THE GENERALIZATION OF THE WIGNER-BACAH ANGULAR MOMENTUM CALCULUS, II

L.C. Biedenharn *)
CERN—Geneva

*) National Science Foundation Senior Postdoctoral Fellow on leave of absence from Duke University, Durham, North Carolina, U.S.A.
The very great importance in all branches of quantum physics of the Wigner-Racah angular momentum calculus has led to many attempts to generalize this structure from the two-dimensional unimodular unitary ("angular momentum") group, where it originated, to the general semi-simple compact Lie group. A solution to the various problems connected with this generalization, in particular the problem of simple reducibility, has been sketched in an earlier note \(^1\), and the detailed proofs of the results stated there have been obtained \(^2\).

The purpose of the present note is to show that this generalization is a canonical resolution of the multiplicity problem, explicitly for all SU\(_n\) and thereby implicitly for all other groups in question by imbedding in SU\(_{\infty}\)\(^3\). The method is especially interesting in that it demonstrates a new significance for the Racah coefficients, and shows the existence of intriguing continuum limit properties for the generalized Racah and Wigner coefficients. Proofs of the assertions to be made below will be given elsewhere, but the structure of the results to be presented is rather elegant and should be easily accessible.

The basic idea in I is to focus attention upon the Wigner operator (the general SU\(_n\) unit tensor operator) which could be uniquely characterized by two Gelfand patterns \(^2\) sharing the same representation labels in U\(_n\). Explicitly one has the Wigner operator \(\langle A \mid \alpha \rangle\) where \([A]\) denotes Young pattern labels \([p\ q\ r\ ...]\), \((\alpha)\) denotes the Gelfand pattern specifying the tensor operator properties (magnetic quantum numbers, ...), and lastly \((a)\) specifies a second Gelfand pattern that explicitly gives the changes in the representation labels induced by the operator. (For example in SU\(_2\), the tensor operator \(\langle p\alpha \mid a\rangle\) transforms under rotations as \(J = p/2, M = \alpha - p/2\) and as an operator induces the change \(\Delta J = a - p/2\) when acting on a given angular momentum state vector.)

A second basic result already obtained \(^2\) is the explicit algebraic form of the fundamental Wigner operators of all SU\(_n\). (The fundamental operator for SU\(_2\) is the spin \(\frac{1}{2}\) operator; and analogously for SU\(_n\).)
Wigner operators appear in two aspects: as unit tensor operators and as coupling coefficients for the group. Using this second property, one may now use the fundamental Wigner coefficients to couple two Wigner operators, that is

\[
\left[ \begin{array}{c} \langle A \rangle \\ \langle A' \rangle \end{array} \right] \times \left[ \begin{array}{c} \langle B \rangle \\ \langle B' \rangle \end{array} \right] \left[ \begin{array}{c} \langle C \rangle \\ \langle C' \rangle \end{array} \right] = \sum \langle C \mid \langle A \rangle \rangle \sum \langle C' \mid \langle A' \rangle \rangle,
\]

where \( \langle A \rangle \) is the fundamental Wigner operator, \( \langle B \rangle \) an arbitrary Wigner operator and the \( \langle C \rangle \) is a unique coupling of the lower Gelfand patterns specified by the fundamental Wigner coefficient in its coupling aspect.

The essential points about Eq. (1) are that

1) the right-hand side consists of several Wigner operators having the same tensor operator properties \((C, J)\) but different upper Gelfand patterns;
2) the coefficients \( \hat{W} \) are (generalized) Racah coefficients.

It is here that the suggestive power of notation makes a contribution: the notation suggests that a unique result for the right-hand side of Eq. (1) would result from the use of "coupling coefficients for upper Gelfand patterns" in direct analogy to the use of Wigner coefficients to couple lower Gelfand patterns.

Using the known results for angular momentum as a model, one sees that it is the Racah coefficients themselves which act as "Wigner coefficients" in coupling upper Gelfand patterns. This establishes a new significance to the generalized Racah coefficients: as the SU\(_2\) example shows already, the group space of upper Gelfand patterns is now, however, a discrete lattice space.
Our discussion so far has been highly implicit and suggestive since aside from the fundamental Wigner coefficients in $SU_n$ (and of course everything in $SU_2$) - the general Wigner and Racah coefficients have not been explicitly defined. To close this gap, let us note that a recursive procedure suffices - knowledge of the $SU_2$ fundamental (spin $\frac{1}{2}$) Racah coefficients suffices to give explicitly a general recursion formula for all $SU_3$ Wigner coefficients and thereby all $SU_3$ Racah coefficients (inner products of four Wigner coefficients). To see this, note that the $SU_2$ Racah coefficient itself allows a (partial) coupling on upper Gelfand patterns - the product $\otimes$, hence Eq. (1) takes the form

$$\langle A \rangle \otimes \langle B \rangle = \# \langle C \rangle,$$

(2)

where now $\#$ denotes a reduced Racah coefficient. That this is a reasonable result follows from noting that no upper pattern coupling ($SU_1$) is necessary in the analogous $SU_2$ form of Eq. (2).

The above result completely determines, recursively, all Wigner and Racah coefficients in $SU_n$. That this is so follows from the fact that Eq. (2) is a recursive definition for $SU_3$ Wigner coefficients (the normalization specifies the value of the reduced Racah coefficient). This, hence, determines the $SU_3$ Racah coefficients, which, in turn, imply a recursive form for $SU_4$ Wigner coefficients, $\ldots$. To be sure, special methods can abbreviate this process but logically this defines everything uniquely.

Let us now indicate why such a construction is canonical. Consider the limit of large quantum numbers. In this (continuum) limit, the general $SU_n$ Wigner operators become particular $SU_n$ representation matrices, and the $SU_n$ Racah coefficients become $SU_n$ Wigner operators. The product construction,
Eq. (2), then becomes, in this limit, the reduction of the direct product of representation matrices. This limit for the basic recursion relation involves only the fundamental Wigner coefficients which are unique. It is the existence of this unique continuum limit which characterizes the proposed generalization of the Wigner–Racah calculus; it is this which we feel justifies our designating the proposed generalization as canonical.

Detailed proofs of the assertions made above will be submitted for publication elsewhere. Thanks are due to Dr. G.E. Baird for his many contributions in developing these ideas, and to Professor G. Racah for a helpful correspondence.
REFERENCES


3) Previous work has shown this explicitly only for SU_3, and then by a rather difficult method (the conjugation classification).

4) This fact accounts for the extra parameters in the Racah coefficient (as contrasted to the Wigner coefficient).

5) The orthonormality of the Wigner operators is now easily seen: coupling either upper or lower Gelfand patterns to the invariant operator \( \bigl( 0 \ldots 0 \bigr) \) implies a delta function in the other pattern space.