Limiting procedures for singular potentials, 1
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Limiting Procedures for Singular Potentials - I.

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(ricevuto l'1 Febbraio 1965)

Summary. — For singular repulsive potentials, as well as for regular potentials, the S -matrix is the limit of the ratio of the two Jost solutions when the radial co-ordinate goes to zero. In the first Part of this paper, it is shown that the S -matrix is the limit of sequences, each term of the sequences being the ratio of the perturbation expansion of the same order of the two Jost solutions. It is shown that these sequences converge to the S -matrix if we connect in a certain manner the order of the perturbation expansion and the radial co-ordinate. In the second Part, some remarks are given about cut-off procedures in potential scattering: a) Can we obtain the physical quantities if we take into account the whole contribution given by the cut-off? b) Can we connect the cut-off and the order of perturbation? c) Is the recent prescription of summing up the first leading singularities a valid approximate method? d) In the Fredholm's type of solution can we have the same kind of divergence both in the numerator and in the denominator when the cut-off goes to zero? e) Can we connect the cut-off going to zero and the order of the Fredholm's determinants going to infinity both in the numerator and in the denominator in such a way that we obtain the physical solution as the limit of convergent sequences?

Introduction.

In previous works ^(1,2) we have studied the problem of the explicit determination of the Jost function for singular potentials where the most singular term is repulsive.

^(1a) H. CORNILLE: *Nuovo Cimento*, **33**, 434 (1964).

^(1b) H. CORNILLE and E. PREDAZZI: *Phys. Lett.*, **10**, 149 (1964); and *Nuovo Cimento*, **35**, 879 (1965).

^(2a) H. CORNILLE: CERN, preprint 9608/TH. 479.

^(2b) H. CORNILLE and E. PREDAZZI: EFINS, preprint 64-61.

The starting point was always the same. We use the fact that the «singular part» (or the «regular part») of the solutions of the Schrödinger equation is k -independent. Thus we define «new Jost solutions» asymptotically ingoing (or outgoing) and such that the singular part has been dropped out near the origin. Doing this, we have solutions which go to constants (Jost functions) near the origin, but this method has the disadvantage that we are led to a complicated interaction (potential) where counter-terms occur. In ref. (2), where the study was in co-ordinate space, it was shown that we can connect in a certain manner the order of the perturbation expansion ($p \rightarrow \infty$) and the radial co-ordinate ($r \rightarrow 0$) in such a way that the Jost function is the limit of convergent sequences. In this paper we study the exact Jost solutions in co-ordinate space and consider their perturbation expansions. These functions develop essential singularities with respect to r near the origin and are infinite when $r \rightarrow 0$. We consider next the ratio of the perturbation expansion of the Jost solution to the singular part of this solution near the origin. The limit of this ratio goes to a constant (Jost function) when $r \rightarrow 0$. In this paper, for this ratio we still connect the order of the perturbation expansion of the Jost solution and the radial co-ordinate and find that generally for the same dependence $r(p) \geq r_L(p)$ as in ref. (2), the Jost function is the limit of convergent sequences.

We recall that it has been shown previously (3) for a very large class of singular potentials where the most singular part is repulsive, that the ratio of the two Jost solutions is convergent when $r \rightarrow 0$ and gives the S -matrix. This reflects the fact that the singular part of the two Jost solutions has the same r -dependence and is k -independent. Now we consider the two ratios of the Jost solutions by their identical singular r -dependent part and next the ratio of these two ratios. For the same order of perturbation expansion, this singular part being factorized in the same way, we get finally

$$S(k) = \lim_{p \rightarrow \infty} \frac{\sum_0^p R_q(k, r(p))}{\sum_0^p R_q(-k, r(p))}, \quad \text{if } r(p) \geq r_L(p),$$

where $R_q(k, r)$ is the q -th term of the perturbation expansion of the Jost solution $R(k, r)$ and $r_L(p)$ is a limiting dependence which depends mainly on the most singular part of the potential.

In the second Part of this paper (for the $l=0, k=0$ case) we introduce an arbitrary cut-off ε on the integration path of the regular integral equation. First, it is shown that if we take into account the *whole* cut-off dependence

(3) N. LIMIC: *Nuovo Cimento*, 26, 581 (1962).

of the solutions, we can formulate the problem in such a way that the physical quantities can always be obtained when $\varepsilon \rightarrow 0$. This result is general and does not depend on the particular kind of singular potentials considered as long as the most singular part is repulsive. These results are obtained without any assumption of analytic continuation when $\varepsilon \rightarrow 0$ because we work directly with quantities which are the analytic continuation when $\varepsilon \rightarrow 0$.

Secondly, in order to obtain an approximate result, we show how we can connect the cut-off going to zero and the order of perturbation going to infinity, as we did in the first Part of the paper in the case of the radial co-ordinate. This works in all cases where the limit $\varepsilon \rightarrow 0$ exists because, as with the use of the radial co-ordinate as a natural cut-off, we take into account the whole singular parts of the solutions.

Thirdly, we investigate the recent prescription of summing up the first leading singularities in the cut-off parameter in each order of the perturbation series which has been extensively studied recently in potential scattering⁽⁴⁻¹¹⁾. We show from the analytic continuation of these first leading singular terms that

i) One obtains always a finite result in the case where the most singular term is a power type.

ii) In general, one obtains nonsensical answers when $\varepsilon \rightarrow 0$. We have shown this explicitly in two different cases. First, we have considered the case when the most singular part of the potential is a general family of logarithmic potentials. In particular, the result first obtained by ALY *et al.* for $G^2(\log r)^2/r^4$ from the perturbation expansion is recovered in the analytic continuation. Secondly, the conclusion is the same for a general family of singular repulsive exponential potentials.

At the end of this second Part, for the same $l=0$, $k=0$ case, we investigate the Fredholm's formulation of the regular (cut-off dependent) solution. First we show that the expansions in the Fredholm's determinants both in the numerator and in the denominator of the solution diverge in exactly the same way when the cut-off goes to zero, such that the limit of the ratio exists.

(4) N. N. KHURI and A. PAIS: *Rev. Mod. Phys.*, **26**, 590 (1964).

(5) G. TIKTOPOULOS and S. B. TREIMAN: *Phys. Rev.* **134**, B 844 (1964).

(6) A. PAIS and T. T. WU: *Journ. Math. Phys.*, **5**, 799 (1964).

(7) A. PAIS and T. T. WU: to be published in *Journ. Math. Phys.*

(8) T. T. WU: *Phys. Rev.*, **136**, B 1176 (1964).

(9) H. H. ALY, RIAZUDDIN and A. H. ZIMERMAN: *Phys. Rev.*, **136**, B 1174 (1964).

(10) H. H. ALY, RIAZUDDIN and A. H. ZIMERMAN: *Scattering by singular logarithmic potential*, preprint.

(11) H. H. ALY, RIAZUDDIN and A. H. ZIMERMAN: *Nuovo Cimento*, **35**, 325 (1965).

Secondly, we show that this limit of the cut-off going to zero in the Fredholm's (cut-off dependent) solution gives indeed the right regular solution of the Schrödinger equation.

Thirdly, we show that we can also connect in a certain manner the cut-off going to zero and the order of Fredholm's determinants both in the numerator and in the denominator in such a way that for any r the regular solution is the limit of convergent sequences.

PART I.

The Radial Co-ordinate as a Natural Cut-Off.

1. - The method.

We consider the Schrödinger equation

$$(1) \quad \left(\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} \right) u = V u,$$

i) $V(r)$ is singular near the origin and the most singular term is repulsive.

ii) When $r \rightarrow \infty$, $V(r)$ is such that the behaviour of the solutions of eq. (1) for asymptotically ingoing or outgoing waves are given by the free terms of (1). For instance, we suppose that the potential decreases at least as fast as $r^{-\lambda}$ ($\lambda > 2$).

We define the Jost solutions $R(\pm k, r)$ of (1), asymptotically ingoing or outgoing, as

$$R(\pm k, r) \underset{r \rightarrow \infty}{\sim} \exp[\mp ikr].$$

Near the origin these solutions behave like

$$(2) \quad R(\pm k, r) \underset{r \rightarrow 0}{\sim} Y_{\text{sing}}(r) R(\pm k) + Y_{\text{reg}}(r) \text{ const.}$$

These «regular» and «singular» parts can be investigated from the differential eq. (1) by the W.K.B. method or other similar methods⁽²⁾. Now, of course, the Jost function is given by

$$(3) \quad R(\pm k) = \lim_{r \rightarrow 0} \frac{R(\pm k, r)}{Y_{\text{sing}}(r)}.$$

From the differential eq. (1) and the boundary condition for R , we can write an *equivalent* integral equation of the Volterra type:

$$(4) \quad R(\pm k, r) = \exp[\mp ikr] + \int_r^\infty \frac{\sin k(r'-r)}{k} \left(V(r') + \frac{l(l+1)}{r'^2} \right) R(\pm k, r') dr'.$$

Because of the equivalence of (4) and (1) $R(\pm k, r)$ given by (4) behaves near the origin like (2). If we iterate (4), calling $R_q(\pm k, r)$ the q -th iteration term, the most singular part of $R_q(\pm k, r)$ is k -independent, and R_q is infinite when r goes to zero; but we can write from (3) and (4)

$$(5) \quad R(\pm k) = \lim_{r \rightarrow 0} \left[[Y_{\text{sing}}(r)]^{-1} \lim_{p \rightarrow \infty} \sum_0^p R_q(k, r) \right].$$

If in (5) we reverse the order of the two limiting procedures, considering first

$$\lim_{r \rightarrow 0} \left[[Y_{\text{sing}}(r)]^{-1} R_q(\pm k, r) \right],$$

we find that for all finite q each term goes to zero.

Similarly to what was done in previous works (^{1b,2}), we connect the two limiting procedures, for instance $r(p) \xrightarrow{p \rightarrow \infty} 0$, and try to find $r(p)$ such that

$$(6) \quad R(\pm k) = \lim_{p \rightarrow \infty} \left[[Y_{\text{sing}}(r(p))]^{-1} \sum_0^p R(\pm k, r(p)) \right],$$

where $r(p)$ cannot be entirely arbitrary because of the fact that in (5) we cannot invert the order of the two limiting procedures.

Now we define

$$(7) \quad \begin{cases} F(k, r) = [Y_{\text{sing}}(r)]^{-1} R(k, r), \\ G(k, r, p) = [Y_{\text{sing}}(r)]^{-1} \sum_0^p R_q(k, r), \\ M(k, r, p) = [Y_{\text{sing}}(r)]^{-1} \sum_{p+1}^\infty R_q(k, r). \end{cases}$$

We have the following identity:

$$(8) \quad F(k, r(p)) = G(k, r(p), p) + M(k, r(p), p).$$

If we consider an arbitrary dependence $r(p) \xrightarrow{p \rightarrow \infty} 0$, we have

$$(9) \quad R(k) = \lim_{p \rightarrow \infty} F(k, r(p)) = \lim_{p \rightarrow \infty} G(k, r(p), p) + \lim_{p \rightarrow \infty} M(k, r(p), p).$$

In order to satisfy (6) it suffices to find $r(p)$ such that

$$(6') \quad R(k) = \lim_{p \rightarrow \infty} G(k, r(p), p)$$

or

$$(6'') \quad \lim_{p \rightarrow \infty} M(k, r(p), p) \rightarrow 0,$$

where $r(p)$ will not be entirely arbitrary.

2. - Inverse power case.

We consider the case where the singularities of the potential V near the origin are like inverse powers. For the behaviour of $V(r)$ we still consider the conditions ii) of Sect. 1; for instance, we can have exponential decreasing behaviour.

2.1. - First, we consider an explicit family for which the behaviour $r \rightarrow 0$ of the solutions of (1) has been previously investigated ^(1b, 2a):

$$(10) \quad V(r) \underset{r \rightarrow 0}{\simeq} \frac{G^2}{r^{2n}} - \sum_{p=1}^{N_0} \frac{A_p}{r^p} \quad (n > 1, G^2 > 0).$$

For potentials like (10) we can explicitly find the singular part and we recall ^(1b, 2a)

$$(11) \quad Y_{\text{sing}}(r) \underset{r \rightarrow 0}{\simeq} \exp \left[\frac{G}{(n-1)r^{n-1}} [1 + O(r)] \right].$$

In order to find $r(p)$ dependences such that (6') or (6'') is satisfied, we only have to investigate the singular part of the perturbation expansion of (4). In the Appendix it is shown that for the singular part of the q -th term $R_q(k, r)$ we have

$$|R_q(k, r)| < \left(\frac{\text{const}}{r^{n-1}} \right)^{2q} [1 + \text{const } r^{2(n-1)} q]^q \frac{\text{const}}{q! \Gamma(q+1-1/2(n-1))}$$

and with the use of Stirling's formula

$$\Gamma(q+1) \underset{q \rightarrow \infty}{\simeq} \sqrt{2\pi q} q^q \exp[-q]$$

we get

$$(12) \quad \begin{cases} \lim_{p \rightarrow \infty} |G(k, r(p), p)| \rightarrow 0 & \text{if } r(p) < \frac{\text{const}}{p^{(1/(n-1))+\varepsilon}}, \\ \lim_{p \rightarrow \infty} |M(k, r(p), p)| \rightarrow 0 & \text{if } r(p) \geq \frac{\text{const}}{p^{(1/(n-1))-\varepsilon}} = r_L(p), \end{cases} \quad \varepsilon \text{ arbitrary small } > 0.$$

Then $\lim_{p \rightarrow \infty} G(k, r(p), p) = R(k)$ if $r(p) \geq r_L(p)$. Note that we find the same limiting dependence as in ref. (2a).

2'2. - Now we consider the case where the potential is singular like the centrifugal barrier near the origin: $V(r) \underset{r \rightarrow 0}{\simeq} G^2/r^2$ ($G^2 \geq 0$). In this case the differential eq. (1) is of the Fuchs type and $Y_{\text{sing}}(r) = r^\lambda$ [where $\lambda = \frac{1}{2}(1 - \sqrt{1 + 4(l(l+1) + G^2)})$]. In the Appendix we obtain

$$|R_q(k, r)| < \frac{\text{const}^q (\log 1/r)^q}{q!}$$

and

$$(13) \quad G(k, r(p), p) \begin{cases} \xrightarrow{p \rightarrow \infty} R(k) & \text{if } r(p) \geq r_L(p) = \frac{1}{\exp[\text{const } p^{1-\varepsilon}]}, \\ \xrightarrow{p \rightarrow \infty} 0 & \text{if } r(p) < \frac{1}{\exp[\text{const } p^{1+\varepsilon}]}, \end{cases} \quad \varepsilon > 0.$$

3. - Logarithmic case.

3'1. - We consider first

$$V(r) \underset{r \rightarrow 0}{\simeq} \frac{G^2 (\log 1/r)^\beta}{r^{2n}} \quad G^2 > 0, n > 1.$$

The exact singular part has been investigated (2.6) and we can write:

$$Y_{\text{sing}}(r) \underset{r \rightarrow 0}{\simeq} \exp \left[G \int_r^{\text{const}} \frac{(\log 1/r')^{\beta/2}}{(r')^n} dr' [1 + O(r)] \right].$$

In the Appendix we find for the singular part of $R_q(k, r)$:

$$|R_q(k, r)| < X_q(r) \frac{q^{\text{const}}}{(q!)^2} [1 + qr^{2(n-1)} \text{const}]^q,$$

$$X_q = \left(\frac{\text{const}(\log 1/r)^{\beta/2}}{r^{n-1}} \right)^{2q} \quad \text{for } \beta > 0$$

and

$$X_q = \left(\frac{\text{const}}{r^{n-1}} \right)^{2q} \quad \text{for } \beta < 0$$

and

$$(14) \quad \lim_{p \rightarrow \infty} G(k, r(p), p) = R(k) \quad \text{if} \quad r(p) \geq \frac{\text{const}}{p^{(1/(n-1))-\epsilon}} = r_L(p).$$

We have then the same (*) limiting dependence as in the power case ^(1b). We have also

$$G(k, r(p), p) \xrightarrow{p \rightarrow \infty} 0 \quad r(p) < \frac{\text{const}}{p^{(1/(n-1))+\epsilon}}.$$

3.2. - Secondly, we consider the delicate case

$$V \underset{r \rightarrow 0}{\simeq} \frac{G^2}{r^2} \left(\log \frac{1}{r} \right)^\beta.$$

With respect to the singular part (**) of R we recall ^(1b) that we are in very different situations in the two cases $\beta > 0$ or $\beta < 0$.

If β is > 0 , the leading singularity comes only from the potential $V(r)$ and we have [ref. ^(1b)]

$$Y_{\text{sing}}(r) \underset{r \rightarrow 0}{\simeq} \exp \left[\frac{G}{(1+\beta/2)} \left(\log \frac{1}{r} \right)^{1+\beta/2} [1 + O(r)] \right].$$

Then in this case it is shown in the Appendix that

$$|R_q(k, r)| < \frac{(\text{const}(\log 1/r)^{(\beta/2)+1} + \text{const})^{2q}}{(2q)!} \left[1 + \frac{q \text{const}}{(\log 1/r)^{\beta+2}} \right]^q$$

(*) (14) is not the most refined limiting dependence but the difference is not significant.

(**) We recall also ^(2b) that when $|\beta| \rightarrow 0$ the number of essential singularities of $Y_{\text{sing}}(r)$ increases and at the limit $|\beta| = 0$ all these essential singularities cancel and sum up to give the right power behaviour corresponding to the inverse square case. In the following we consider always $|\beta|$ finite.

and always with the use of Stirling's formula

$$(15) \quad G(p, r(p), k) \begin{cases} \rightarrow R(k) & \text{if } r(p) \geq \frac{1}{\exp[\text{const } p^{(1/(1+\beta/2))-\epsilon}]} = r_L(p), \\ \rightarrow 0 & \text{if } r(p) < \frac{1}{\exp[\text{const } p^{(1/(1+\beta/2))+\epsilon}]} \end{cases}$$

In the case $\beta < 0$, we consider only $-1 < \beta < 0$ because we recall ^(1b) that for $\beta < -1$ the singular part $Y_{\text{sing}}(r)$ near the origin is given only by the centrifugal potential. For $-1 < \beta < 0$ we recall

$$Y_{\text{sing}}(r) \underset{r \rightarrow 0}{\simeq} \exp \left[\left[-l \log r + \frac{G^2}{2(l + \frac{1}{2})(1 - |\beta|)} \left(\log \frac{1}{r} \right)^{1-|\beta|} \right] [1 + O(r)] \right].$$

Thus for $l \neq 0$ the situation is like that of the inverse square case; it is shown in the Appendix that

$$(16) \quad \lim_{p \rightarrow \infty} G(k, r(p), p) \begin{cases} \rightarrow R(k) & \text{if } r(p) \geq \frac{1}{\exp[\text{const } p^{1-\epsilon}]} = r_L(p), \\ \rightarrow 0 & \text{if } r(p) < \frac{1}{\exp[\text{const } p^{1+\epsilon}]}, \end{cases}$$

whereas for $l = 0$

$$(17) \quad \lim_{p \rightarrow \infty} G(k, r(p), p) \begin{cases} \rightarrow R(k) & \text{if } r(p) \geq \frac{1}{\exp[\text{const } p^{(1/(1-|\beta|))-\epsilon}]} = r_L(p), \\ \rightarrow 0 & \text{if } r(p) < \frac{1}{\exp[\text{const } p^{(1/(1-|\beta|))+\epsilon}]} \end{cases}$$

4. - Exponential case.

We consider now the more singular case:

$$V \underset{r \rightarrow 0}{\simeq} \frac{G^2}{r^{2n}} \exp[\eta/r^\gamma] + \frac{A}{r^m}, \quad \eta > 0, \gamma > 0, G^2 > 0.$$

The exact singular part has been investigated ^(1b), we recall that we can write

$$Y_{\text{sing}}(r) \underset{r \rightarrow 0}{\simeq} \exp \left[\int_r^{\text{const}} \frac{G}{r'^n} \exp[\eta/2r'^\gamma] dr' [1 + O(r)] \right].$$

In the Appendix we get

$$(18) \quad |R_q(k, r)| < \frac{[(\exp[\eta/2r^\gamma])(\text{const}/r^{n-1} + \text{const})]^{2q}}{2q!} \left[1 + \frac{q \text{ const}}{r^{\text{const}} \exp[\eta/4r^\gamma]} \right]^q$$

and

$$(19) \quad G(p, r(p), k) \begin{cases} \xrightarrow{p \rightarrow \infty} R(k) & \text{if } r(p) > \frac{1}{(\text{const} \log p^{2(1-\epsilon)/\eta})^{1/\gamma}} = r_L(p), \\ \xrightarrow{p \rightarrow \infty} 0 & \text{if } r(p) \leq \frac{1}{(\text{const} \log p^{2(1+\epsilon)/\eta})^{1/\gamma}}. \end{cases}$$

5. - More general singular potentials.

Suppose now that we consider more general families of singular potentials

$$V_i(r) \underset{r \rightarrow 0}{\simeq} V_{0,i}(r)[1 + \varphi_i(r)],$$

$i=1, 2, 3$; $\varphi_i(r) \xrightarrow{r \rightarrow 0} 0$, $|\varphi_i(r)| < \text{const} < 1$ for r small; where

$$V_{0,1} = \frac{G^2}{r^{2n}} \quad (n > 1), \quad V_{0,2} = \frac{G^2}{r^{2n}} \left(\log \frac{1}{r} \right)^\beta \quad (n > 1), \quad V_{0,3} = \frac{G^2}{r^{2n}} \exp[\eta/r^\gamma]$$

$(\eta > 0, \gamma > 0).$

For simplicity, we have considered only $V_i(r) \approx V_{0,i}(r)$, but in fact all previous results can be easily extended to the case $V_i(r)$, and in particular the same limiting dependences $r_L(p)$ hold. We give briefly some arguments why this arises.

First we must find bounds for the singular part of $R_q(k, r)$ and in the Appendix, where these bounds are obtained, we have always considered potentials $V_i(r)$ such that $|V_{0,i}[1 + \varphi_i(r)]| < \text{const} |V_{0,i}(r)|$ for small r . We recall that ^(2b) we can have in particular cases for $Y_{\text{sing}}(r)$ an infinity of essential singularities coming when we expand the square root $\sqrt{V_{0,i}[1 + \varphi_i]}$. But on the one hand ^(2b) they can be grouped in $Y_{\text{sing}}(r)$ and on the other hand the above inequality holds.

Secondly, for $V_i(r)$ the singularities of $Y_{\text{sing}}(r)$ are

$$\underset{r \rightarrow 0}{\simeq} \exp \left[\int_r^{\text{const}} \sqrt{V_{0,i}(1 + \varphi_i(r'))} dr' [1 + O(r)] \right]$$

[see the method given in ref. ^(1b)]. It is then easy to see that $G(p, r(p), k) \underset{p \rightarrow \infty}{\simeq} R(k)$ or 0 with the same restrictions on the $r(p)$ as in the $V_{0,i}(r)$ cases.

6. - *S*-matrix.

It has been shown by LIMIC ⁽³⁾ for a very large class of singular potentials, where the most singular part is repulsive, that

$$S(k) = \lim_{r \rightarrow 0} \frac{R(k, r)}{R(-k, r)}.$$

This reflects the fact that $R(\pm k, r)$ have the same k -independent singular part near the origin, $Y_{\text{sing}}(r)$. We remark also that this is true as well for the most singular part of $R_q(\pm k, r)$. [This is a secondary point, as we shall see explicitly in the second Part of this paper, where we consider the finite ratio when $r \rightarrow 0$ of two functions which have the same type of singularities when $r \rightarrow 0$, but such that their singularities are not the same for the same order of perturbation.] The important fact is that the whole perturbation expansion of the two Jost solutions develops the same r singularities when $r \rightarrow 0$. Now, from the results obtained in the above Sections,

$$S(k) = \frac{R(k)}{R(-k)} = \frac{\lim_{p_1 \rightarrow \infty} G(p_1, r(p_1), k)}{\lim_{p_2 \rightarrow \infty} G(p_2, r(p_2), k)},$$

where we suppose, of course, that $r(p) \geq r_L(p)$. Because of the fact that for each p finite or infinite $G(p, r(p), k)$ is finite, we can take $p_1 = p_2$ and factorize for the numerator and the denominator the same $[Y_{\text{sing}}(r(p))]^{-1}$. Finally, we have

$$(20) \quad S(k) = \lim_{p \rightarrow \infty} \frac{\sum_0^p R_q(k, r(p))}{\sum_0^p R_q(-k, r(p))} \quad \text{if } r(p) \geq r_L(p),$$

these $r_L(p)$ being the same as for the Jost functions $R(k)$.

Now we ask, what arises when $r(p) \rightarrow 0$ more rapidly than the limiting value, for instance, for $V \underset{r \rightarrow 0}{\simeq} G^2/r^n$ ($n > 1$) suppose we take $r(p) \leq \text{const}/p^{(1/(n-1))+\epsilon}$? In this case, from the identity (9) and the result that $\lim_{p \rightarrow \infty} G(p, r(p), k) = 0$ we get $R(k) = \lim_{p \rightarrow \infty} M(k, r(p), p)$ and

$$S(k) = \lim_{p \rightarrow \infty} \frac{\sum_{p+1}^{\infty} R_q(k, r(p))}{\sum_{p+1}^{\infty} R_q(-k, r(p))} \quad r(p) \leq r_L(p).$$

But this is not convenient for the explicit calculation of the S -matrix. We have shown above that $G(p, r(p), \pm k) \xrightarrow{p \rightarrow \infty} 0$ if $r(p) < r_L(p)$. Because of the fact that $R_a(\pm k, r)$ develop their r -singularities in exactly the same way, the two sequences $G(p, r(p), \pm k)$ [for the same $r(p) < r_L(p)$] go to zero when $p \rightarrow \infty$ with the same p -dependence (k -independent). Then in this case the ratio in (20) goes to 1 when $p \rightarrow \infty$.

Finally we want to remark that this method can be extended to other finite quantities of the type $\lim_{r \rightarrow 0} N(k, r)/D(k, r)$ where the numerator and the denominator develop the same divergent singularities when $r \rightarrow 0$. For instance

$$(21) \quad i \operatorname{tg} \delta(k) = \lim_{p \rightarrow \infty} \frac{\sum_0^p (R_a(k, r(p)) - R_a(-k, r(p)))}{\sum_0^p (R_a(k, r(p)) + R_a(k, r(p)))} \quad \text{if } r(p) \geq r_L(p),$$

$r_L(p)$ being the same as for the S -matrix.

PART II.

Some Remarks About the Cut-Off Procedure.

So far, in order to obtain the S -matrix or the Jost function, we have never used a cut-off procedure; in fact, the radial co-ordinate going to zero worked as a natural cut-off. We have also taken great advantage of the fact that in potential scattering all physical quantities can be obtained as the limit when $r \rightarrow 0$ of the ratio of two functions with the same k -independent singularities. For the S -matrix this follows from the work of LIMIO (*), and for the Jost function we have used $\lim_{r \rightarrow 0} (R(k, r)/Y_{\text{sing}}(r))$. But in the relativistic case, for instance in the Bethe-Salpeter equation, it is often necessary to introduce regulators or cut-offs.

Thus we think it will be useful to see, first by introduction of an arbitrary cut-off in potential scattering, whether we can still use our method of connecting the cut-off going to zero and the order of perturbation going to infinity. In fact, we will see that this is possible if we have a situation similar to the previous one: *i.e.*, if we have a finite quantity like $\lim_{\varepsilon \rightarrow 0} (N(k, \varepsilon)/D(k, \varepsilon))$ where N and D develop the same divergent ε singularity.

(*) See also the more recent discussion given by PAIS and WU (?).

Similarly to many other cut-off workers, we restrict ourselves for simplicity to the case $k=0$, $l=0$ (*). We consider also a potential decreasing more rapidly than r^{-3} when $r \rightarrow \infty$, in order to avoid difficulties not connected with the use of singular potentials. We assume also that the most singular part of the potential is repulsive. We replace the potential by a cut-off potential (**) $V_\varepsilon(r) = \theta(r-\varepsilon)G^2 V(r)$, ($G^2 > 0$), where $V(r)$ is singular when $r \rightarrow 0$ as in the previous Sections.

We define the «regular solution» $\psi(r, \varepsilon)$ for $r \geq \varepsilon > 0$ by the following integral equation:

$$(22) \quad \psi(r, \varepsilon) = r - r \int_r^\infty V_\varepsilon(r') \psi(r', \varepsilon) dr' - \int_\varepsilon^r r' V_\varepsilon(r') \psi(r', \varepsilon) dr'$$

and we put formally

$$(23) \quad \frac{\text{tg } \delta(k, \varepsilon)}{k} = - \int_\varepsilon^\infty V_\varepsilon(r) r \psi(r, \varepsilon) dr.$$

The problem is to see if the $\lim_{\varepsilon \rightarrow 0} (\text{tg } \delta(k, \varepsilon)/k)$ exists and if the limit is really the physical quantity $(\text{tg } \delta/k)$. Another linked problem is to see if $\lim_{\varepsilon \rightarrow 0} \psi(r, \varepsilon)$ gives the right regular physical solution $\psi(r)$, (see below). We want to study the solutions of (22) in two different manners.

First, we consider the iterative solution of (22). From the structure of the kernels of (22) we see that we can write, when $r \geq \varepsilon$ for each q -th iteration $\psi_q(r, \varepsilon) = \psi_q^{(1)}(r) + \psi_q^{(2)}(r, \varepsilon)$. This can be seen explicitly by iterating (22). Because $r \geq \varepsilon$ in (22), ε appears only as a limit of the integration path. Then the first iterative term $\psi_1(r, \varepsilon)$ is a sum of two terms, the first depending only on r , the second only on ε . For the second iterative term $\psi_2(r, \varepsilon)$ we group on the one hand terms depending only on r and on the other hand terms depending on (r, ε) or ε only. For the whole solution in an obvious notation we get

$$(24) \quad \psi(r, \varepsilon) = (\psi^{(1)}(r) + \psi^{(2)}(r, \varepsilon)) \theta(r - \varepsilon),$$

where $\psi^{(1)}$ and $\psi^{(2)}$ are power series in G^2 . If we substitute this perturbation solution of ψ , given by (24), in (23), we get

$$(25) \quad \frac{\text{tg } \delta(k, \varepsilon)}{k} = \sum_q G^{2q} a_q(\varepsilon).$$

(*) The $k \neq 0$, $l \neq 0$ case will be published in another paper.

(**) See the foot-note at the end of this paper.

From (25) can we go to the limit $\varepsilon \rightarrow 0$? For the families of potentials which have been investigated (power, logarithms), one finds that the series expansion has a finite circle of convergence such that in (25) the sum is defined only for $\varepsilon > \varepsilon_0$. For instance ⁽⁵⁾ (*) for $G^2 V(r) = G^2/r^4$ we get

$$\frac{\operatorname{tg} \delta(k, \varepsilon)}{k} = -G \left[\frac{G}{\varepsilon} - \left(\frac{G}{\varepsilon} \right)^3 \frac{1}{3} + \frac{2}{15} \left(\frac{G}{\varepsilon} \right)^5 + \dots = -G \operatorname{tgh} \left(\frac{G}{\varepsilon} \right) \right] \quad \text{if } \varepsilon > \varepsilon_0 = \frac{2G}{\pi}.$$

Because of the fact that $\varepsilon > \varepsilon_0$, ε cannot go to zero and from the series (25) we cannot connect the two limiting dependences $q \rightarrow \infty$ and $\varepsilon \rightarrow 0$; at this stage our method cannot be applied.

Cut-off workers generally sum (25) for $\varepsilon > \varepsilon_0$ and take the prescription that the analytic continuation of the sum is valid. Sometimes, in special cases where the exact solution can be explicitly obtained, this prescription can be justified by showing that the limit $\varepsilon \rightarrow 0$ of the continuation coincides with the exact result obtained independently. ALY *et al.* ⁽⁹⁾, considering the sum of the leading singularities in (25) for the potential $(G^2(\log r)^2)/r^4$, have found that the continuation of the sum when $\varepsilon \rightarrow 0$ has no meaning, $\operatorname{tg} \delta(k, \varepsilon)/k \underset{\varepsilon \rightarrow 0}{\sim} -G \log \varepsilon$. We shall come back to this important result later.

Because the problem is mainly a problem of analytic continuation we must find another expression of $(\operatorname{tg} \delta(k, \varepsilon)/k)$ which reduces to (23) given by (24) for ε finite, and which has a meaning when $\varepsilon \rightarrow 0$.

Secondly, we want to investigate the solution of (22) in another manner by a change of the integration path in (22), $[\varepsilon, r] \rightarrow [\varepsilon, \infty] + [\infty, r]$, we see that $\psi(r, \varepsilon)$ satisfies another equivalent integral equation:

$$(26) \quad \psi(r, \varepsilon) = r + \frac{\operatorname{tg} \delta(k, \varepsilon)}{k} + \int_r^\infty (r' - r) V_\varepsilon(r') \psi(r', \varepsilon) \theta(r' - \varepsilon) dr'.$$

Consider the integral equation

$$(27) \quad \varphi(r) = \varphi_0(r) + \int_r^\infty (r' - r) G^2 V \varphi(r') dr'.$$

If we call $R(r)$ and $T(r)$ the solutions of (27) corresponding respectively to taking φ_0 equal to 1 and r , we get for $\psi(r, \varepsilon)$

$$(28) \quad \psi(r, \varepsilon) = \left[T(r) + \frac{\operatorname{tg} \delta(k, \varepsilon)}{k} R(r) \right] \theta(r - \varepsilon).$$

(*) We note the misprint in the paper of TIKTOPOULOS and TREIMAN ⁽⁵⁾ who have given the expansion of $\operatorname{tgh}^{-1}(G/\varepsilon)$ instead of the correct one of $\operatorname{tgh}(G/\varepsilon)$.

Now, of course, (28) is equivalent to (24) such that $T(r) = \psi^{(1)}(r)$ and $R(r) \cdot (\operatorname{tg} \delta(k, \varepsilon)/k) = \psi^{(2)}(r, \varepsilon)$. But from (27) we can obtain the solutions R and T for r as small as we want (for instance for $r < \varepsilon$ or $r < \varepsilon_0$). If we substitute (28) into (23), we get

$$(29) \quad \frac{\operatorname{tg} \delta(k, \varepsilon)}{k} = \frac{-\int_{\varepsilon}^{\infty} G^2 T(r) r V(r) dr}{1 + \int_{\varepsilon}^{\infty} G^2 R(r) r V(r) dr}.$$

If in (29) the ratio has a meaning when $\varepsilon \rightarrow 0$, both in the numerator and the denominator, then (29) will be the continuation of (25). But $R(r)$ and $T(r)$ are the two independent solutions of the Schrödinger equation defined by their asymptotic behaviour 1 and r . Now, when $r \rightarrow 0$, these two solutions behave like

$$(30) \quad \begin{pmatrix} R(r) \\ T(r) \end{pmatrix} \underset{r \rightarrow 0}{\simeq} Y_{\text{sing}}(r) \begin{pmatrix} \text{const}_1 \\ \text{const}_2 \end{pmatrix} + Y_{\text{reg}}(r) \begin{pmatrix} \text{const}_3 \\ \text{const}_4 \end{pmatrix}.$$

Now for any singular potential where the most singular part is repulsive, we get from (29) and (30)

$$(31) \quad \lim_{\varepsilon \rightarrow 0} \frac{\operatorname{tg} \delta(k, \varepsilon)}{k} = \lim_{\varepsilon \rightarrow 0} \frac{-\int_{\varepsilon}^{\infty} T(r) r V(r) dr}{\int_{\varepsilon}^{\infty} R(r) r V(r) dr} = \lim_{\varepsilon = r \rightarrow 0} -\frac{T(r)}{R(r)} = -\frac{\text{const}_2}{\text{const}_1} = \frac{\operatorname{tg} \delta(k)}{k}.$$

As an illustration, consider the potential ⁽⁵⁾ $G^2 V = G^2/r^4$, $T = r \cosh(G/r)$, $R = (r/G) \sinh(G/r)$, from (29): $(\operatorname{tg} \delta(k, \varepsilon)/k) = -G(\sinh(G/\varepsilon)/\cosh G/\varepsilon)$ for any ε finite or arbitrarily small, thus (29) is indeed the continuation of the corresponding series (25) and we can verify with this example the result (31).

We want to show now that $\lim_{\varepsilon \rightarrow 0} \psi(r, \varepsilon)$ is indeed the regular solution near the origin. For this we report (29) in (28)

$$(28a) \quad \psi(r, \varepsilon) = T(r) + R(r) \left(\frac{-\int_{\varepsilon}^{\infty} G^2 T r V dr}{1 + \int_{\varepsilon}^{\infty} G^2 R r V dr} \right).$$

Following (30) and (31), we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \psi(r, \varepsilon) &\underset{r \rightarrow 0}{\simeq} Y_{\text{sing}}(r) \left[\text{const}_2 - \frac{\text{const}_2 \cdot \text{const}_1}{\text{const}_1} \right] + \\ &+ \left[\text{const}_4 - \frac{\text{const}_2 \cdot \text{const}_3}{\text{const}_1} \right] Y_{\text{reg}}(r) \simeq \text{const } Y_{\text{reg}}(r). \end{aligned}$$

For instance for G^2/r^4 we get

$$\lim_{\varepsilon \rightarrow 0} \psi(r, \varepsilon) \underset{r \rightarrow 0}{\simeq} r \exp[-G/r].$$

Thus we see that this introduction of the cut-off is always valid and gives the correct results if we take into account the whole singular part of the solutions. We see also that the analytic continuation for $(\text{tg } \delta(k, \varepsilon)/k)$ is also correct if we consider the *whole* solution (24) or the *whole* series (25). It is then also not surprising that people who have considered special cases where the whole solution can be obtained in a closed form have found that the cut-off procedure works by taking into account the whole cut-off dependence.

We remark that (29) gives a finite result when $\varepsilon \rightarrow 0$ because both numerator and denominator diverge in exactly the same way when $\varepsilon \rightarrow 0$. Similar advantageous situations have been considered by KHURI and PAIS (⁴). We want to remark also that the assumption that G be small or not plays no role in the fact that the analytic continuation of $(\text{tg } \delta(k, \varepsilon)/k)$, given by (25), is valid when $\varepsilon \rightarrow 0$. This result is also independent of the magnitude of the radius of convergence of the series (25). The only important thing is that G^2 or G is positive.

We consider now approximate methods in order to calculate the limit $\varepsilon \rightarrow 0$ in (31).

Firstly, can we apply our method of connection of two limiting procedures in order to obtain (31)? The answer is yes. Consider $R_q(r)$ and $T_q(r)$, the corresponding q -th iterative terms of the perturbation expansion of the integral eq. (27). $R(r)$ is the Jost solution $R(k=0, r)$ previously studied and $T(r)$ is a solution of the same integral equation with a different free term. Both expansions $\sum T_q(r)$ and $\sum R_q(r)$ developed the same divergent singularities $Y_{\text{sing}}(r)$ when $r \rightarrow 0$ and consequently the bounds obtained in the Appendix for the whole singular part of $|R|$ are essentially the same for the whole singular part of $|T|$; thus we have for instance [see (30)]

$$\begin{aligned} \text{const}_2 &= \lim_{p \rightarrow \infty} \sum_{q=0}^p T_q(r(p)) (Y_{\text{sing}}(r(p)))^{-1} & \text{if } r(p) \geq r_L(p), \\ \text{const}_2 &= \lim_{p \rightarrow \infty} \sum_{q=0}^p \left(\int_{\varepsilon(p)}^{\infty} T_q(r) r V(r) dr \right) \left(\int_{\varepsilon(p)}^{\text{const}} Y_{\text{sing}}(r) r V(r) dr \right)^{-1} & \text{if } \varepsilon(p) \geq \varepsilon_L(p), \end{aligned}$$

where $r_L(p) = \varepsilon_L(p)$ has been previously obtained for $R(k, r)/Y_{\text{sing}}(r)$ and depends mainly on the most singular part of the potential.

We get also [see (30)]

$$-\frac{\text{const}_2}{\text{const}_1} = \lim_{r \rightarrow 0} -\frac{T(r)}{R(r)} = \frac{\lim_{r \rightarrow 0} \left(\sum_0^\infty T_q(r) Y_{\text{sing}}^{-1}(r) \right)}{\lim_{r \rightarrow 0} \left(\sum_0^\infty R_q(r) Y_{\text{sing}}^{-1}(r) \right)}$$

and

$$(32) \quad \text{tg } \delta/k = \lim_{p \rightarrow \infty} -\frac{\sum_0^p T_q(r(p))}{\sum_0^p R_q(r(p))} = \lim_{p \rightarrow \infty} -\frac{\sum_0^p \int_{\varepsilon(p)}^\infty T_q(r) r V(r) dr}{\sum_0^p \int_{\varepsilon(p)}^\infty R_q(r) r V(r) dr}$$

if $r(p) = r_L(p)$ and $\varepsilon(p) \geq \varepsilon_L(p) = r_L(p)$.

The important points are that these results are valid although $R_q(r)$ and $T_q(r)$ have not the same r singularities and even if the leading singularities for $\sum R_q(q)$ and $\sum T_q(r)$ do not develop the same type of singularities. This is because in (32) we take into account the whole singular part of T and R .

In order to obtain approximate methods with the use of the cut-off procedure, recent prescriptions have been given. We want now to examine these prescriptions (so-called peratization⁽¹²⁾) in the present context of potential scattering. The programme is first to isolate the leading singular part of each term in the expansion, then to sum these parts and to see if one can give a finite meaning to the sum as $\varepsilon \rightarrow 0$, and so on for the next to leading singular part. We note that the series of the first leading singular terms in the series (25) come from the ratio of the first leading singular terms both in the numerator and in the denominator in (31). In other words, the analytic continuation of the first leading singular terms in (25) is given by considering both leading terms in R and T in (29), (31), and it is only from (29), (31) where the limit $\varepsilon \rightarrow 0$ exists that we can test the validity of the prescription.

We recall that the analytic extension of (25) has a meaning when $\varepsilon \rightarrow 0$ because [see (31)] the whole $R(r)$ and $T(r)$ develop the same divergent singularity when $r \rightarrow 0$. Then we see that here the prescription of peratization for the first leading terms will be valid only if for the family of potentials considered both the sums of the leading terms $\sum \bar{R}_q(r)$ and $\sum \bar{T}_q(r)$ diverge in exactly the same way when $r \rightarrow 0$ [where $\bar{R}_q(r)$ and $\bar{T}_q(r)$ are the first leading terms of $R_q(q)$ and $T_q(r)$]; in the following we write $\bar{X}$ for any first leading singular part of X .

We will show now that this is the case if the most singular part is of power type. In other cases (for instance when the most singular term is a logarithm

⁽¹²⁾ G. FEINBERG and A. PAIS: *Phys. Rev.*, **131**, 2724 (1963).

or an exponential) we will see that $\sum \bar{R}_q$ and $\sum \bar{T}_q$ do not develop the same kind of singularities near the origin when $r \rightarrow 0$. We begin with potentials of the pure power type which have been extensively studied (^{4,6}):

$$(33) \quad V(r) = \frac{G^2}{r^{2n}}, \quad n \text{ arbitrarily } > \frac{3}{2}.$$

In this case $R_q = \bar{R}_q$, $T_q = \bar{T}_q$ and we get from the perturbation solution of (27)

$$(34) \quad \begin{cases} \bar{T}(r) = \Gamma(1-\alpha) r \sum_0^\infty \left(\frac{G\alpha}{r^{n-1}} \right)^{2q} \frac{1}{q! \Gamma(q+1-\alpha)}, \\ \bar{R}(r) = \Gamma(1+\alpha) \sum_0^\infty \left(\frac{G\alpha}{r^{n-1}} \right)^{2q} \frac{1}{q! \Gamma(q+1+\alpha)}, \end{cases}$$

where $\alpha = (1/2(n-1))$. With the use of Stirling's formula, we get

$$2^{2q} q! \Gamma(q+1 \pm \alpha) \underset{q \rightarrow \infty}{\simeq} (2q)! \pi^{\frac{1}{2} \pm \alpha}.$$

We want to obtain the singular part of $\bar{R}$ and $\bar{T}$ when r is small. Then in (34) we investigate for q large and r small. First, we make the change $2q = m$ and recall (^{1b}) that it is possible to obtain a correspondence between m and X when both go to infinity in an exponential-type expansion:

$$(35) \quad \sum_m \frac{X^m m^z}{\Gamma(m+1)} \underset{m \rightarrow \infty}{\simeq} \sum_m \frac{X^m}{\Gamma(m+1)} X^z \left(1 + \frac{z(z-1)}{2m} \right), \quad z \text{ arbitrary.}$$

Then we get for the singular parts of R and T when r is small and m large

$$(36) \quad \begin{cases} \bar{R}(r) \simeq \frac{\Gamma(1+\alpha)}{\pi^{\frac{1}{2}}} \sum_m \left(\frac{2G\alpha}{r^{n-1}} \right)^m \frac{1}{m!} \left(\frac{2}{m} \right)^{\frac{1}{2}+\alpha} \simeq \\ \quad \simeq \frac{\Gamma(1+\alpha)}{\pi^{\frac{1}{2}}} \frac{r^{n/2}}{(G\alpha)^{\frac{1}{2}+\alpha}} \exp [(G/(n-1)r^{n-1})], \\ \bar{T}(r) \underset{r \rightarrow 0}{\simeq} r \frac{\Gamma(1-\alpha)}{\pi^{\frac{1}{2}}} \sum_m \left(\frac{2G\alpha}{r^{n-1}} \right)^m \frac{1}{m!} \left(\frac{2}{m} \right)^{\frac{1}{2}-\alpha} \simeq \\ \quad \simeq \frac{\Gamma(1-\alpha)}{\pi^{\frac{1}{2}}} \frac{r^{n/2}}{(G\alpha)^{\frac{1}{2}-\alpha}} \exp [(G/(n-1)r^{n-1})], \end{cases}$$

and we get in (29), (31)

$$(37) \quad \frac{\overline{\text{tg } \delta(k)}}{k} = \lim_{\varepsilon \rightarrow 0} \frac{\text{tg } \delta(k, \varepsilon)}{k} = \lim_{r \rightarrow 0} -\frac{T(r)}{R(r)} = -\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} (G\alpha)^{1/n-1} = \frac{\text{tg } \delta(k)}{k}.$$

Thus peratization works for this restricted class of singular potentials (33). This is not a new result ^(4,5) but here it is shown that really the analytic continuation of the series (25) gives the correct result. Note from (36) that $\bar{R}$ and $\bar{T}$ give the same singular r -dependence which is the singular dependence of the whole R and T functions. Note that for the family (33) it is obvious that the peratization does work because $R = \bar{R}$, $T = \bar{T}$; then, evidently, $\bar{R}$ and $\bar{T}$, being the whole solutions, have the same singular part [see (30)] when $r \rightarrow 0$. Then the result given here is nothing other than the verification in a particular case of the result (31) of the validity of the whole cut-off procedures.

Suppose now that we add another potential in (33) less singular than (33), but sufficiently singular such that it modifies the behaviour of the solutions of the Schrödinger equation near the origin [see, *e.g.*, ref. ^(1,2)]. Then the behaviour of the whole R and T near the origin is not given by the r -dependence of (36); *i.e.*, only the leading exponential is the same ($\exp[G/(n-1)r^{n-1}]$), but there can appear other essential singularities ^(1,2). Does the prescription of the first leading singularities always give a finite result in (31), which now is only an approximation of ($\text{tg } \delta/k$)? We remark from the integral eq. (27) that the leading term in each q -th iterative term always comes only from the leading singular part of the potential, such that $\bar{T}_q$ and $\bar{R}_q$ are still the same as (34)–(36) and lead to a finite ratio in (31) when $\varepsilon \rightarrow 0$. Peratization for power potentials has been previously studied and the results well established; we have considered here this case because we use the same method for the logarithmic case where the situation does not seem so clear ^(8,10). We thus consider now a pure logarithmic singular potential:

$$(38) \quad V = \frac{G^2 (\log 1/r)^\beta}{r^{2n}}, \quad \beta \text{ arbitrarily } > 0, n > \frac{3}{2}, G^2 > 0.$$

In this case we get from (27) for the leading terms $\bar{R}$ and $\bar{T}$

$$(39) \quad \begin{cases} \bar{R}(r) = \Gamma(1+\alpha) \sum_0^\infty \left(\frac{G\alpha}{r^{n-1}} \left(\log \frac{1}{r} \right)^{\beta/2} \right)^{2q} \frac{1}{q! \Gamma(q+1+\alpha)}, \\ \bar{T}(r) = \Gamma(1-\alpha) \sum_0^\infty r \left(\frac{G\alpha}{r^{n-1}} \left(\log \frac{1}{r} \right)^{\beta/2} \right)^{2q} \frac{1}{q! \Gamma(q+1-\alpha)}. \end{cases}$$

By investigating the singular part of $\bar{R}(r)$ and $\bar{T}(r)$ when $r \rightarrow 0$, in the same way as above in the power case, we get

$$(40) \quad \begin{cases} \bar{R}(r) \underset{r \rightarrow 0}{\simeq} \frac{\Gamma(1+\alpha)}{\pi^{\frac{1}{2}} (G\alpha)^{\frac{1}{2}+\alpha}} \frac{r^{n/2} \exp[2G\alpha/r^{n-1}]}{(\log 1/r)^{\beta/2(\frac{1}{2}+\alpha)}}, \\ \bar{T}(r) \underset{r \rightarrow 0}{\simeq} \frac{\Gamma(1-\alpha)}{\pi^{\frac{1}{2}} (G\alpha)^{\frac{1}{2}-\alpha}} \frac{r^{n/2} \exp[2G\alpha/r^{n-1}]}{(\log 1/r)^{\beta/2(\frac{1}{2}-\alpha)}}. \end{cases}$$

Thus we see that in the logarithmic case $\bar{T}$ and $\bar{R}$ do not develop the same r singularities when r goes to zero. Following what we have said above, the prescription of peratization about the first leading singularity does not work and the ratio (29), (31) has no meaning when the cut-off goes to zero:

$$(41) \quad \frac{\overline{\text{tg } \delta(k)}}{k} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{\text{tg } \delta(k, \varepsilon)}}{k} = \lim_{\varepsilon \rightarrow 0} -\frac{\bar{T}(r)}{\bar{R}(r)} =$$

$$= -\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} (G\alpha)^{2\alpha} \left(\log \frac{1}{r}\right)^{\beta\alpha} \simeq \text{const} \left(\log \frac{1}{\varepsilon}\right)^{\beta/2(n-1)}.$$

If we consider the case $n=2$, $\beta=2$ investigated by ALY *et al.*, we get

$$\lim_{\varepsilon \rightarrow 0} \frac{\overline{\text{tg } \delta(k, \varepsilon)}}{k} = G \log \varepsilon.$$

ALY *et al.*, have obtained this result from the series (25); we see that the analytic continuation (29)–(31) of this series gives the same result. We note that the connection with their results can be extended. In fact, their investigations correspond to the first study of eq. (22), the iteration solution of (22). They have obtained by considering only the leading singularities both in r and ε an approximate solution $\bar{\psi}(r, \varepsilon)$ given in the form (24), which still does not exist when $\varepsilon \rightarrow 0$:

$$(42) \quad \bar{\psi}(r, \varepsilon) \sim r \cosh \left(\frac{G |\log r|}{r} \right) + r \frac{\log \varepsilon}{|\log r|} \sinh (G |\log r|) \text{tgh} \left(\frac{G |\log \varepsilon|}{\varepsilon} \right).$$

We consider now the second formulation of the study of eq. (22), *i.e.*, we consider $\bar{\psi}(r, \varepsilon)$ in (28) where $(\text{tg } \delta(k, \varepsilon)/k)$ is given by (29)–(31), when $\varepsilon \rightarrow 0$. In this case $n=2$, $\beta=2$ from (39) one has

$$\bar{T}(r) = r \cosh \left(\frac{G \log r}{r} \right), \quad \bar{R}(r) = \frac{r}{G \log r} \sinh \left(\frac{G \log r}{r} \right)$$

and then the result (42).

If we return to the general case $n > \frac{3}{2}$, $\beta > 0$, we see from the results (40)–(41) for $\bar{T}$, $\bar{R}$, $(\text{tg } \delta(k, \varepsilon)/k)$ that the first leading approximate solution $\bar{\psi}(r, \varepsilon)$ has no meaning when $\varepsilon \rightarrow 0$. If we now add any other less singular potential, we find that, because the leading singular terms of $\bar{R}$ and $\bar{T}$ are not modified, the breakdown will be the same for the prescription of the first leading singular terms of $(\text{tg } \delta(k, \varepsilon)/k)$. Of course, we can imagine adding judicious terms such that we can group together *many* singularities in each iterative term in such a way that they develop the same kinds of singularities

near the origin both for the corresponding approximations of R and T . But this has nothing to do with the prescription of first summing the leading singularities in each perturbation term. In connection with the results obtained for the first leading terms, we can ask: is it the power case or the logarithmic case that is the exception? For the answer we consider another family of repulsive singular potentials of the exponential type

$$(43) \quad V \simeq G^2 \frac{\exp[\eta/r^\gamma]}{r^{2n}}, \quad \eta > 0, \gamma > 0, n > \frac{3}{2}.$$

We find for the leading part of R and T

$$\begin{aligned} \bar{R}(r) &= \sum_0^\infty \left(\frac{G}{\eta\gamma} \frac{\exp[\eta/2r^\gamma]}{r^{n-(1+\gamma)}} \right)^{2q} \frac{1}{(q!)^2} \underset{r \rightarrow 0}{\simeq} \left(\frac{\eta\gamma}{\pi G} \right)^{\frac{1}{2}} r^{n/2 - (\frac{1}{2} + \gamma/2)} \\ &\quad \cdot \exp[-\eta/4r^\gamma] \exp\left(\frac{2G}{\eta\gamma} \frac{\exp[\eta/2r^\gamma]}{r^{n-(1+\gamma)}} \right), \\ \bar{T} = r\bar{R} \quad \text{and} \quad \frac{\text{tg } \delta(k, \varepsilon)}{k} \underset{\varepsilon \rightarrow 0}{\simeq} -\varepsilon \rightarrow 0. \end{aligned}$$

For the particular case (*) ($G^2 e^{\varepsilon/r}/r^4$) ALY *et al.* (11) have found the same result from the perturbation expansion of the leading term. Thus it is the power case which is the peculiar case and this is due to the peculiar form of the kernel in the integral eq. (27):

$$F(r) = \int_r^\infty (r-r')f(r')dr':$$

Taking $f(r')$ as a sufficiently singular power, this kernel generates a single term of the power type. Then the approximation of the first leading singular terms corresponds always to treating the problem of the leading singular term of the potential exactly. Equivalently, $\bar{R}$ and $\bar{T}$ are the exact solutions corresponding to this most singular term of the potential.

Now for a term which is not of the power type from the kernel we obtain in general *many* terms such that the leading term corresponds only to a part of the singularity. It will be only fortuitous if $\bar{R}$ and $\bar{T}$ develop the same type of singularities when $r \rightarrow 0$ because R and T are two independent solutions of the same integral equation, but with different behaviour at infinity.

We want to add that there is perhaps a possibility that the situation will be more favourable. For instance we can consider the case of the two Jost solutions $R(\pm k, r)$ such that their perturbation expansions develop their r singularities (when $r \rightarrow 0$) in exactly the same way.

(*) See the foot-note at the end of this paper.

Finally, we note that our method ($\varepsilon(p) \rightarrow 0$, $\varepsilon(p) \geq \varepsilon_L(p)$) avoids all these difficulties because we take into account the whole singular part of the solutions independently of the particular features of the singular repulsive interaction.

7. - Numerical applications.

1) In order to illustrate numerically this kind of convergence, we take for instance the case $G^2 V(r) = G^2/r^4$ where the Jost function is $R(r, k=0) = r/G \sin(G/r)$ and the Jost function $R(k=0)$ theoretically, such that $2GR \cdot (k=0) = 1$. Then from the perturbative expansion of $R(r, k=0)$ we apply our method:

$$\lim_{p \rightarrow \infty} G(p, r(p)) = \lim_{p \rightarrow \infty} [\exp[-G/r(p)]] \sum_{n=0}^p (G/r(p))^{2n+1} \frac{1}{(2n+1)!} \rightarrow \begin{cases} 1 & \text{if } r(p) \geq \frac{\text{const}}{p^{1-\varepsilon}}, \\ 0 & \text{if } r(p) < \frac{\text{const}}{p^{1+\varepsilon}}. \end{cases}$$

The results are given in Table I.

TABLE I. - Potential G^2/r^4 , $l=0$, $k=0$; $2G \times (\text{Jost function}) = 1$.

p	$G(p, Gp^{-1})$	$G(p, G/\log p)$	$G(p, Gp^{-2})$	$G(p, Gp^{-3})$
3	0.968	0.888	0.387	0.000008
6	0.992	0.972	0.00001	0
10	0.998	0.99	0	0
∞	1		0	

2) For the same potential $T(r) = r \cosh(G/r)$ and, theoretically $-(1/G)(\text{tg } \delta/k) = 1$. Then from the perturbative expansion of R and T we, still apply our method:

$$\lim_{p \rightarrow \infty} X(p, r(p)) = \lim_{p \rightarrow \infty} \frac{\sum_{n=0}^p (G/r(p))^{2n} (1/2n!)}{\sum_{n=0}^p (G/r(p))^{2n+1} (1/(2n+1)!)} \rightarrow \begin{cases} 1 & \text{if } r(p) \geq \frac{\text{const}}{p^{1-\varepsilon}}, \\ 0 & \text{if } r(p) < \frac{\text{const}}{p^{1+\varepsilon}}. \end{cases}$$

The results are given in Table II.

TABLE II. — $(1/G) \operatorname{tg} \delta(k)/k = 1$.

p	$X(p, Gp^{-\frac{1}{2}})$	$X(p, G/\log p)$	$X(p, Gp^{-3})$	$X(p, Ge^{-p})$
3	1.064	1.25	0.255	0.338
6	1.015	1.057	0.058	0.032
10	1.003	1.02	0.021	0.001
∞	1		0	

In the above work (first and second Part), we have always considered the case where only one parameter (radial co-ordinate or cut-off) occurs. But, generally [see the regular solution $\psi(r, \varepsilon)$, eq. (22), or the Lippmann-Schwinger equation, or the Bethe-Salpeter equation, or the dispersion relation, and so on] there is at least one other variable. Then we think it will be interesting to see if in the case of singular interactions we can formulate the problem in similar ways as in the above work. For instance, it will be interesting to see if there exist cases where the denominator and the numerator in the Fredholm's determinants give rise to the same type of singularities when the cut-off goes to zero (or other limiting value) and if then a similar treatment to the above one can be used. This is what we want now to investigate. We consider the same solution $\psi(r, \varepsilon)$ defined in eq. (22) that we can write

$$(22') \quad \psi(r, \varepsilon) = r + \lambda \int_{\varepsilon}^{\infty} k(r, r') \psi(r', \varepsilon) dr',$$

where $k(r, r') = -V(r')g(r, r')$, $g(r, r') = r\theta(r' - r) + r'\theta(r - r') = g(r', r)$ and $\lambda = G^2 > 0$.

We have shown that the solution can be written see eq. (23a)]

$$(28b) \quad \psi(r, \varepsilon) = r + \frac{N(r, \varepsilon, \lambda)}{D(\varepsilon, \lambda)},$$

where

$$(44) \quad N = (T(r) - r) \left(1 + \int_{\varepsilon}^{\infty} \lambda x R(x) V(x) dx \right) - \\ - R(r) \int_{\varepsilon}^{\infty} \lambda T(x) x V(x) dx = \sum_{q=0}^{\infty} \lambda^{q+1} N_q(r, \varepsilon),$$

$$(45) \quad D = 1 + \int_{\varepsilon}^{\infty} \lambda x R(x) V(x) dx = 1 + \sum_{q=1}^{\infty} \lambda^q D_q(\varepsilon).$$

But we have shown above that $\sum_q \lambda^{q+1} N_q(r, \varepsilon)$ and $\sum_q \lambda^q D_q(\varepsilon)$ develop the same ε singularity when ε is small, this singularity being $\int_{\varepsilon}^{\text{const}} Y_{\text{sing}}(x) V(x) dx$. Then, if we apply the results obtained above, we have that in (28b) $\lim_{\varepsilon \rightarrow 0} \psi(r, \varepsilon)$ is indeed the regular solution $\psi(r)$ and for any r we get

$$(46) \quad \psi(r) = \lim_{\varepsilon \rightarrow 0} \psi(r, \varepsilon) = r + \lim_{p \rightarrow \infty} \frac{\sum_0^p \lambda^{q+1} N_q(r, \varepsilon(p))}{1 + \sum_1^p \lambda^q D_q(\varepsilon(p))} \quad \text{if } \varepsilon(p) \geq \varepsilon_L(p),$$

where $\varepsilon_L(p)$ is the same limiting dependence as $r_L(p)$ investigated in the first Part for different families of most singular repulsive potentials. But (28b) is not the only formulation of the solution of (22') with a N/D type. We know from the Fredholm's theory of integral equations that the solution of (22') can also be written as a ratio ($\varepsilon \neq 0$)

$$(47) \quad \psi(r, \varepsilon) = r + \frac{\int_{\varepsilon}^{\infty} \Delta(r, r', \varepsilon, \lambda) r' dr'}{\Delta(\varepsilon, \lambda)} = r + \frac{\sum_0^{\infty} \lambda^{q+1} B_q(r, \varepsilon)}{1 + \sum_1^{\infty} \lambda^q A_q(\varepsilon)},$$

$$(28c) \quad \psi(r, \varepsilon) = r + \frac{B(r, \varepsilon, \lambda)}{\Delta(\varepsilon, \lambda)}.$$

$\Delta(r, r', \varepsilon, \lambda)$ and $\Delta(\varepsilon, \lambda)$ are called, respectively, Fredholm's first minor and Fredholm's determinant trace of $k(r, r')$. In the following we shall call B_q and A_q Fredholm's q -th determinants of the numerator and of the denominator and

$$(48) \quad A_q(\varepsilon) = \frac{(-1)^q}{q!} \int_{\varepsilon}^{\infty} dr_1 \dots \int_{\varepsilon}^{\infty} dr_q \begin{vmatrix} k(r_1, r_1) & k(r_1, r_2) & \dots & k(r_1, r_q) \\ \vdots & \vdots & \ddots & \vdots \\ k(r_q, r_1) & \dots & \dots & k(r_q, r_q) \end{vmatrix},$$

$$(49) \quad B_q(r, \varepsilon) = \frac{(-1)^q}{q!} \int_{\varepsilon}^{\infty} dr' \int_{\varepsilon}^{\infty} dr_1 \dots \int_{\varepsilon}^{\infty} dr_q r' \begin{vmatrix} k(r, r') & k(r, r_1) & \dots & k(r, r_q) \\ k(r_1, r') & k(r_1, r_1) & \dots & k(r_1, r_q) \\ \vdots & \vdots & \ddots & \vdots \\ k(r_q, r') & k(r_q, r_1) & \dots & k(r_q, r_q) \end{vmatrix}.$$

Note that B_q depends on the free term (here r) of the integral of eq. (22'). On the contrary, A_q depends only on the kernel $k(r, r')\theta(r' - \varepsilon)$. We recall that

JOST and PAIS ⁽¹³⁾, many years ago, in a very important work in potential scattering, have studied the problem of the properties of $\psi(r, \lambda, k)$ with respect to λ (for regular potentials); by using Fredholm's theory of integral equations.

For ε finite, the two series (*) $\sum \lambda^q A_q$ and $\sum \lambda^{q+1} B_q$ converge absolutely for any λ . We assume $f(\lambda, \varepsilon) \neq 0$. From the unicity of the solution of Fredholm's integral equation ($\varepsilon \neq 0$) we get from (29b) and (29c)

$$(50) \quad N/D = B/\Delta.$$

In fact, we want to prove now that

$$(51) \quad \Delta(\varepsilon, \lambda) = D(\varepsilon, \lambda) \quad \text{or} \quad D_q(\varepsilon) = A_q(\varepsilon).$$

First, we consider

$$D(\varepsilon, \lambda) = 1 + \lambda \int_{\varepsilon}^{\infty} r_1 V(r_1) \left(1 + \sum_{q=1}^{\infty} \lambda^q R_q(r_1) \right) dr_1,$$

where $R_q(r_1)$ is the q -th term of the perturbative expansion of $R(r)$ given by (27). We get

$$(52) \quad R_q(r_1) = \int_{r_1}^{\infty} dr_2 \int_{r_1}^{\infty} dr_3 \dots \int_{r_q}^{\infty} dr_{q+1} \prod_{j=1}^q (r_{j+1} - r_j) V(r_{j+1})$$

with $r_q \geq r_{q-1} \geq \dots \geq r_1$,

$$(53) \quad D_q(\varepsilon) = \int_{\varepsilon}^{\infty} r_1 V(r_1) R_{q-1}(r_1) dr_1 = \int_{\varepsilon}^{\infty} dr_1 \int_{r_1}^{\infty} dr_2 \dots \int_{r_{q-1}}^{\infty} \prod_{j=1}^q (r_j - r_{j-1}) V(r_j),$$

where $r_0 = 0$ and $\varepsilon \leq r_1 \leq r_2 \leq \dots \leq r_{q-1}$.

Secondly, we consider $A_q(\varepsilon)$ given by (48), the integrand in (48) being sym-

⁽¹³⁾ R. JOST and A. PAIS: *Phys. Rev.*, **82**, 840 (1951).

(*) For the corresponding conditions about the potentials ($\varepsilon \neq 0$) for the validity of the Fredholm's type of solutions in potential scattering see JOST and PAIS ⁽¹³⁾, R. G. NEWTON: *Journ. Math. Phys.*, **1**, 319 (1960) and the recent work of S. WEINBERG: *Phys. Rev.*, **131**, 443 (1963).

metric in all the variables, we get

$$(54) \quad A_q(\varepsilon) = \int_{\varepsilon}^{\infty} dr_1 \int_{r_1}^{\infty} dr_2 \dots \int_{r_{q-1}}^{\infty} dr_q \mathcal{D}(r_1, r_2, \dots, r_q) \prod_{j=1}^q V(r_j),$$

where $\varepsilon < r_1 < r_2 < \dots < r_q$ and $\mathcal{D}(r_j)$ is the determinant

$$\mathcal{D}(\dots r_j \dots) = \begin{vmatrix} g(r_1, r_1) & \dots & g(r_1, r_q) \\ \cdot & \cdot & \cdot \\ g(r_q, r_1) & \dots & g(r_q, r_q) \end{vmatrix}.$$

But if $x > y$, $g(x, y) = g(y, x) = y$ and $g(x, x) = x$. From this we get

$$\mathcal{D}(\dots r_j \dots) = \begin{vmatrix} r_1 & r_1 & r_1 & \dots & r_1 \\ r_1 & r_2 & r_2 & \dots & r_1 \\ \cdot & r_2 & r_3 & \dots & r_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_1 & r_2 & r_3 & \dots & r_q \end{vmatrix}$$

We see that $\mathcal{D}$ is quadratic in $r_1, r_2, \dots, r_{q-1}$, linear in r_q . Furthermore, $\mathcal{D} = 0$ if $r_1 = 0$ or $r_j = r_{j-1}$ ($j = 2, \dots, q$). Then it is easy to get

$$\mathcal{D}(\dots r_j \dots) = \prod_{j=1}^q (r_j - r_{j-1}),$$

(with $r_0 = 0$) and

$$A_q(\varepsilon) = D_q(\varepsilon).$$

Thus (*) we have proved [eq. (51)]: $\Delta(\varepsilon, \lambda) = D(\varepsilon, \lambda)$. From (50) it follows that $N(r, \varepsilon, \lambda) = B(r, \varepsilon, \lambda)$ or

$$(55) \quad B_q(r, \varepsilon) = N_q(r, \varepsilon).$$

(*) We recall that JOST and PAIS⁽¹³⁾ have shown for $l=0$, $k \neq 0$ (regular potentials) that the denominator in Fredholm's solution of the regular solution is equal to the Jost function. The result we obtain $\Delta(\varepsilon, \lambda) = D(\varepsilon, \lambda)$ can be considered as an extension of their result for singular potentials. In fact, if we consider usual regular potentials ($V \sim \text{const } r^{-\eta}$, $\eta < 2$) we can put $\varepsilon = 0$ and from the perturbative expansion (52) we get $R(0) = 1 + \int_0^{\infty} \lambda r R(r) V(r) dr = D(\varepsilon = 0, \lambda) = \Delta(\varepsilon = 0, \lambda)$, $R(0)$ being the Jost function for $k=0$, or the Jost solution for $k=0$, $r=0$.

We want now to verify this relation for $q=0$ and $q=1$. We investigate the λ expansion given by (44). For this we substitute R and T by their perturbative expansion. For R it is given in (52) and for T we get from the integral eq. (27)

$$T(r) = \sum_0^{\infty} \lambda^q T_q(r), \quad T_0(r) = r,$$

and

$$(56) \quad T_q(r) = \int_r^{\infty} dr_1 \int_{r_1}^{\infty} dr_2 \dots \int_{r_{q-1}}^{\infty} dr_q r_q \prod_{j=1}^q V(r_j) (r_j - r_{j-1}) \quad (r_0 = r),$$

$$(57) \quad N(r, \varepsilon, \lambda) = \left[\sum_1^{\infty} \lambda^m T_m(r) \right] \left[1 + \sum_1^{\infty} \lambda^n D_n(\varepsilon) \right] - \left[1 + \sum_1^{\infty} \lambda^m R_m(r) \right] \cdot \left[\sum_1^{\infty} \lambda^n \int_{\varepsilon}^{\infty} T_{n-1}(r) r V(r) dr \right].$$

First, we consider the case $q=0$. From (49) and (57), we get

$$B_0(r, \varepsilon) = \int_{\varepsilon}^{\infty} k(r, r') r' dr',$$

$$N_0(r, \varepsilon) = T_1(r) - \int_{\varepsilon}^{\infty} T_0 r' V(r') dr' = - \int_{\varepsilon}^{\infty} r' V(r') [r' + (r - r') \theta(r' - r)] dr' = B_0(r, \varepsilon).$$

Secondly, we consider the case $q=1$. Still from (49), we get

$$B_1(r, \varepsilon) = \int_{\varepsilon}^{\infty} dr' \int_{\varepsilon}^{\infty} dr'' r'' [k(r, r'') k(r'', r') - k(r, r') k(r'', r'')]$$

and, after some algebra,

$$B_1(r, \varepsilon) = \int_{\varepsilon}^{\infty} dr' \int_{\varepsilon}^{\infty} dr'' k(r, r') r'' (r' - r'') V(r'').$$

From eq. (57) we get

$$(58) \quad N_1(r, \varepsilon) = \left(T_2(r) - \int_{\varepsilon}^{\infty} x V(x) T_1(x) dx \right) + \left(T_1(r) D_1(\varepsilon) - R_1(r) \int_{\varepsilon}^{\infty} x^2 V(x) dx \right).$$

By re-arrangement, the two first terms in the right-hand side of (58) can be written

$$-\int_{\varepsilon}^{\infty} dr' \int_{r'}^{\infty} dr'' k(r, r') r'' (r' - r'') V(r'')$$

and, by re-arrangement, the two last terms are written

$$\int_{\varepsilon}^{\infty} dr' \int_{\varepsilon}^{\infty} dr'' k(r, r') r'' (r' - r'') V(r''),$$

such that we have also $B_1(r, \varepsilon) = N_1(r, \varepsilon)$. The proof can be extended for $q = 2 \dots$.

What are the consequences of the identification $N \equiv B$ and $D \equiv A$? All results obtained previously for $\psi(r, \varepsilon)$ with N/D , (29b), are still valid with the Fredholm's type solution B/A , [eq. (29c)]. We consider always singular potentials where the most singular part is repulsive and of the families considered previously.

1) The whole numerator and the whole denominator in the Fredholm's type solution develop the same cut-off singularities when $\varepsilon \rightarrow 0$, such that they diverge in the same way and the limit $\varepsilon \rightarrow 0$ of the ratio has always a meaning.

2) The limit $\varepsilon \rightarrow 0$ of this ratio is indeed the right regular solution of the Schrödinger equation:

$$\lim_{\varepsilon \rightarrow 0} \psi(r, \varepsilon) = \psi(r) = r + \lim_{\varepsilon \rightarrow 0} \frac{\int_{\varepsilon}^{\infty} \Delta(r, r', \varepsilon, \lambda) r' dr'}{\Delta(\varepsilon, \lambda)}.$$

3) We can connect the cut-off going to zero and the order of Fredholm's determinants both in the numerator and in the denominator in such a way that we have for the regular solution

$$(59) \quad \psi(r) = r + \lim_{p \rightarrow \infty} \frac{\sum_0^p \lambda^{q+1} B_q(r, \varepsilon(p))}{1 + \sum_0^p \lambda^q A_q(\varepsilon(p))} \quad \text{if} \quad \varepsilon(p) \geq \varepsilon_L(p),$$

where the limiting dependence $\varepsilon_L(p)$ is the same as for the S -matrix (first Part of this paper) and has been determined for different families of repulsive singular potentials.

In the first Part, as well as in the second Part of this paper, we have never discussed the problem of bound states.

In the first Part where we give a method in order to calculate the Jost function, it is sufficient to impose that the limit of the sequences corresponding to the available $r(p)$ dependences ($r(p) \geq r_L(p)$) is zero ($R(k = -ix) = 0$, $x > 0$).

In the second Part (for $k=0$) we recall that we have identified the expansion of Fredholm's denominator $\Delta(\varepsilon, \lambda)$ with an expression including the Jost solution:

$$1 + \sum_1^{\infty} \lambda^q A_q(\varepsilon) = 1 + \lambda \int_{\varepsilon}^{\infty} r R(k=0, r) V(r) dr,$$

so that the singular ε part of Δ is

$$F(\varepsilon) = \int_{\varepsilon}^{\text{const}} Y_{\text{sing}}(r) r V(r) dr.$$

For the Jost function we get

$$R(k=0) = \lim_{\varepsilon \rightarrow 0} \frac{\Delta(\varepsilon, \lambda)}{F(\varepsilon)}$$

and

$$R(k=0) = \lim_{p \rightarrow \infty} \left(\sum_1^p \lambda^q A_q(\varepsilon(p)) F^{-1}(\varepsilon(p)) \right) \varepsilon(p) \geq \varepsilon_L(p).$$

$\varepsilon_L(p)$ is the same as $r_L(p)$ previously discussed and for the bound state condition we impose that the limit is zero.

8. - Conclusions.

In the first Part, as well as in the second Part of this paper, we have considered physical quantities and physical solutions which are the finite ratio of two functions when a parameter (radial co-ordinate or cut-off) goes to zero. We have seen that if both functions develop the same type of singularities when the parameter goes to zero, we are in a favourable situation in order to obtain approximate methods for the explicit calculation of the finite ratio. For instance, for the cases investigated, we have shown that a powerful method consists in connecting the order of perturbation expansions of the two functions and the parameter going to zero; this method having the great advantage of taking into account the whole singular part of the two functions.

For the Jost function, we have used the finite ratio of the Jost solution and its singular part when $r \rightarrow 0$. We recall that in quantum field theory the vertex renormalization constant can be defined, at least in ladder approximation, as the limit of the vertex function for large momentum. Our method could be perhaps useful to pick out the finite (physically meaningful) part of this constant defined as the ratio between the vertex and the singular part at $p^2 \rightarrow \infty$ [see, e.g., the work of FURLAN and MAHOUX ⁽¹⁴⁾].

We have also shown that the prescription of summing up the first leading singularities of the perturbation expansion in the parameter for the ratio can be very dangerous if the leading singular terms in the perturbation expansion of the two functions, giving the ratio, do not develop the same kind of singularities.

Although these results have been obtained in potential scattering explicitly, it is our hope that they are not restricted to this particular case, but are perhaps more general features for the finite ratio of two singular functions diverging in the same way, both given by singular expansions.

For instance, concerning the results obtained with Fredholm's formulation of the regular solution in co-ordinate space ($l=0$, $k=0$), we think it will be interesting to see if they can be extended to other types of integral equation with singular kernels ($l \neq 0$, $k \neq 0$, in co-ordinate space and Lippmann-Schwinger ⁽¹⁵⁾ equation, momentum space, Bethe-Salpeter equation, dispersion relation, etc.). After all, the first result obtained that the Fredholm's numerator and denominator diverge in the same way and that the finite limit of the ratio is the right solution can be considered as a justification *a posteriori* of the introduction of a regulator or a cut-off. When this is proved, because of the fact that in the q -th determinant, generally, for singular interaction we cannot put $\varepsilon=0$ (or regulator $=\infty$), we have the feeling that, by connecting the cut-off and the order of expansion in the Fredholm's determinants, there exists perhaps available dependence in order to obtain the right finite ratio. But certainly there exist also conditions similar to the Schrödinger case where the most singular part of the potential must be repulsive, in order to obtain finite ratio from the Fredholm's numerator and denominator.

* * *

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⁽¹⁴⁾ G. FURLAN and G. MAHOUX: *Nuovo Cimento*, 36, 215 (1965).

⁽¹⁵⁾ H. CORNILLE: *Limiting procedures for singular potentials*, II, CERN preprint TH. 544.

APPENDIX

We want to find bounds for the singular part of $R_q(k, r)$ when r is small and also bounds for the singular parts of $M(k, r, p)$ and $G(k, r, p)$ defined by (7) when r is small and p large. For this we iterate (4) and find for the singular part of the q -th term

$$(A.1) \quad R_q(r) \simeq \int_r^a (r' - r) \left(\bar{V}(r') + \frac{l(l+1)}{r'^2} \right) R_{q-1}(r') dr',$$

where $\bar{V}(r)$ is the singular part of the potential near the origin and a is a small finite constant.

A) Firstly we assume

$$(A.2) \quad \bar{V}(r) \underset{r \rightarrow 0}{\simeq} \frac{G^2}{r^{2n}} [1 + \varphi(r)] \begin{cases} \varphi \xrightarrow{r \rightarrow 0} 0, & G^2 > 0, n > 1, \\ |\varphi(r)| < \text{const}, \\ r \in [0, a]. \end{cases}$$

We put formally $R_q = X_q f_q$ where

$$(A.3) \quad X_q = \left(\frac{G}{2(n-1)r^{n-1}} \right)^{2q} \frac{1}{\Gamma(q+1)\Gamma((q+1-1/2(n-1)))}.$$

We substitute (A.2), (A.3) in (A.1) and get

$$(A.4) \quad |f_q(r)| < \text{const} \max_{r' \in [r, a]} |f_{q-1}(r')| [1 + \text{const} q r^{2(n-1)q}].$$

All const are positive finite constants in this Appendix. We iterate (A.4) and we get, with the use of Stirling's formula,

$$|f_q| < \max_{r' \in [r, a]} |f_1(r')| [\text{const} + \text{const} q r^{2(n-1)q}],$$

$$|R_q| < \left(\frac{\text{const}}{r^{n-1}} \right)^{2q} \frac{1}{(q!)^2} q^{\text{const}} [1 + \text{const} q r^{2(n-1)q}].$$

Suppose that we consider another solution of (4) defined with a different free term [e.g., $T(r)$ defined in the second Part of the paper for the case $k=0$ by the free term r]. It is sufficient in order to obtain bounds for the perturbation expansion to modify $\max_{r' \in [r, a]} |f_1(r')|$ and it is easily seen that this does not modify the r -dependence of the bound for the function $T(r)$ defined in the second Part of the paper. We now give an arbitrary $r(p)$ -dependence

$r(p) = (\text{const}/p^{1/(n-1)(1+\lambda)})$ where γ is arbitrarily > -1 , and we get for p large

$$\sum_{p+1}^{\infty} |R_q(k, r(p))| < \sum_{p+1}^{\infty} \left(\frac{\text{const } p^{1+\gamma}}{q} \right)^{2q} \left(1 + \frac{\text{const } q}{p^{2+2\gamma}} \right)^q q^{\text{const}}.$$

$$\cdot |M(k, r(p), p)| < \exp[-\text{const } p^{1+\gamma}] \sum_{p+1}^{\infty} \left(\frac{\text{const}}{q} + \frac{\text{const}}{p^{2\gamma}} \right)^q.$$

If $-1 < \gamma < 0$ we can choose p large enough, such that $|M| \xrightarrow{p \rightarrow \infty} 0$. Thus

$$(A.5) \quad |G(k, r(p), p)| \xrightarrow{p \rightarrow \infty} R(k) \quad \text{if} \quad r(p) \geq \text{const}/p^{(1n-1)-\varepsilon}.$$

Now we assume $\gamma > 0$ and consider the sum of the p first terms R_q and get

$$\sum_1^p |R_q(k, r(p))| < \sum_1^p \frac{(\text{const } p^{1+\gamma})^{2q}}{(q!)^2}.$$

In the last sum, for p large enough, the maximum term is the last one. The

$$(A.6) \quad |G(k, r(p), p)| < \text{const}^{2p} \exp[-\text{const } p^{1+\gamma} + 2p\gamma \log p] \xrightarrow{p \rightarrow \infty} 0.$$

In order to understand more rapidly these results about $\gamma \leq 0$ we can consider the $\lim_{p \rightarrow \infty} \exp[-1/(r(p))^{n-1}] \sum_0^p u_q(r(p))$ with

$$u_q = \frac{1}{q! r^{(n-1)q}} = \frac{p^{(1+\gamma)q}}{q!}$$

for the previous limiting dependence. Because of the fact that $u_{q+1}/u_q = p^{1+\gamma}/(q+1)$ we see that for $-1 < \gamma < 0$ the main terms of the sum are inside the range $[1, p]$, whereas for $\gamma > 0$ they are outside. Then in the second case $k > 0$ it is the remaining sum $\sum_{p+1}^{\infty}$ which is the important one.

B) Now we consider

$$\bar{V}(r) \underset{r \rightarrow 0}{\sim} \frac{G^2}{r^2} [1 + \varphi(r)] \begin{cases} \varphi(r) \xrightarrow{r \rightarrow 0} 0 \text{ as a power } r^{\text{const}}, \\ G^2 \geq 0. \end{cases}$$

[In this case $Y_{\text{sing}}(r) = r^\lambda$ with $\lambda = \frac{1}{2}\{1 - \sqrt{1 + 4(l(l+1) + G^2)}\}$ for $G^2 \neq 0$, $\lambda = -l$ for $G^2 = 0$. We take $X_q = G^{2q}((\log 1/r)^q/q!)$ and we find

$$|f_q| < \text{const}^q \max_{r \in [ar, l]} |f_1| \quad \text{and} \quad |R_q(k, r)| < \text{const}^q (\log 1/r)^q/q!.$$

From this we obtain in the same way as previously

$$(A.7) \quad \begin{cases} |G(k, r(p), p)| \xrightarrow{p \rightarrow \infty} 0 & \text{if } r(p) \leq \frac{1}{\exp[\text{const } p^{1+\varepsilon}]}, \\ |M(k, r(p), p)| \xrightarrow{p \rightarrow \infty} 0 & \text{if } r(p)' \geq r_L(p) = \frac{1}{\exp[\text{const } p^{1-\varepsilon}]}, \end{cases} \quad \varepsilon \text{ small } > 0.$$

C) *Logarithmic case*: we assume

$$\bar{V}(r) \underset{r \rightarrow 0}{\simeq} \frac{G^2 (\log 1/r)^\beta}{r^{2n}} [1 + \varphi(r)] \quad \begin{cases} \varphi \xrightarrow{r \rightarrow 0} 0, \quad n > 1, \beta \text{ arbitrary}, \\ |\varphi| < \text{const}, \quad r \in (r, a], \quad G^2 > 0, \end{cases}$$

We put

$$X_q = \left(\frac{G (\log 1/r)^{\beta/2}}{(n-1)r^{n-1}} \right)^{2q} \frac{1}{q! \Gamma(q+1-1/2(n-1))} \quad \text{in the case } \beta > 0$$

and

$$X_q = \left(\frac{G}{(n-1)r^{n-1}} \right)^{2q} \frac{1}{q! \Gamma(q+1-1/2(n-1))} \quad \text{in the case } \beta < 0.$$

We find

$$|f_q| < \text{const}^q [1 + q \text{ const } r^{2(n-1)}]^q \text{Max}_{r' \in (r, a]} |f_1(r')|.$$

1) $\beta > 0$.

$$|R_q| < \left(\frac{(\log 1/r)^{\beta/2} \text{const}}{r^{n-1}} \right)^{2q} \frac{q^{\text{const}}}{(q!)^2} [1 + \text{const } q r^{2(n-1)}]^q;$$

we put $r(p) = (\text{const}/p^{1/(n-1)-\varepsilon})$ and we find (with $\varepsilon > 0$, $1/(n-1) - \varepsilon > 0$)

$$|M(k, r(p), p)| < \text{const} \exp[-\text{const } p^{1-\varepsilon}] \sum_{p+1}^{\infty} \cdot \left(\frac{\text{const} (\log p)^{\text{const}}}{q} + \frac{\text{const} (\log p)^{\text{const}}}{p^{2\varepsilon}} \right)^q \xrightarrow{p \rightarrow \infty} 0.$$

For the dependence $r(p) = (\text{const } p^{1/(n-1)+\varepsilon})$, ($\varepsilon > 0$) we get

$$|G| < \text{const} \exp[-\text{const } p^{1+\varepsilon}] \sum_1^p \frac{(\text{const } p^{1+\varepsilon} (\log p)^{\text{const}})^{2q}}{(q!)^2} \xrightarrow{p \rightarrow \infty} 0.$$

2) $\beta < 0$.

$$|R_q| < \left(\frac{\text{const}}{r^{n-1}} \right)^{2q} \frac{q^{\text{const}}}{(q!)^2} [1 + q r^{2(n-1)}]^q,$$

and we get similarly to the $\beta > 0$ case

$$|M| < \text{const} \exp \left[-\frac{\text{const } p^{1-\varepsilon}}{(\log p)^{\beta/2}} \right] \sum_{p+1}^{\infty} |R_q| \xrightarrow{p \rightarrow \infty} 0 \quad (\varepsilon > 0).$$

$$|G| < \text{const} \exp \left[-\frac{\text{const } p^{1+\varepsilon}}{(\log p)^{\beta/2}} \right] \sum_0^p |R_q| \xrightarrow{p \rightarrow \infty} 0$$

D) *Transition case:*

$$\bar{V}(r) \simeq \frac{G^2}{r^2} \left(\log \frac{1}{r} \right)^{\beta} [1 + \varphi(r)] \quad \varphi(r) \xrightarrow{r \rightarrow 0} 0 \text{ as a power } r^{\text{const}}.$$

1) $\beta > 0$.

We put

$$X_q = \left[G \int_r^a \frac{(\log 1/r')^{\beta/2}}{r'} dr' \right]^{2q} \frac{1}{(2q)!}.$$

We get

$$|f_q| < \text{const}^q \max_{r' \in [r, a]} |f_1| \left[1 + \frac{q \text{ const}}{(\log 1/r)^{\beta+2}} \right]^q,$$

then

$$|R_q| < \frac{(\text{const}(\log 1/r)^{(\beta/2)+1} + \text{const})^{2q}}{(2q)!} \left[1 + \frac{q \text{ const}}{(\log 1/r)^{\beta+2}} \right]^q.$$

We put $r(p) = 1/\exp[\text{const } p^{(1/1+(\beta/2))-\varepsilon}]$, where $\varepsilon > 0$ and we find

$$|M| < \text{const} \exp \left[-\text{const } p^{1-\varepsilon} \right] \sum_{p+1}^{\infty} \left[\left(1 + \frac{\text{const}}{p^{2\varepsilon}} \right)^2 \left(\frac{\text{const}}{q} + \frac{\text{const}}{p^{2\varepsilon}} \right) \right]^q \xrightarrow{p \rightarrow \infty} 0.$$

For $r(p) = 1/\exp[\text{const}/p^{(1/1+(\beta/2))+\varepsilon}]$, we get

$$|G| < \text{const} \exp \left[-\text{const } p^{1+\varepsilon} \right] \sum_1^p \frac{(\text{const} + \text{const } p^{1+\varepsilon})^{2q}}{(2q)!} \xrightarrow{p \rightarrow \infty} 0.$$

2) If $\beta < 0$, $|\beta| < 1$, $l = 0$, we put

$$X_q = \left(\frac{G^2}{1-|\beta|} \left(\log \frac{1}{r} \right)^{1-|\beta|} \right)^q \frac{1}{q!}.$$

and get

$$|f_a| \leq \text{const}^a \left[1 + \frac{\text{const}}{(\log 1/r)^{1-|\beta|}} \right]^a \max_{r' \in [r, a]} |f_1(r')|,$$

$$|R_a| \leq \frac{\text{const}^a}{q!} \left(\log \frac{1}{r} \right)^{(1-|\beta|)a} \left[1 + \frac{\text{const}}{(\log 1/r)^{1-|\beta|}} \right]^a.$$

For $r(p) = 1/\exp[\text{const } p^{(1-|\beta|)-\varepsilon}]$, we get

$$|M| < \text{const} \exp[-\text{const } p^{1-\varepsilon}] \sum_{p+1}^{\infty} \left(\frac{\text{const}}{q} \frac{\text{const}}{p^\varepsilon} \right)^q \xrightarrow{p \rightarrow \infty} 0.$$

For $r(p) = 1/\exp[\text{const } p^{(1-|\beta|)+\varepsilon}]$, we get

$$|G| < \text{const} \exp[-\text{const } p^{1+\varepsilon}] \sum_1^p \frac{(p^{1+\varepsilon} \text{const} + \text{const})^q}{q!} \xrightarrow{p \rightarrow \infty} 0.$$

E) Exponential case

$$\bar{V}(r) \underset{r \rightarrow 0}{\simeq} \left[\frac{G^2}{r^{2n}} \exp[\eta/r^\gamma] \right] [1 + \varphi(r)], \quad \eta > 0, \gamma > 0, G^2 > 0.$$

We put

$$X_a = \left[G \int_r^{\text{const}} \frac{\exp[\eta/2r'^\gamma]}{r'^n} dr' \right] \frac{1}{(2q)!}.$$

We get

$$|f_a| < \text{const}^a \max_{r' \in [r, a]} |f_1| \left[\text{const} + \frac{q \text{const}}{\left\{ \int_r^{\text{const}} (\exp[\eta/2r'^\gamma]/r'^n) dr' \right\}^2} \right]^a,$$

then

$$|R_a| \leq \frac{(\exp[\eta/2r^\gamma][\text{const } r^{1-n} + \text{const}])^{2q}}{(2q)!} \left(1 + \frac{q \text{const}}{\exp[\eta/2r^\gamma] r^{\text{const}}} \right)^q.$$

We put $r(p) = 1/(\text{const} \log p^{(2(1-\varepsilon)/\eta)})^{1/\gamma}$, and we get

$$|M| < \text{const} \exp[-\text{const } p^{1-\varepsilon} (\log p)^{\text{const}}].$$

$$\sum_{p+1}^{\infty} \left[\frac{\text{const}(\log p)^{\text{const}}}{q} + \frac{\text{const}(\log p)^{\text{const}}}{p^\varepsilon} \right]^q \xrightarrow{p \rightarrow \infty} 0.$$

For $r(p) = 1/(\text{const} \log p^{(2(1+\varepsilon)/\eta)})^{1/\gamma}$, we get

$$|G| < \text{const} \exp[-\text{const } p^{1-\varepsilon} (\log p)^{\text{const}}] \sum_1^p \frac{(\text{const}(\log p)^{\text{const}} p^{1+\varepsilon})^{2q}}{(2q)!} \xrightarrow{p \rightarrow \infty} 0.$$

Footnote (*).

In order to avoid any misunderstanding we want to repeat that in the second part of this work we have defined at the beginning the cut-off potential as $V_\varepsilon(r) = \theta(r - \varepsilon) G^2 V(r)$, ($G^2 > 0$) and afterwards we have never changed this family of cut-off potentials. We repeat also that the corresponding perturbation expansion of $(\text{tg } \delta/k)(k, \varepsilon)$ is defined by (22), (23), (24), (25) where the q -th order term is $G^{2q} a_q(\varepsilon)$ and $a_q(\varepsilon)$ is G^2 -independent.

It is clear, of course, that these definitions have excluded families of potentials with more than one coupling constant or also potentials like $V_\varepsilon = \theta(r - \varepsilon) G^2 [V_1(r) + V_2(r)/G^2]$ where the corresponding a_q is $a_q(\varepsilon, G^2)$, G^2 -dependent. This implies, for instance, that the sentence « For the particular case ($G^2 \exp [2/r]/r^4$) ... term » means « For the particular case $V = G^2 \exp [2/r]/r^4$... term », [see ALY et al. ⁽¹¹⁾, p. 328], and cannot mean « For the particular case $V = G^2(\exp [2/r]/r^4 + 1/4G^2 r^4)$... term ».

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RIASSUNTO (*)

Per potenziali ripulsivi singolari, come anche per i potenziali regolari, la matrice S è il limite del rapporto delle due soluzioni di Jost quando la coordinata radiale tende a zero. Nella prima Parte di questo articolo si dimostra che la matrice S è il limite di successioni, ciascun termine delle quali è il rapporto dello sviluppo perturbativo dello stesso ordine delle due soluzioni di Jost. Si dimostra che queste successioni convergono alla matrice S se colleghiamo opportunamente l'ordine dello sviluppo perturbativo e la coordinata radiale. Nella seconda Parte, si espongono alcune osservazioni sui procedimenti di taglio nello scattering di potenziale: a) Si possono ottenere le grandezze fisiche se si tien conto di tutto il contributo dato dal taglio? b) Si può collegare il taglio con l'ordine della perturbazione? c) La recente indicazione di sommare le prime singolarità principali è un metodo approssimato valido? d) Nelle soluzioni di Fredholm si possono avere gli stessi tipi di divergenze sia nel numeratore che nel denominatore quando il taglio tende a zero? e) Possiamo collegare il taglio che tende a zero con l'ordine dei determinanti di Fredholm che tendono all'infinito sia nel numeratore che nel denominatore in modo da ottenere la soluzione fisica come limite di successioni convergenti?

(*) Traduzione a cura della Redazione.