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SOLUTIONS OF SINGULAR N/D EQUATIONSII. APPLICATIONS

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A B S T R A C T

The exact solutions of the singular N/D equations that arise from the assumption of saturated unitarity are applied to three problems of current interest in the physics of strong interactions.

In the first, related to the bootstrap calculation of the parameters of a vector meson, an exact solution for the undamped vector exchange between scalars is given. This solution contains an arbitrary parameter implying that, in general, either the vector - scalar mass ratio or coupling constant must be given a priori. However, with an extra requirement (which is supported by a field-theoretic argument) a solution completely free of arbitrary parameters is singled out.

In the second, the basic equations and conclusions of the new strip approximation of Chew are rederived by means of the authors' formalism: this gives the resolvent of the basic singular integral equation in the form of a single contour integral which allows a direct study of certain of its analytic and asymptotic properties. A physical assumption on the phase shift is shown to specify a unique solution.

The third problem is the investigation of the Gribov-Pomeranchuk singularity. The continuation of a scattering amplitude to $J = -1$ is defined by an N/D equation and it is shown that: a) there is an accumulation of poles at $J = -1$, b) there is also a branch point, and c) the amplitude remains bounded for physical energies, so that continued unitarity is not violated.

1. INTRODUCTION

In a previous paper by the authors ¹⁾ [hereafter referred to as (I)], the solutions of certain singular integral equations were given. These arose from the N/D decomposition of an elastic partial wave amplitude $A_J(\nu)$, in which the asymptotic behaviour was as large as is consistent with unitarity. Certain properties of the resolvent kernel were discussed in (I).

The purpose of the present work is to apply the results of (I) to certain problems of current interest in strong interaction physics.

In the first application (Section 2), the P wave amplitude of pseudoscalar-pseudoscalar scattering is considered, the main force being assumed to be due to the exchange of a vector meson in the crossed channel. This was the first application of the bootstrap idea ²⁾, recently extended to the study of strong interaction symmetries ^{3),4)}.

As is shown in Section 2 and elsewhere ⁵⁾, the undamped exchange of a vector meson leads to a singular integral equation for N (or for D). The usual treatment in the literature : i) approximates the exact solution of this equation by its first iteration ⁴⁾, and ii) often introduces an arbitrary finite cut-off, which changes the singular into a Fredholm equation ⁶⁾. Physically, the cut-off is introduced to approximate a high energy mechanism that damps out the one-particle exchange. However, since certain low energy features of a scattering amplitude depend critically upon its asymptotic behaviour, and since it is by no means certain that the vector exchange is in fact damped, it is of interest to consider vector exchange exactly.

In Section 2, a cut-off free solution is given, based on an explicit expression for the resolvent of the singular kernel. A similar solution could be obtained in the approximation $\nu_L = 0$ [Section 3 of (I)], i.e., with the gap between the left- and right-hand cuts contracted to zero.

However, there are some important differences between $\nu_L = 0$ and $\nu_L \neq 0$ relevant to the present application. As it was shown in (I), the singular integral equation in this approximation has two independent homogeneous solutions; therefore, its general solution introduces two arbitrary constants. In contrast, the general solution given in Section 2 of this paper arises from a singular equation with only one homogeneous solution and thus introduces only one arbitrary constant. Moreover, the approximation $\nu_L = 0$ is expected to seriously distort some of the low energy features of the solution. Furthermore, the present treatment shows that a solution can be obtained without introducing any more parameters: in particular, no CDD pole need be introduced in the D function. Lastly, it is shown that, by requiring that the solution be analytic in the pseudoscalar-vector meson coupling constant at the origin, a parameter-free solution can be singled out from the manifold.

The second application (Section 3) is the Reggeized strip approximation ^{7),8)}. A set of approximate dynamical equations, based on the strip concept, is given for an amplitude that has Regge asymptotic behaviour.

This method, the basic steps of which are repeated in Section 3, leads to a system of two coupled integral equations, one with a singular and one with a Fredholm kernel. By means of Wiener-Hopf techniques ⁹⁾, Chew has given the resolvent of the singular kernel as a double Fourier transform, involving two contour integrations ⁸⁾. In Section 3, the same singular equation is written as one of the standard forms treated in (I), so that the resolvent can be written as a single contour integral. With an appropriate choice of contour, it is shown that this resolvent is equivalent to Chew's.

The last application (Section 4) is an investigation of the nature of the singularity of a continued partial wave amplitude at $J = -1$.

On the basis of partial wave dispersion relations, Gribov and Pomeranchuk have shown that, if an analytic continuation can be made to the neighbourhood of a negative integer, there must be a singularity there

of a particular type ¹⁰⁾. This is due to the fact that the presence of a third double spectral function introduces fixed (i.e., energy independent) poles in the left-hand discontinuity $\text{Im} A_J(\nu)$, at $J = -1, -2, \dots$. However, the unitarity condition (for any ν in the elastic region) can be continued in J into any region where the amplitude itself can be continued; and it is inconsistent with any fixed singularity at which A_J becomes unbounded.

A possible solution of the difficulty is that as A_J is continued towards $J = -1$ (say) an infinite number of poles moves through the right-hand cut on to the physical sheet of ν . In the J plane these poles accumulate at $J = -1$, providing a mechanism whereby $A_J(\nu)$ can remain bounded, for physical ν . In this way, however, A_J appears to have an essential singularity at $J = -1$ (and similarly at the other negative integers).

An accumulation point of poles at $J = -1$ is also exhibited by the Bethe-Salpeter partial wave amplitude in the ϕ^3 theory, provided that the corresponding kernel is defined in terms of a non-planar Feynman graph (i.e., one with crossed lines) ^{11), 12)}.

On the other hand, Squires ¹³⁾ has examined the nature of the singularity at $J = -1$, by means of an approximate solution of an N/D equation, in which the left-hand discontinuity is assumed to be proportional to $|\nu|^{-\frac{1}{2}}(J-J_0)^{-1}$. He finds that the solution has a ν dependent branch point at $J = -1$, but no essential singularity.

In Section 4 of the present paper the method of Squires is used, but employing the exact resolvents of (I). His finding that there is a branch point at $J = -1$ is explicitly verified; but it is also shown that a certain residual Fredholm kernel that Squires neglects contributes an accumulation of poles. Hence, there is an essential singularity superimposed upon the branch point. It is demonstrated, by referring to certain results in (I), that $A_J(\nu)$ remains bounded for physical ν .

2. VECTOR BOSON EXCHANGE AND BOOTSTRAPS

Consider a simple bootstrap calculation of the parameters (mass and coupling constant) of a vector boson. In this, the N/D equations for scattering in the P state of the s channel are written under the assumption that the main "force" arises from the exchange of an undamped vector meson in the crossed t channel. An example is $\pi-\pi$ elastic scattering with the exchange of a ρ meson (although in this case the S wave interaction in the t channel is not negligible). In what follows, possible isospin dependence will not be displayed explicitly.

Suppose that

$$A_1(\nu) = N(\nu) / D(\nu) \quad (2.1)$$

is the partial P wave amplitude for the elastic scattering of two scalar or pseudoscalar mesons of mass 1. The variable ν ($=$ square of the c.m. momentum) is related to the square of the invariant energy s by :

$$s = 4(\nu + 1)$$

A once-subtracted representation for N and D [as in Eq. (2.1b) of (I)] leads to the following integral equation :

$$N(\nu) = N(\nu_0) + F(\nu) - F(\nu_0) + \frac{\nu - \nu_0}{\pi} \int d\nu' \rho(\nu') \frac{F(\nu') - F(\nu)}{\nu' - \nu} \frac{N(\nu')}{\nu' - \nu_0} \quad (2.2)$$

where $\rho(\nu)$ is the usual phase-space factor and $F(\nu)$ the contribution of the left-hand cut, given explicitly below. However, $A_1(\nu)$ has a zero at the physical threshold $\nu = 0$, hence, by taking $\nu_0 = 0$ one has :

$$N(\nu_0) = N(0) = 0 \quad (2.3)$$

The function $F(\nu)$ represents the "force" responsible for the interaction. According to the assumption, this is predominantly due to the exchange of a vector meson in the t channel and is given by

$$F(\nu) = \frac{\lambda}{4\pi} \frac{8\nu + 4 + m^2}{\nu} Q_1\left(1 + \frac{m^2}{2\nu}\right) \quad (2.4)$$

Here λ is the product of the square of the vector-scalar meson coupling constant and the relevant crossing matrix element, m is the mass of the vector meson and Q_1 a Legendre function of the second kind. Using the fact that $Q_l(z) \rightarrow z^{-l-1}$ for $z \rightarrow \infty$, one easily sees that also

$$F(\nu_0) = F(0) = 0 \quad (2.5)$$

On the other hand :

$$Q_1(z) = \frac{z}{2} \log \frac{z+1}{z-1} - 1 \quad (2.6)$$

thus, for $|\nu| \rightarrow \infty$: $Q_1(1 + m^2/2\nu) \rightarrow \frac{1}{2} \log \nu$, which implies that the kernel of (2.2) is singular. Finally, it follows from (2.6) that $Q_1(z)$ has a branch cut $-1 \leq z \leq 1$ (i.e., $-\infty < \nu \leq -m^2/4$): therefore :

$$F(\nu) = \frac{\lambda}{\pi} \log(\nu + m^2/4) + O(1) \quad (2.7)$$

The singular (and symmetric) part of (2.2) can be singled out. Firstly, the phase-space factor is expanded :

$$\rho(\nu) = \left(\frac{\nu}{\nu+1}\right)^{1/2} = 1 + O\left(\frac{1}{\nu}\right) \quad (2.8)$$

Secondly, a new variable x and a new unknown function $f(x)$ are introduced :

$$x = 4\nu/m^2 + 1 \qquad f(x) = \frac{\pi}{\lambda} \frac{N(\nu)}{\nu} \qquad (2.9)$$

Then, by inserting in (2.2) only the leading terms of (2.7) and (2.8), and by making use of (2.3) and (2.5), one obtains the following (symmetric) singular integral equation :

$$f(x) = \frac{\log x}{x} + \frac{\lambda}{\pi^2} \int_1^\infty dx' \frac{\log x'/x}{x' - x} f(x) \qquad (2.10)$$

The terms neglected contribute a non-singular (Fredholm) part. It is possible to give a treatment of the complete integral equation for $f(x)$ (i.e., including the Fredholm part) by splitting (2.2) into two coupled integral equations, the one singular and the other Fredholm, as in Section 2 of (I). However, here it will be sufficient to neglect the non-singular part, since the purpose is simply to investigate the consequences of admitting a singular kernel. As has been demonstrated in Section 5 of (I), the essential features of the solution are undisturbed by the presence of the Fredholm part.

Equation (2.10) has precisely the form (4.1b) of (I), in which the inhomogeneous term is $f_0(x) = \frac{1}{x} \log x$. In (I) it was shown that a one-parameter infinity of solutions exists, which can be written in the form

$$f(x) = \frac{\log x}{x} + \lambda \int_1^\infty dx' R(x, x'; \lambda) \frac{\log x'}{x'} + \qquad (2.11)$$

$$+ A P_{-s_0}(2x-1)$$

where

$$s_0 = \frac{1}{\pi} \arcsin \sqrt{\lambda} \qquad (0 < \arcsin \sqrt{\lambda} < \frac{\pi}{2})$$

and the resolvent kernel, R , is

$$R(x, x'; \lambda) = \frac{1}{2} \int_C ds \frac{s \tanh \pi s}{\cosh^2 \pi s - \lambda} P_{-\frac{1}{2}+is}(2x-1) P_{-\frac{1}{2}+is}(2x'-1) \quad (2.12)$$

Possible choices of the contour C are shown in Fig. 1. The term $P_{-s_0}(2x-1)$ in (2.11) is the general homogeneous solution of (2.10) and A is an arbitrary multiplicative constant.

Now, in a vector-meson bootstrap calculation, the procedure will be to solve (2.10) for a particular coupling constant λ and vector meson mass m . Next, $D(\nu)$ will be calculated in terms of $f(x)$ or $N(\nu)$ from

$$D(\nu) = 1 - \frac{\nu}{\pi} \int_0^\infty d\nu' \rho(\nu') \frac{N(\nu')}{\nu'(\nu' - \nu)}$$

The vector particle should appear in the s channel as a zero of $D(\nu)$. Then, according to the bootstrap principle, the position and residue of the pole of N/D should match the mass and coupling constant of the exchanged vector meson. However, since the multiplicative constant A is indeterminate, it appears that, in general, m and λ cannot both be bootstrapped (i.e., that the value of one will have to be taken a priori).

It is possible, however, to single out a particular solution of (2.11) by imposing an additional restriction. As discussed in Section 5 of (I), the homogeneous solution $P_{-s_0}(2x-1)$ has branch points at both $\lambda = 0$ and $\lambda = 1$. On the other hand, by choosing in (2.12) the contour $C \equiv C_0$ (Fig. 1) the resolvent $R(x, x'; \lambda)$ was found to have only one finite branch point, at $\lambda = 1$; the same will be true for $f(x)$ if the

homogeneous solution in (2.11) is neglected (i.e., take $A=0$). Thus, by requiring that $f(x)$ be analytic in λ at $\lambda = 0$, one may specify a unique, parameter-free solution. Clearly, for $|\lambda| < 1$, this solution can be expanded in powers of λ . From the point of view of S matrix theory, this is a rather ad hoc requirement; however, from the conventional field theory point of view, it would correspond to supposing that the solution has certain features characteristic of renormalized interactions.

The requirement of the possibility of expanding $f(x)$ in powers of λ near $\lambda = 0$ could be argued on the following field theoretic grounds ¹⁴⁾: in general, exchange of a massive vector boson is described by a propagator $(g_{\mu\nu} - (q_\mu q_\nu / m^2)) / (q^2 - m^2)$, where q and m are the four-momentum and the mass of the vector boson. In the Born approximation of the scattering of two pseudoscalar mesons via exchange of a vector boson, the four-momenta of the scattered mesons are on the mass shell; as a result, the term $q_\mu q_\nu / m^2$ of the propagator does not contribute. In this respect, then, exchange of a massive vector boson does not differ from exchange of a photon. Furthermore, in an approach based on the N/D equations and on the assumption that the force arises from the left-hand cut discontinuity due to the Born term alone, the term $q_\mu q_\nu / m^2$ is not introduced in any further stage. Thus, in this respect, it makes no difference whether one has pseudoscalar mesons interacting through exchange of a massive vector boson, or through a photon. However, the photon-pseudoscalar meson interaction is known to lead to a renormalizable scattering theory, with solutions expandable in powers of the coupling constant at the origin.

As was mentioned in the introduction, in certain bootstrap calculations ⁴⁾ only one iteration of the N/D equations is completed, in the hope that this gives a reasonable approximation to the complete solution. Since an iterative method gives a series expansion in λ , this can only be a reasonable approximation to a solution which is regular at $\lambda = 0$. Hence, if one imposes the requirement of analyticity of $f(x)$ in the origin of λ and if the coupling constant $\lambda = \lambda_0$ which satisfies

the bootstrap is relatively small, it is expected that a calculation using the exact solution $f(x)$ will give results at least in qualitative agreement with those using approximate N/D iterative solutions.

Moreover, in some cases the vector exchange has been cut off ⁶⁾, which has the effect of replacing the singular by a Fredholm kernel, and hence substituting a sequence of poles for the cut in the λ plane. Again this procedure can be expected to be a reasonable approximation to the exact solution only for $\lambda_0 \ll 1$.

The problem of vector boson exchange and the singular integral equation to which it leads have also been treated by Dietz and Domokos ⁵⁾ in the approximation $\nu_L = 0$, with special interest in the asymptotic ($\nu \rightarrow \infty$) part of the solution. The same problem (but for $\nu_L \neq 0$) has also been studied by Morgan ¹⁵⁾, who applies Wiener-Hopf techniques ⁹⁾ and discusses the general solution with branch points at both $\lambda = 0$ and $\lambda = 1$, as well as certain of its physical implications.

3. THE NEW STRIP APPROXIMATION OF CHEW

In a series of papers ^{7),8),17)} Chew and his co-workers have developed a new set of dynamical equations based on a modification of the N/D method, with use of the strip concept. In this, only a finite part of the right-hand cut is apportioned to D. The N function carries the left-hand cut and the remainder of the right-hand cut, the assumption being that the high energy region is dominated by a (small) number of Regge poles.

The purpose of this Section is to rederive the basic results of Refs. ^{7),8)}, using the formulae developed in (I). To make the present work self-contained, the basic steps leading to the N/D equations, and some of the most important concepts connected with this new strip approximation, will be redeveloped within the formalism of (I).

In what follows, it will be required to make a continuation into complex angular momentum, J . Although the N/D equations give a non-Fredholm equation for N , it will be assumed that by continuing the singular kernel in J one obtains the continuation of the solution. However, near the physical threshold $\nu = 0$, the partial wave amplitude $A_J(\nu)$ behaves like ν^J ; and for complex J it develops an additional branch point of a kinematic type ¹⁶⁾. To remove it, it is necessary to divide the amplitude by ν^J , defining :

$$A_J(\nu) / \nu^J \equiv a_J(\nu) = N_J(\nu) / D_J(\nu) \quad (3.1)$$

Then the coupled integral equations for N_J and D_J are :

$$\begin{aligned} N_J(\nu) &= \frac{1}{\pi} \left[\int_{-\infty}^{-1} + \int_{\nu_1}^{\infty} \right] d\nu' \frac{\text{Im } a_J(\nu') D_J(\nu')}{\nu' - \nu} \\ D_J(\nu) &= 1 - \frac{1}{\pi} \int_0^{\nu_1} d\nu' \frac{\nu'^J \rho(\nu')}{\nu' - \nu} N_J(\nu') \end{aligned} \quad (3.2)$$

with $\rho(\nu) = [\nu/(\nu+1)]^{\frac{1}{2}}$. Here the high energy region which has been allocated to N is $\nu_1 \leq \nu < \infty$. Upon defining

$$B_J(\nu) = \frac{1}{\pi} \left[\int_{-\infty}^{\nu_1} + \int_{\nu_1}^{\infty} \right] d\nu' \frac{\text{Im } a_J(\nu')}{\nu' - \nu} \quad (3.3)$$

(3.2) implies the following integral equation :

$$N_J(\nu) = B_J(\nu) + \frac{1}{\pi} \int_0^{\nu_1} d\nu' \frac{B_J(\nu') - B_J(\nu)}{\nu' - \nu} \nu'^{\frac{1}{2}} \rho(\nu') N_J(\nu') \quad (3.4)$$

From (3.3) it can be seen that $B_J(\nu)$ is analytic on the interval $0 \leq \nu \leq \nu_1$, except for a logarithmic branch point at $\nu = \nu_1$. One has in fact

$$B_J(\nu) = -\frac{\lambda_J}{\pi \nu_1^{\frac{1}{2}} \rho(\nu_1)} \log(\nu_1 - \nu) + H_J(\nu) \quad (3.5)$$

where $H_J(\nu)$ is analytic, and therefore bounded and differentiable, on $0 \leq \nu \leq \nu_1$, and

$$\lambda_J = \rho(\nu_1) \text{Im } A_J(\nu_1)$$

Substituting (3.5) into the kernel of (3.4), one has

$$\begin{aligned} N_J(\nu) = B_J(\nu) - \frac{\lambda_J}{\pi^2} \int_0^{\nu_1} d\nu' \frac{\log[(\nu_1 - \nu')/(\nu_1 - \nu)]}{\nu' - \nu} N_J(\nu') \\ + \frac{\lambda_J}{\pi^2} \int_0^{\nu_1} d\nu' F_J(\nu, \nu') N_J(\nu') \end{aligned} \quad (3.6)$$

it can be easily seen that $F_J(\nu, \nu')$ is a Fredholm kernel. Then, with the substitution $x = \nu_1/(\nu_1 - \nu)$ and the definitions :

$$f_J(x) = \frac{\nu_1 - \nu}{\nu_1} N_J(\nu) \quad b_J(x) = \frac{\nu_1 - \nu}{\nu_1} B_J(\nu) \quad (3.7)$$

one transforms (3.6) into the following system of coupled equations :

$$f_J(x) = f_J^0(x) + \frac{\lambda_J}{\pi^2} \int_1^\infty dx' \frac{\log x'/x}{x' - x} f_J(x') \quad (3.8)$$

$$f_J^0(x) = b_J(x) + \frac{\lambda_J}{\pi^2} \int_1^\infty dx' K_J(x, x') f_J(x') \quad (3.9)$$

In (3.9), $K_J(x, x')$ is the transform of $F_J(\nu, \nu')$. Now, (3.8) has precisely the form (4.1b) of (I), so that it can be solved by the Mehler transform method given in that paper. As shown in (I), the general resolvent of (3.8) must have the form :

$$R(x, x'; \lambda_J) = R(x, x'; \lambda_J) + \frac{P_{-\frac{1}{2}+p_0}(2x-1)F(x'; \lambda_J)}{-\frac{1}{2}+p_0} \quad (3.10)$$

where $R(x, x'; \lambda_J)$ is the resolvent given in Eq. (4.3b) of (I), which has a cut for $1 \leq \lambda < \infty$, but is analytic at $\lambda = 0$ (contour C of Fig.1); and $P_{-\frac{1}{2}+p_0}(2x-1)$ is a homogeneous solution of (3.8) ($p_0 = \frac{1}{\pi} \arccos \sqrt{\lambda_J}$, $0 < \arccos < \pi/2$).

The first objective is to show that, if one makes the reasonable assumption that for large enough J the modulus of the phase shift

$\delta_J(\nu_1)$ is smaller than $\pi/2$, then for these J one must omit the

homogeneous term of (3.9). In these cases, then, a unique solution $f_J(x)$ is defined, at least in so far as an analytic continuation in J is possible at all. Finally, it will be shown that the resolvent $R(x, x'; \lambda_J)$, which corresponds to this solution, reproduces correctly the basic conclusions of Ref. 7).

For a physical value of J , the decomposition (3.1)-(3.2) implies that $D_J(\nu)$ has a cut ($0 < \nu < \nu_1$); moreover, if the phase of $A_J(\nu)$ is $\delta_J(\nu)$, then $D_J(\nu)$ has the phase $-\delta_J(\nu)$ on this cut. For simplicity, elastic unitarity will be assumed, so that $\delta_J(\nu)$ is also the phase shift. Now, consider the function

$$\mathcal{D}_J(\nu) = \exp\left(-\frac{i}{\pi} \int_0^{\nu_1} \frac{\delta_J(\nu')}{\nu' - \nu} d\nu'\right)$$

which tends to unity as $\nu \rightarrow \infty$. Then, $D_J(\nu)/\mathcal{D}_J(\nu)$ is holomorphic in the finite ν plane and therefore it must be a polynomial. However, from (3.2) $D_J(\nu) \rightarrow 1$ for $\nu \rightarrow \infty$. Thus, it follows that

$$D_J(\nu) \equiv \mathcal{D}_J(\nu)$$

One can therefore write

$$\begin{aligned} D_J(\nu) &= \exp\left(-\frac{\delta_J(\nu_1)}{\pi} \int_0^{\nu_1} \frac{d\nu'}{\nu' - \nu}\right) \\ &\quad \cdot \exp\left(-\frac{i}{\pi} \int_0^{\nu_1} \frac{\delta_J(\nu') - \delta_J(\nu_1)}{\nu' - \nu} d\nu'\right) \\ &= (\nu_1 - \nu)^{-\delta_J(\nu_1)/\pi} \cdot F_D(\nu) \end{aligned}$$

where $F_D(\nu)$ is regular at $\nu = \nu_1$ [for this it is sufficient that $\delta_J(\nu)$ be differentiable at $\nu = \nu_1$]. Since $A_J(\nu)$ does not, in general, have a singularity at ν_1 , one must have also

$$N_J(\nu) = (\nu_1 - \nu)^{-\delta_J(\nu_1)/\pi} \cdot F_N(\nu)$$

where $F_N(\nu)$ is also regular at ν_1 . Thus, for $\nu \rightarrow \nu_1$

$$N_J(\nu) \sim (\nu_1 - \nu)^{-\delta_J(\nu_1)/\pi}$$

More generally, if $A_J(\nu)$ has a zero of order m at $\nu = \nu_1$:

$$N_J(\nu) \sim (\nu_1 - \nu)^{m - \delta_J(\nu_1)/\pi}$$

With the substitution of (3.7), this implies

$$f(x) \sim x^{-m-1 + \delta_J(\nu_1)/\pi} \quad (x \rightarrow \infty) \quad (3.11)$$

Now, it will be shown that by choosing from the manifold (3.10) the resolvent $\bar{R}(x, x'; \lambda_J) = R(x, x'; \lambda_J)$ one obtains a solution $f_J(x)$ compatible with the asymptotic form (3.11). In (I), [Eq. (5.4)], it was shown that $f_J(x)$ satisfies the Kreps form:

$$\begin{aligned} f_J(x) = & b_J(x) + \lambda_J \int_1^\infty dx' R(x, x'; \lambda_J) b_J(x') + \\ & + \frac{\lambda_J}{\pi^2} \int_1^\infty dx' \left[K_J(x, x') + \lambda_J \int_1^\infty dy R(x, y; \lambda_J) K_J(y, x') \right] f_J(x') \end{aligned} \quad (3.12)$$

Next, it was shown [Eq. (5.9) of (I)] that $R(x, x'; \lambda_J)$ satisfies the family of majorizations

$$|R_J(x, x'; \lambda_J)| \leq B x^{-\frac{1}{2}-p} x'^{-\frac{1}{2}+p} \quad (3.13)$$

for any $-p_0 \leq p \leq p_0$ and all $1 \leq x, x' < \infty$. Moreover, (3.5)-(3.7) give

$$b_J(x) < C_1 \log x / x \quad |K_J(x, y)| < C_2 / (x+y)y \quad (3.14)$$

for all $1 \leq x, y < \infty$. From (3.13) and (3.14) it can be concluded that (3.12) is a Fredholm equation for $f_J(x)$, so that one has certainly $|f_J(x)| < C_0 x^{-\frac{1}{2}}$. Inserting this into the right-hand side of (3.12), one has finally

$$|f_J(x)| \leq C x^{-\frac{1}{2} - p_0} \quad (3.15)$$

To compare this with (3.11) note that, according to the definition of in (3.5),

$$\lambda_J = \sin^2 \delta_J(\nu_1) = \cos^2 \pi p_0$$

and since $\delta_J(\nu_1)$ is the (elastic unitary) phase shift, the physically important values of λ_J satisfy $0 < \lambda_J < 1$. Moreover:

$$p_0 = \pm \frac{\delta_J(\nu_1)}{\pi} + \frac{2n+1}{2} \quad (3.16)$$

where n is an integer. Now, it is reasonable to assume that for large enough J $|\delta_J(\nu_1)|$ decreases and for $J > J_0$, say, it satisfies $|\delta_J(\nu_1)| < \pi/2$; Jones¹⁷⁾ supports this assumption with a Regge pole model. Then (3.16) gives

$$\frac{1}{2} - p_0 = |\delta_J(\nu_1)| / \pi < \frac{1}{2} \quad (3.17)$$

With this (3.15) becomes

$$|f_J(x)| \leq C x^{-1 + |\delta_J(\nu_1)| / \pi}$$

which is certainly compatible with the asymptotic behaviour of (3.11).

Now it can be shown that a solution $f_J(x)$ of (3.8)-(3.9) exists even when the general resolvent (3.10) is used¹⁸⁾. Then it can easily be seen that such a solution is inconsistent with (3.11). For $x \rightarrow \infty$, $P_{-\frac{1}{2}+p_0}(2x-1) \sim x^{-\frac{1}{2}+p_0}$, so that, by means of the Kreps form (3.12), this leads to an asymptotic behaviour

$$f_J(x) \sim x^{-\frac{1}{2}+p_0}$$

Use of the assumption $|\delta_J(\nu_1)| < \pi/2$ for $J \geq J_0$ leads in the same way to

$$f_J(x) \sim x^{-|\delta_J(\nu_1)|/\pi}$$

which is certainly inconsistent with (3.11).

In this way, the resolvent $\mathcal{R}(x, x'; \lambda_J) = R(x, x'; \lambda_J)$ is defined for all $\text{Re } J > J_0$. If it is possible at all, the analytic continuation of the solution into the whole J plane will be unique, although there could be cuts. A possible mechanism through which the cuts could be generated has been described in the literature^{19), 20)}. Jones¹⁷⁾, following Chew, invokes the "Principle of Maximal Analyticity of the Second Degree", according to which this unique continuation is supposed to give the physical solutions at the integer values of J .

Finally, the asymptotic properties of $R(x, x'; \lambda_J)$ in x, x' will be compared to those of the resolvent of Ref.⁸⁾. There, the solution of (3.4) was written in the form [Eq. (I.9) of Ref.⁸⁾]:

$$N_J(\nu) = \int_0^{\nu} d\nu' O_J(\nu, \nu') N_J^0(\nu') \quad (3.18)$$

where $N_J^0(\nu)$ was shown to satisfy [Eqs. (I.10), (I.11) of Ref.⁸⁾]:

$$N_J^0(\nu) = B_J(\nu) + \int_0^{\nu} d\nu' \int_0^{\nu'} d\nu'' K_J(\nu, \nu'') O_J(\nu'', \nu') N_J^0(\nu') \quad (3.19)$$

It was concluded [Eq. (II.23) of Ref. ⁸] that :

$$O_J(\nu, \nu') \rightarrow \begin{cases} (\nu_1 - \nu)^{-a} & \text{for } \nu \rightarrow \nu_1 \text{ and } \nu' \text{ fixed} \\ (\nu_1 - \nu')^{-a} & \text{for } \nu' \rightarrow \nu_1 \text{ and } \nu \text{ fixed} \end{cases} \quad (3.20)$$

where

$$a = |\delta_J(\nu_1)| / \pi \quad (3.21)$$

Now, by means of Eqs. (3.6)-(3.9), it follows that the Chew kernel $O_J(\nu, \nu')$ corresponds to

$$\frac{\pi^2}{\lambda_J} \delta(\nu - \nu') + \nu_1 \frac{R\left(\frac{\nu_1}{\nu_1 - \nu}, \frac{\nu_1}{\nu_1 - \nu'}; \lambda_J\right)}{(\nu_1 - \nu)(\nu_1 - \nu')}$$

From (3.13) one has

$$R(x, x'; \lambda_J) \rightarrow \begin{cases} x^{-\frac{1}{2} - p_0} & \text{for } x \rightarrow \infty \text{ and } x' \text{ fixed} \\ x'^{-\frac{1}{2} - p_0} & \text{for } x' \rightarrow \infty \text{ and } x \text{ fixed} \end{cases}$$

Hence

$$\frac{R\left(\frac{\nu_1}{\nu_1 - \nu}, \frac{\nu_1}{\nu_1 - \nu'}; \lambda_J\right)}{(\nu_1 - \nu)(\nu_1 - \nu')} \rightarrow \begin{cases} (\nu_1 - \nu)^{-\frac{1}{2} + p_0} & \text{for } \nu \rightarrow \nu_1 \text{ and } \nu' \text{ fixed} \\ (\nu_1 - \nu')^{-\frac{1}{2} + p_0} & \text{for } \nu' \rightarrow \nu_1 \text{ and } \nu \text{ fixed} \end{cases}$$

Clearly, because of (3.17) and (3.21), this asymptotic behaviour is equivalent to (3.20).

In conclusion, it is perhaps of interest to reiterate that, for the particular resolvent $R(x, x'; \lambda)$, both the Chew form (3.19) and the Kreps form (3.12) are Fredholm equations.

4. THE NATURE OF THE GRIBOV-POMERANCHUK SINGULARITY

In this Section, the continuation of a scattering amplitude to the point $J = -1$ will be considered. As in the last Section, the kinematical singularity at $\nu = 0$ must be factored out, so that one writes

$$A_J(\nu) / \nu^J \equiv a_J(\nu) = N_J(\nu) / D_J(\nu) \quad (4.1)$$

However, here $D_J(\nu)$ will carry the whole of the right-hand cut, $N_J(\nu)$ the left-hand cut.

Both the N and the D equations will be subtracted at the same point $\nu = \nu_0$, $D(\nu_0)$ being normalized to unity, giving

$$\begin{aligned} N_J(\nu) &= N_J(\nu_0) + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{-1} d\nu' \frac{\text{Im } a_J(\nu') D_J(\nu')}{(\nu' - \nu)(\nu' - \nu_0)} \\ D_J(\nu) &= 1 - \frac{\nu - \nu_0}{\pi} \int_0^{\infty} d\nu' \nu'^J \rho(\nu') \frac{N_J(\nu')}{(\nu' - \nu)(\nu' - \nu_0)} \end{aligned} \quad (4.2)$$

The relativistic phase space, $\rho(\nu) = [\nu/(\nu+1)]^{\frac{1}{2}}$, will be used.

By substituting the second equation into the first, one has

$$\psi_J(\nu) = \frac{N_J(\nu_0)}{\nu - \nu_0} + \bar{\Phi}_J(\nu, \nu_0) + \frac{1}{\pi} \int_0^{\infty} d\nu' \nu'^J \rho(\nu') \bar{\Phi}_J(\nu, \nu') \psi_J(\nu') \quad (4.3)$$

where

$$\psi_J(\nu) = N_J(\nu) / (\nu - \nu_0)$$

and

$$\bar{\Phi}_J(\nu, \nu_0) = \frac{1}{\pi} \int_{-\infty}^{-1} d\nu' \frac{\text{Im } a_J(\nu')}{(\nu' - \nu)(\nu' - \nu_0)} \quad (4.4)$$

As was shown in Section 2 of (I), the assumption that unitarity is saturated leads to $A_J(\nu) \rightarrow \text{const}$ for $\nu \rightarrow -\infty$. As before, this saturating behaviour gives a singular integral equation. For simplicity in writing, the discontinuity on the left-hand cut will be taken to be constant (independent of ν); but a ν dependent discontinuity could be treated by the methods of Section 5 of (I); and it will become evident that such a behaviour does not alter any of the considerations of this Section. On the other hand, Gribov and Pomeranchuk have shown that, as a function of J , $\text{Im } A_J(\nu)$, for $-\infty < \nu \leq -\nu_L$, has simple poles at $J = -1, -2, \dots$. Accordingly, in order to investigate $J = -1$, $\text{Im } A_J(\nu)$ will be taken to be just a pole at $J = -1$. This inessential simplification could also be removed by the methods of Section 5 of (I). Finally, then,

$$\text{Im } a_J(\nu) = \frac{g}{J+1} \nu^{-J} \quad (4.5)$$

where g is a constant.

With this choice of $\text{Im } a_J(\nu)$, the integral (4.4) giving $\bar{\Phi}_J(\nu, \nu_0)$ is not defined at $J = -1$. Moreover, in the half-plane $\text{Re } J > -1$ in which it is defined, the equation (4.3) is not Fredholm. Nevertheless, $\bar{\Phi}_J(\nu, \nu_0)$ will be analytically continued to $J = -1$; and it will be assumed that this continuation does in fact give the continuation of $\psi_J(\nu)$. With a change of integration variable $\nu' = -x$, one has formally

$$\bar{\Phi}_J(\nu, \nu_0) = \frac{1}{\nu - \nu_0} \frac{1}{\pi} \frac{g}{J+1} \left[\int_0^\infty - \int_0^1 \right] dx x^{-J} \left(\frac{1}{x+\nu} - \frac{1}{x+\nu_0} \right) \quad (4.6)$$

Since for $0 < \operatorname{Re} J < 1$:

$$\int_0^{\infty} \frac{dx \, x^{-J}}{x+\nu} = \frac{\pi \nu^{-J}}{\sin \pi J}$$

it may easily be verified that the continuation of $\bar{\Phi}_J(\nu, \nu_0)$ to the half-plane $\operatorname{Re} J < 1$ is

$$\bar{\Phi}_J(\nu, \nu_0) = \frac{g}{J+1} \left[\frac{\nu^{-J} - \nu_0^{-J}}{(\nu - \nu_0) \sin \pi J} - \int_0^1 dx \frac{x^{-J}}{(x+\nu)(x+\nu_0)} \right] \quad (4.7)$$

The expression (4.7) shows that $\bar{\Phi}_J(\nu, \nu_0)$ has poles at $J = -1, -2, \dots$. In particular, setting $J = -1 + \varepsilon$ in (4.7), and using $\nu^{-J} = \nu(1 - \varepsilon \log \nu + O(\varepsilon^2))$, one has

$$\bar{\Phi}_J(\nu, \nu_0) = -\frac{g}{\varepsilon} \left[-\frac{1}{\varepsilon} + \frac{\nu \log(\nu+1) - \nu_0 \log(\nu_0+1)}{\nu - \nu_0} \right] + O(1)$$

Inserting this into (4.3), and neglecting higher order terms in ε ,

$$\psi_J(\nu) = \frac{1}{\varepsilon^2} b(J, \nu+1) + \frac{g}{\pi \varepsilon} \int_0^{\infty} d\nu' \rho(\nu') \frac{\log \frac{\nu'+1}{\nu+1}}{\nu' - \nu} \psi_J(\nu') \quad (4.8)$$

where

$$b(J, \nu+1) = \varepsilon^2 \frac{N_J(\nu_0)}{\nu - \nu_0} - g \left[1 + \frac{1}{\pi} \int_0^{\infty} \frac{d\xi}{\xi} \rho(\xi) \psi_J(\xi) \right] \quad (4.9)$$

If $\psi_J(\nu)$ exists, the integral in (4.9) certainly converges, since $\psi_J(\nu)$ satisfies (4.8).

Introducing the variable $x = \nu + 1$ and

$$f(x) = \varepsilon^2 \psi_J(\nu) \quad (4.10)$$

one has

$$f(x) = b(J, x) + \frac{\lambda}{\pi^2} \int_1^\infty dx' \frac{\log x'/x}{x' - x} f(x') + \frac{\lambda}{\pi^2} \int_1^\infty dx' K(x, x') f(x') \quad (4.11)$$

where

$$\lambda = \frac{g\pi}{J+1} \quad (4.12)$$

and

$$K(x, x') = [\rho(x') - 1] \frac{\log x'/x}{x' - x} \quad (4.13)$$

In a full treatment, the residual Fredholm kernel $K(x, x')$ would also contain all the contributions that have been neglected here.

As in Section 5 of (I), (4.11) can be split into a system of two coupled integral equations, the one with the kernel $(\log(x'/x))/(x'-x)$, the other with the kernel $K(x, x')$. In that Section, it was shown that the solution $f(x)$ exists; and, because of the structure of the singular resolvent $R(x, x'; \lambda)$, it was shown that $\lambda = \infty$, which corresponds here to $J = -1$, is a branch point of the solution, with an infinite number of sheets in general. The branch point was proved to be an energy

dependent power. For real x , the solution $f(x)$ was shown to be bounded on the first sheet of λ (or J): but for complex x it is unbounded. It can easily be shown, using a contradiction argument, as in (I), that the fact that $b(J, x)$ in (4.11) also has a branch point at $J = -1$ cannot lead to a cancellation of the branch point of the singular resolvent, $R(x, x'; \lambda)$. Thus the foregoing conclusions, concerning the branch structure of the solution, are in agreement with those of Squires¹³.

However, it will now be shown that the Fredholm remnant $K(x, x')$ superimposes on the branch point at $J = -1$ an accumulation point of poles. This fact is not trivial, since the Fredholm kernel obtained in Section 5 of (I) by contracting the singular resolvent $R(x, x'; \lambda)$ with $K(x, x')$ is not symmetric, and so might not have any eigenvalues. However, a symmetric Fredholm equation of the second kind can be constructed, using a different decomposition of (4.11). The proof to follow relies heavily upon the fact that it is known from (I) that a solution $f(x)$ of (4.11) exists. In order to make the inhomogeneous term square integrable, (4.11) is first "subtracted" at $x = x_1$, say, giving :

$$F(x) = B(x) + \frac{\lambda}{\pi^2} \int_1^\infty dx' \frac{\log x'/x}{x' - x} F(x') + \frac{\lambda}{\pi^2} \int_1^\infty dx' K(x, x') F(x') \quad (4.14)$$

where

$$F(x) = \frac{1}{x^2} \frac{f(x)}{x - x_1} = \frac{\psi_J(x-1)}{x - x_1} = \frac{N_J(x-1)}{(x - x_1)(x + 1 - \nu_0)} \quad (4.15)$$

and

$$B(x) = \frac{\psi_J(x_1)}{x - x_1} - \frac{N_J(x_0)}{(x + 1 - \nu_0)(x_1 + 1 - \nu_0)} - \frac{\lambda}{\pi} \frac{\log x/x_1}{x - x_1} \int_1^\infty dx' \left(1 - \frac{1}{x'}\right)^{1/2} \frac{\psi_J(x')}{x' - x_1} \quad (4.16)$$

Next, Eq. (4.14) is written :

$$F(x) = F_0(x) + \frac{\lambda}{\pi^2} \int_1^{\infty} dx' K(x, x') F(x') \quad (4.17)$$

with

$$F_0(x) = B(x) + \frac{\lambda}{\pi^2} \int_1^{\infty} dx' \frac{\log x'/x}{x' - x} F(x') \quad (4.18)$$

The kernel $K(x, x')$, defined by (4.13), can be symmetrized by multiplying (4.17) by $[\rho(x)-1]^{\frac{1}{2}}$ and defining

$$F_s(x) = [\rho(x) - 1]^{\frac{1}{2}} F(x) \quad (4.19)$$

This gives

$$F_s(x) = [\rho(x) - 1]^{\frac{1}{2}} F_0(x) + \frac{\lambda}{\pi^2} \int_1^{\infty} dx' K_s(x, x') F_s(x') \quad (4.20)$$

with the symmetric Fredholm kernel

$$K_s(x, x') = \left[\frac{\rho(x) - 1}{\rho(x') - 1} \right]^{\frac{1}{2}} K(x, x') = \frac{\log x'/x}{x' - x} \cdot [\rho(x) - 1]^{\frac{1}{2}} [\rho(x') - 1]^{\frac{1}{2}} \quad (4.21)$$

To show that (4.20) is a Fredholm equation, it is still necessary to demonstrate that the inhomogeneous term is square integrable. It was shown in (I) that a unique square integrable solution, $F(x)$, of (4.14) exists. This was done by using the special resolvent which is regular at $\lambda = 0$: and it was shown that, in the case $\text{Re } \lambda > 1$, if λ is given a small imaginary part, one can find a majorization

$$|F(x)| < A x^{-1/2-\gamma} \quad \text{with } \gamma > 0 \quad (4.22)$$

where A is a constant.

The singular kernel can be majorized as follows :

$$\begin{aligned} \frac{\log x'/x}{x' - x} &= \int_0^\infty \frac{d\zeta}{(\zeta + x)(\zeta + x')} \\ &< \int_0^\infty \frac{d\zeta}{\zeta^{-\frac{1}{2}-\varepsilon} x^{\frac{1}{2}+\varepsilon} (\zeta + x')} \\ &< B x^{-\frac{1}{2}-\varepsilon} x'^{-\frac{1}{2}+\varepsilon} \end{aligned} \quad (4.23)$$

for any $0 < \varepsilon < 1$ and for all $1 \leq x, x' < \infty$. Here B is a constant. (4.22) and (4.23) can be used to majorize the integral in (4.18) :

$$\begin{aligned} \left| \int_1^\infty dx' \frac{\log x'/x}{x' - x} F(x') \right| &< A B x^{-\frac{1}{2}-\varepsilon} \int_1^\infty dx' x'^{-1+\varepsilon-\gamma} \\ &< C x^{-\frac{1}{2}-\varepsilon} \end{aligned} \quad (4.24)$$

if one chooses $\varepsilon < \gamma$, with C a constant. From (4.16) it can be seen that $B(x)$ is squareintegrable, so that (4.18) shows that $F_0(x)$ is square integrable, and, a fortiori, that $[\rho(x)-1]^{\frac{1}{2}} F_0(x)$ is also square integrable. Hence (4.20) is a symmetric Fredholm equation of the second kind for $F_s(x)$.

Since $K_s(x, x')$ is symmetric between x and x' , it must have at least one eigenvalue. As it is also non-separable, there are in fact an infinity of real eigenvalues, with an accumulation point at $\lambda = \infty$ ²¹⁾. The solution of (4.20) can be written

$$F_s(x) = [\rho(x) - i]^{1/2} F_0(x) + \frac{\lambda}{\pi^2} \int_1^\infty dx' S(x, x'; \lambda) [\rho(x') - i]^{1/2} F_0(x') \quad (4.25)$$

for any λ which is not an eigenvalue of $K(x, x')$. Here $S(x, x'; \lambda)$ is the resolvent of $K(x, x')$, and it has a pole at each $\lambda = \lambda_n$ which is an eigenvalue of $K(x, x')$. It follows from the preceding remarks that $S(x, x'; \lambda)$ has an essential singularity at $\lambda = \infty$. Then (4.25) implies that $F_s(x)$, as a function of λ , also has an essential singularity at $\lambda = \infty$. The possibility that an essential singularity in $F_0(x)$ cancels that in $S(x, x'; \lambda)$ can easily be ruled out. For suppose this to be the case, i.e., that $F_0(x)$ has an essential singularity in λ , but $F_s(x)$ does not. Then the integral term in (4.20) would have no essential singularity, and so could not cancel that of the inhomogeneous term. Hence $F_s(x)$ has an essential singularity, which is a contradiction.

Thus $F_s(x)$ has an essential singularity at $\lambda = \infty$, which implies, through (4.19) and (4.15), that $N_J(x)$ has an essential singularity at $J = -1$, which is superimposed upon the branch point.

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FIG.1