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ON THE METHOD OF DASHEN AND GELL-MANN

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A B S T R A C T

It is demonstrated that the symmetry scheme proposed by Dashen and Gell-Mann contains an internal contradiction. In particular, it is shown that the assumption that the hybrid collinear group $SU(6)_W$ leaves the subspace of one-particle states with appropriate momentum invariant, implies that the entire group $U(6) \otimes U(6)$ commutes with the Hamiltonian.

The contradiction disappears if we sacrifice the identification of the generators of $U(6) \otimes U(6)$ with the integrals of current components. None of the desirable consequences of the theory are lost by such a retreat.

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1. Dashen and Gell-Mann have recently proposed ¹⁾ an ingenious method for obtaining many of the desirable consequences of $SU(6)$ invariance while avoiding the notorious difficulties ²⁾ of a fully relativistic $SU(6)$ invariant theory. They begin by considering the group isomorphic to $SU(6) \otimes SU(6)$, generated by the space integrals of the positive-parity components of vector, axial vector, scalar, and tensor currents, and assuming that this group has the property of turning one-particle states at rest into one-particle states at rest. The one-particle rest states then form representations of this group. If all the members of a representation have the same mass, it is possible to calculate the matrix elements of the group between moving states. They find that there is a subgroup of the original group, dependent on the direction of motion, whose matrix elements have the same value for states in motion as they have for states at rest. Dashen and Gell-Mann assume that this group, called the hybrid collinear group ³⁾, turns one-particle states with the appropriate direction of motion into one-particle states. They phrase this by saying that the other elements of the original group "leak" while the elements of the collinear group do not. The collinear group is the symmetry group of processes in which all momenta are aligned. For processes in which there are two independent directions of motion, the appropriate group is the intersection of the two collinear groups, a still smaller group called the hybrid coplanar group. The only group that is a symmetry of the Hamiltonian in the usual sense is the intersection of all the collinear groups for all directions. This is just $SU(3)$.

The purpose of this note is to prove that if the hybrid collinear group does not leak, it commutes with the Hamiltonian. Thus the Dashen-Gell-Mann theory contains an internal contradiction. The argument is stable, in the sense that the contradiction persists if we allow small leakage.

2.

This does not mean that we must sacrifice the idea of a diminishing chain of symmetry groups. The argument depends on the identification of the group generators with integrals of local currents ; only this need be abandoned. This is a minimal retreat : none of the agreements of the theory with experiment depend on this identification (with the possible exception of the ratio $G_A/G_V = 5/3$).

2. We now proceed with the argument. Our assumption is that there is no leakage for the elements of the collinear hybrid group. Since the integral of the z component of the axial-vector current is in this group for motion in the z direction, this means

$$\langle n | \int A_z(x, t) d^3x | 1 \rangle = 0, \quad (1)$$

where $|1\rangle$ is a one-particle state moving in the z direction, and $|n\rangle$ is a many-particle state. Let $|n\rangle$ be a state with the same three momentum as $|1\rangle$. Then Eq. (1) is equivalent to

$$\langle n | A_z(x) | 1 \rangle = 0, \quad (2)$$

which, in turn, is the same as

$$\langle n | (\Box^2 A_\mu - \partial_\mu \partial^\nu A_\nu) P^\mu | 1 \rangle = 0, \quad (3)$$

where P_μ is the total four-momentum operator. If this equation is true in any one Lorentz frame, it is true in all Lorentz frames. If $|n\rangle$ and $|1\rangle$ are any one particle momentum eigenstate and any many particle momentum eigenstate, such that their momentum difference is

timelike, there exists a Lorentz transformation that transforms them into states whose three-momentum difference is zero and whose common three-momentum points in the z direction. Thus Eq. (3) is valid for any such pair of states.

To proceed, we need an identity ⁴⁾ which expresses the fact that the interaction between distant particles goes to zero as the particles become infinitely separated, as does the interaction between particles and fields. Let a^{in} be the annihilation operator for an incoming particle in some normalizable state and let $a^{\text{in}}(\lambda)$ be defined by

$$a^{\text{in}}(\lambda) = \exp(i P_z \lambda) a^{\text{in}} \exp(-i P_z \lambda). \quad (4)$$

Likewise, let b^{out} be the annihilation operator for an outgoing particle in some other normalizable state, and let $b^{\text{out}}(\lambda)$ be defined in the same way. Let $|\alpha\rangle$ and $|\beta\rangle$ be any two normalizable states, and let $A(x)$ be any local operator. Then

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \langle \alpha | b^{\text{out}}(\lambda) A(x) a^{\text{in}}(\lambda)^\dagger | \beta \rangle \\ = \langle \alpha | A(x) | \beta \rangle \langle 0 | b^{\text{out}} a^{\text{in} \dagger} | 0 \rangle. \end{aligned} \quad (5)$$

Now let $|\alpha\rangle$ be a normalizable state orthogonal to the vacuum, such that, when decomposed into a sum of four-momentum eigenstates, it only contains states whose four momenta lie within a small neighbourhood. (We will specify the size of the neighbourhood shortly.) Likewise, let a^{in} and a^{out} be annihilation operators for incoming and outgoing one-particle states subject to the same restriction. Then it is clear that it is possible to choose the neighbourhoods such that all possible momentum transfers between $a^{\text{out} \dagger} |\alpha\rangle$ and $a^{\text{in} \dagger} |0\rangle$ are timelike. Therefore, as a consequence of Eq. (3),

4.

$$\begin{aligned} \langle \alpha | \bar{a}^{\text{out}}(\lambda) (\Box^2 A_\mu - \partial_\mu \partial^\nu A_\nu) [P^\mu, a^{\text{in}}(\lambda)^\dagger] | 0 \rangle \\ = 0. \end{aligned} \quad (6)$$

The four operators $[P_\mu, a^{\text{in}}(\lambda)^\dagger]$ are creation operators for normalizable states, and therefore we may apply the identity (5). Taking the limit, we find

$$\langle \alpha | (\Box^2 A_\mu - \partial_\mu \partial^\nu A_\nu) | 0 \rangle \langle 0 | \bar{a}^{\text{out}} P^\mu a^{\text{in} \dagger} | 0 \rangle = 0. \quad (7)$$

By suitably varying a^{in} , we may let the second term range through an open set in four space. Therefore,

$$\langle \alpha | (\Box^2 A_\mu - \partial_\mu \partial^\nu A_\nu) | 0 \rangle = 0. \quad (8)$$

Taking the curl, we find,

$$\langle \alpha | \Box^2 (\partial_\mu A_\nu - \partial_\nu A_\mu) | 0 \rangle = 0, \quad (9)$$

which is equivalent to

$$\langle \alpha | (\partial_\mu A_\nu - \partial_\nu A_\mu) | 0 \rangle = 0. \quad (10)$$

However, the $|\alpha\rangle$'s form a complete (in fact, overcomplete) set of states. Thus,

$$(\partial_\mu A_\nu - \partial_\nu A_\mu) | 0 \rangle = 0. \quad (11)$$

It is known ⁵⁾ that any local operator which annihilates the vacuum is zero ; therefore,

$$\partial_\mu A_\nu - \partial_\nu A_\mu = 0. \quad (12)$$

This implies that

$$\int A_z(x, t) d^3x$$

is independent of time, i.e., commutes with the Hamiltonian.

This completes the argument. Similar arguments may be constructed for the other elements of the collinear group, and therefore for the entire group $U(6) \otimes U(6)$.

3. The argument presented above shows that if there is no leakage (i.e., if Eq. (1) is exact), then the symmetry is exact, and therefore there is an internal contradiction in the Dashen-Gell-Mann scheme. However, it involves an analytic continuation [in going from Eq. (11) to Eq. (12)]. We know that the continuation of a function small in one region can be large in another, and therefore the argument does not settle the question of whether there is a contradiction if Eq. (1) is only approximately valid. We will now show that there is a contradiction even in this case.

Let us consider the matrix element of the axial-vector current between the vacuum and a state of one nucleon and one antinucleon. This may be expressed in terms of two invariant form factors,

$$\begin{aligned} & \langle N\bar{N} | A_\mu(x) | 0 \rangle \\ &= \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} [\bar{u}' \gamma_5 u F_P(k^2) + \bar{u}' \gamma_5 \gamma_\mu u F_A(k^2)], \quad (13) \end{aligned}$$

where u and u' are appropriate spinors, and where we choose to normalize our currents such that $G_V = 1$. Equation (11) is derived from Eq. (1) without analytic continuations. If we place a nucleon-antinucleon pair state on the left, it tells us that F_A vanishes for $k^2 \geq 4m_N^2$. If we place a three-pion state on the left, and exploit unitarity, it tells us that F_A vanishes for $k^2 \geq 9m_\pi^2$. If Eq. (1) is only approximately true, these relations should also be only approximately true. Let us give a precise meaning to "approximately true", and assume that $|F_A|$ is bounded above ⁶⁾ by some small number ϵ :

$$|F_A| < \epsilon \quad \text{for} \quad k^2 \geq 9m_\pi^2. \quad (14)$$

Now, it is known that F_A is an analytic function of k^2 in the entire complex plane, except for a cut extending from $9m_\pi^2$ to infinity along the positive real axis. It is also known that F_A has no essential singularities at infinity. Therefore, applying the maximum-modulus principle to the Riemann sphere, we find

$$F_A(0) < \epsilon \quad (15)$$

But it is known that the Dashen-Gell-Mann assumptions lead to the famous relation

$$F_A(0) = 5/3. \quad (16)$$

Equations (15) and (16) are not consistent unless $\epsilon \geq 5/3$, which is not in the usual sense of the word approximate, "approximately zero leakage".

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REFERENCES AND FOOTNOTES

- 1) R. Dashen and M. Gell-Mann, Phys.Letters 17, 142, 145 (1965).
In so far as it applies to particles at rest, the theory was developed independently by B.W. Lee, Phys.Rev.Letters 14, 676 (1965).
- 2) S. Coleman, Phys.Rev. 138, B1262 (1965).
- 3) This is the $SU(6)_W$ group, discovered in another context by H. Lipkin and S. Meshkov, Phys.Rev.Letters 14, 670 (1965).
- 4) D.W. Robinson assures me that this identity can be proved by the usual methods of axiomatic field theory, provided that the annihilation operators are associated with quasi-local states. This is the case in our application.
- 5) P. Federbusch and K. Johnson, Phys.Rev. 120, 1926 (1960).
- 6) We choose a constant bound only for convenience. A similar contradiction could be obtained with any polynomial bound.