E and W differ only by $2M$, this is roughly equivalent to a relation

$$E \sim T^2 \quad (\text{A3.3})$$

The entropy becomes instead of (3.8)

$$S = 2 (2M n r^3)^{1/4} \{E(E+2M)\}^{1/4} \quad (\text{A3.4})$$

The normalization procedure is now slightly more complicated because we cannot simply extrapolate the formula (A2.4) to the temperature corresponding to the ground state of the boson system. A more detailed consideration can easily be achieved.

The results of the calculations with the energy dependent volume of interaction are given in Fig. 4 for $r = 1/m_\pi$, i.e., the same as that used by Fast and Hagedorn. We used the value of y which they introduced

$$y = \frac{4M^2}{W^2} \quad (\text{A3.5})$$

and an interaction volume for strange particles different from that of pions

$$(V)_{st+} = \frac{1}{2} (V)_{\text{pions}} \quad (\text{A3.6})$$

In this way we come as close as possible in our model to Hagedorn's assumptions⁵⁾.

For comparison we plotted the curve given in Hagedorn's paper⁵⁾ in the same figure. The agreement seems to be remarkably good if we take into account that our calculations were done under several simplifying assumptions. This shows that our approximation methods are sufficient to estimate the result of more accurate phase-space calculations. It also shows that both the constant interaction volume and the volume (A3.1) lead to reasonable representation of the observed values and energy dependences. The experiments are not good enough to decide which assumption regarding the volume V is the better one.

REFERENCES AND FOOTNOTES

- 1) G. Cocconi, V.T. Cocconi, A.D. Krish, J. Orear, R. Rubinstein, D.B. Scarl, W.F. Baker, E.W. Jenkins and A.L. Read, Phys. Rev. Letters 11, 499 (1963), 12, 132 (1964).
- 2) G. Cocconi, The proton-proton elastic scattering at high energies, CERN preprint, 5 March 1964.
- 3) G. Fast, R. Hagedorn and L.W. Jones, Nuovo Cimento 27, 856 (1963).
- 4) L.W. Jones, Phys. Letters 8, 287 (1964).
- 5) R. Hagedorn, Large angle proton-proton elastic scattering at high energies, CERN preprint, January 1964.
- 6) R. Hagedorn, Large angle cross-sections $p+p \rightarrow A+B$ and $\pi^+p \rightarrow A+B$ at high energies predicted by the statistical model, CERN preprint, 11 June 1964.
- 7) G. Fast and R. Hagedorn, Nuovo Cimento 27, 208 (1963).
- 8) The neglecting of other bosons such as η, ω, ρ mesons is made on the basis that they would influence the statistical weight of the states only as a "binding energy" between pions. Hence it should not have an important effect on the situation.
- 9) E. Fermi, Progr. Theoret. Phys. 5, 570 (1950).

FIGURE CAPTIONS

- Figure 1. Comparison of our calculations with experimental data of Cocconi et al.
 a) the curve without strange particles $y = 1/20$, $r = 0.9/m_K$.
 b) the curve with kaons $y = 1/20$, $r = 0.8/m_K$.
- Figure 2. Correction due to the rest mass of π mesons.
- Figure 3. Correction due to the kaons.
- Figure 4. Comparison of our calculation with that of Hagedorn.
 a) our calculation as described in Appendix 3.
 b) curve from the paper by Hagedorn⁵⁾.

FIG. 1

FIG.2

FIG. 3

FIG. 4