CM-P00057165

LIMITING PROCEDURES FOR SINGULAR POTENTIALS - II

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ABSTRACT

In a preceding paper, a method was given to construct explicitly the zero energy physical solution of the Schrödinger equation for singular potentials where the most singular part is repulsive. Here we extend the previous results both for the physical solution in co-ordinate space and for the solution of the Lippman-Schwinger equation. We introduce an arbitrary cut-off, study the Fredholm type of solution and show that :

- i) both the numerator and the denominator diverge in the same way when the cut-off goes to zero and the limit of their ratio is indeed the right solution ;
- ii) if we consider all variables and constants fixed and take the cut-off going to zero, then the solution is the limit of convergent sequences if we connect in a certain manner the cut-off going to zero and the order of Fredholm's determinants going to infinity, both in the numerator and in the denominator.

THE HISTORY OF THE UNITED STATES OF AMERICA

The history of the United States of America is a story of a young nation that grew from a small group of colonies on the eastern coast of North America into a powerful, independent country. The story begins with the first European settlers, who came to the New World in search of new opportunities and a better life. They found a land of vast natural resources and a people who were different from them, but they also found a land of freedom and opportunity. The story of the United States is a story of the struggle for independence, the struggle for equality, and the struggle for a better future for all.

The story of the United States is a story of the struggle for independence. In 1776, the thirteen colonies declared their independence from Great Britain. They fought a war that lasted for eight years, and they won. The United States was born. The story of the United States is a story of the struggle for equality. In 1787, the Constitution was written. It gave the people the right to elect their representatives to Congress. It gave the people the right to elect a President. It gave the people the right to elect a Supreme Court. The story of the United States is a story of the struggle for a better future for all. In 1863, the Emancipation Proclamation was issued. It declared that all slaves were free. It was a great step forward, but it was not the end of the struggle. The story of the United States is a story of the struggle for a better future for all.

I. INTRODUCTION

In a preceding paper ¹⁾, for singular repulsive potentials and energy equal to zero in the Schrödinger equation, we have investigated the Fredholm formulation of the physical solution. For this we have introduced an arbitrary cut-off on the integration path of the integral equation of the solution in co-ordinate space. The main results were :

- 1) First, we have shown that the expansions in the Fredholm determinants both in the numerator and in the denominator of the solution diverge in exactly the same way when the cut-off goes to zero such that the limit of the ratio exists.
- 2) Secondly, we have shown that this limit of the cut-off going to zero in the Fredholm (cut-off dependent) solution gives indeed the right physical solution of the Schrödinger equation.
- 3) Thirdly, we have shown that we can connect in a certain manner the cut-off going to zero and the order of the Fredholm determinants both in the numerator and in the denominator in such a way that for any value of the radial co-ordinate the physical solution is the limit of convergent sequences.

In supplement to these properties 1), 2), 3), it was also shown that the prescription of summing up the first leading singularities can be very dangerous when the most singular part of the potential is not of the power type. On the contrary, properties 1), 2) hold for any singular repulsive potentials and property 3) was explicitly investigated for arbitrary singular potentials where the most singular repulsive part near the origin is either of the inverse power type or the logarithmic type or either of the exponential type.

It is the aim of this paper to extend these results for non-zero energy. On the one hand, we still consider the problem in co-ordinate space and show that properties 1), 2), 3), are still valid.

On the other hand, we consider the Lippman-Schwinger formulation of potential scattering. We recall that, at least formally, there is some analogy between the Lippman-Schwinger equation and the Bethe-Salpeter equation in ladder approximation. It is shown in this paper that for the Lippman-Schwinger equation properties similar to 1), 2), 3), can be found.

II. THE PHYSICAL SOLUTION IN CO-ORDINATE SPACE

We consider the Schrödinger equation

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - \lambda V \right) \phi = 0 \quad (1)$$

We want to extend results which have been obtained previously ¹⁾ for $k = 0$. We assume for the potential

- i) $V(r)$ is singular near the origin and the most singular part is repulsive ($\lambda > 0$);
- ii) when $r \rightarrow \infty$, $V(r)$ decreases more rapidly than r^{-3} , ($\gamma > 3$).

A. S wave ($\ell = 0$)

We introduce an arbitrary cut-off $\epsilon > 0$, ($\epsilon \neq 0$) such that the potential is $V_\epsilon = \theta(r-\epsilon) V(r)$ and we define

$$\psi(r, k, \epsilon) = e^{-ikr} - e^{ikr} + \lambda \int_{\epsilon}^{\infty} K(r, r', k) \psi(r', k, \epsilon) dr' \quad (2)$$

where $K(r, r', k) = -V(r') k^{-1} g(r, r', k)$ and g/k is the free Green function

$$g(r, r', k) = \sin kr' e^{ikr} \theta(r-r') + \sin kr e^{ikr'} \theta(r'-r) \quad (3)$$

From (2) we get

$$\psi(r, k, \epsilon) \underset{r \rightarrow \infty}{\sim} e^{-ikr} - S(k, \epsilon) e^{ikr} \quad (4)$$

$$S(k, \epsilon) = 1 + \frac{\lambda}{k} \int_{\epsilon}^{\infty} \sin kr V(r) \Psi(r, k, \epsilon) dr \quad (5)$$

The problem is to see if $\lim_{\epsilon \rightarrow 0} \Psi(r, k, \epsilon)$ and $\lim_{\epsilon \rightarrow 0} S(k, \epsilon)$ exist and if these limits are really the physical solution $\Psi(r, k)$ and the physical $S(k)$ matrix.

1) Solution of Equation (2) by successive substitutions (Born series)

From the structure of the kernel in (2) we see that Ψ can be written

$$\Psi(r, k, \epsilon) = (\Psi^1(r, k) + \Psi^2(r, k, \epsilon)) \theta(r - \epsilon) \quad (6)$$

If we substitute this perturbative solution (6) of Ψ in (5) we get

$$S(k, \epsilon) = 1 + \sum_q \lambda^q \mathcal{A}_q(k, \epsilon) \quad (7)$$

But we have seen in Ref. ¹⁾ that it is not so easy to go to the limit $\epsilon \rightarrow 0$ in $\Psi(r, k, \epsilon)$ and $S(k, \epsilon)$ given by (6) and (7). We have shown also in Ref. ¹⁾ that it is more convenient to use a formulation such that Ψ and S are the ratios of two functions diverging in the same way when the cut-off goes to zero.

2) Solution of Equation (2) in terms of Jost solutions

Thus, secondly, we investigate the solution of (2) in another manner by a change of the integration path in (2); $[\epsilon, r] \rightarrow [\epsilon, \omega] + [\omega, r]$ such that $\Psi(r, k, \epsilon)$ satisfies another equivalent integral equation

$$\psi(r, k, \epsilon) = e^{-ikr} - e^{ikr} S(k, \epsilon) + \frac{\lambda}{k} \int_r^\infty \sin k(r'-r) V(r') \psi(r', k, \epsilon) \theta(r'-\epsilon) dr' \quad (8)$$

Consider the integral equation :

$$\phi(r, k) = \phi_0(r, k) + \frac{\lambda}{k} \int_r^\infty \sin k(r'-r) V(r') \phi(r', k) dr' \quad (9)$$

The solutions of (9) with ϕ_0 equal respectively to $e^{\pm ikr}$ are the Jost solutions $R(\pm k, r)$ such that the solution of (8) can be written

$$\psi(r, k, \epsilon) = (R(k, r) - R(-k, r) S(k, \epsilon)) \theta(r - \epsilon) \quad (10)$$

From (6) and (10) we see that $\psi^1 = R(k, r)$ and $\psi^2 = -R(-k, r) \times S(k, \epsilon)$. But from (9) we can obtain the solutions for r as small as we want. If we substitute (10) into (5) we get

$$S(k, \epsilon) = \frac{1 + \frac{\lambda}{k} \int_\epsilon^\infty \sin kr V(r) R(k, r) dr}{1 + \frac{\lambda}{k} \int_\epsilon^\infty \sin kr V(r) R(-k, r) dr} \quad (11)$$

If in (11) both the numerator and the denominator exist when ϵ is very small and if the ratio exists when $\epsilon \rightarrow 0$ then (11) will be the continuation of (7). But $R(\pm k, r)$ are two independent solutions of (1) behaving near the origin like

$$R(\pm k, r) \underset{r \rightarrow 0}{\sim} \gamma_{\text{sing}}(r) R(\pm k) + \gamma_{\text{reg}}(r) \text{Const}(\pm k) \quad (12)$$

where $R(\pm k)$ are the Jost functions and the singular part $Y_{\text{sing}}(r)$, k independent, can be investigated from (1) [see the previous references ^{*} quoted in Ref. ¹⁾].

Then for any singular potential where the most singular part is repulsive we get from (12)

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} S(k, \epsilon) &= \lim_{\epsilon \rightarrow 0} \frac{\int_{\epsilon}^{\infty} \sin kr V(r) R(k, r) dr}{\int_{\epsilon}^{\infty} \sin kr V(r) R(-k, r) dr} \\ &= \lim_{\epsilon \rightarrow 0} \frac{R(k, r)}{R(-k, r)} = \frac{R(k)}{R(-k)} = S(k) \end{aligned} \quad (13)$$

We want to show now that $\lim_{\epsilon \rightarrow 0} \Psi(r, k, \epsilon)$ given by (10), (11), is indeed the right physical solution

$$\begin{aligned} \Psi(r, k, \epsilon) &= R(k, r) - R(-k, r) \left(\frac{1 + \frac{\lambda}{k} \int_{\epsilon}^{\infty} \sin kr V(r) R(k, r) dr}{1 + \frac{\lambda}{k} \int_{\epsilon}^{\infty} \sin kr V(r) R(-k, r) dr} \right) \\ \lim_{\epsilon \rightarrow 0} \Psi(r, k, \epsilon) &= U_{\text{sing}}(r, k) \left[R(k) - R(-k) \frac{R(k)}{R(-k)} \right] \\ &\quad + U_{\text{reg}}(r, k) \left[\text{const}(k) - \text{const}(-k) \frac{R(k)}{R(-k)} \right] \\ &= \text{const } U_{\text{reg}}(r, k) \end{aligned} \quad (14)$$

where $U_{\text{reg}}(r, k)$ and $U_{\text{sing}}(r, k)$ are the two independent solutions of (1) such that $U_{\text{reg}}(r, k) \underset{r \rightarrow 0}{\sim} Y_{\text{reg}}(r)$ and $U_{\text{sing}}(r, k) \underset{r \rightarrow 0}{\sim} Y_{\text{sing}}(r)$.

^{*}) We note also that it is possible to investigate the behaviour of the solution $\Psi(r, k, \epsilon)$ when $\epsilon = 0$ directly from the corresponding integral Eq. (2). See, V. de Alfaro and E. Predazzi, preprint.

Thus we see that this introduction of the cut-off gives always the correct result if we take into account the whole singular part (cut-off dependent) of the solutions. Note that other physical quantities can also be obtained in this way. For instance, instead of the S matrix we can consider

$$F(k, k, \epsilon) = \frac{S(k, \epsilon) - 1}{2i} = \lambda \int_{\epsilon}^{\infty} \sin kr V(r) \frac{\Psi(r; k, \epsilon)}{2ik} dr \quad (5')$$

From (5') and the solution (6) obtained by successive substitutions we get

$$F(k, k, \epsilon) = \sum_q \lambda^q F_q(k, k, \epsilon) \quad (7')$$

We define $T(k, r) = [\bar{R}(-k, r) - R(k, r)]/2ik$, solution of (9) with $\phi_0 = \sin kr/k$. For the second formulation of Ψ we consider an integral equation like (8) but with free term $e^{-ikr} - e^{ikr}(2iF(k, k, \epsilon) + 1)$ and we get for the corresponding Ψ

$$\frac{\Psi(r; k, \epsilon)}{-2ik} = (T(k, r) + R(-k, r)k^{-1}F(k, k, \epsilon)) \theta(r - \epsilon) \quad (10')$$

From (5') and (10') we get

$$k^{-1}F(k, k, \epsilon) = \frac{-\lambda k^{-1} \int_{\epsilon}^{\infty} \sin kr V(r) T(k, r) dr}{1 + \lambda k^{-1} \int_{\epsilon}^{\infty} \sin kr V(r) R(-k, r) dr} \quad (11')$$

Here also (11') is the continuation of (7') and the limit $\epsilon \rightarrow 0$ in (11') exists and gives the right physical quantity

$$\lim_{\epsilon \rightarrow 0} F(k, k, \epsilon) = - \lim_{\epsilon \rightarrow 0} \frac{k T(k, r)}{R(-k, r)} = \frac{R(k) - R(-k)}{2 R(-k)} = F(k, k) \quad (13')$$

We note that if we take the limit $k \rightarrow 0$ in (5'), (7'), (10'), (11'), (13'), we get the results obtained in Ref. ¹⁾.

3) Approximate methods

We give briefly the results because they are straightforward extensions of those obtained in Ref. ¹⁾ for $k = 0$.

First, we consider the prescription ^{*}) of summing up the first leading singularities in $F(k, k, \epsilon)$ given by (7'). If we call $T = \sum_q T_q$ and $R = \sum_q R_q$ the iterate solution of the Volterra equation (9), the first leading singularities when $\epsilon \rightarrow 0$ in (11') come from the first leading singularities $\bar{R}_q$ and $\bar{T}_q$ in each q^{th} iterate term R_q and T_q . We recall that (11') is the continuation of (7') and the limit $\epsilon \rightarrow 0$ in (11') exists (for this approximation) only if $\sum_q \bar{R}_q(-k, r)$ and $\sum_q \bar{T}_q(k, r)$ develop the same type of singularities when $r \rightarrow 0$. This is the case ¹⁾ if the most singular part of the potential is of the inverse power type ^{**)}. On the contrary, if the most singular part of the potential is of logarithmic type or exponential type, then $\sum \bar{R}_q$ and $\sum \bar{T}_q$ do not develop the same singularities and the corresponding ratio in (11') has no meaning when $\epsilon \rightarrow 0$.

Secondly, if we connect the cut-off going to zero and the order P of perturbation expansion $\sum_0^P R_q$ and $\sum_0^P T_q$ we get [see Ref. ¹⁾]

^{*}) See, G. Feinberg and A. Pais, Phys. Rev. 131, 2724 (1963).

^{**)} We note that in the inverse power case the analogy between the singularities of the solutions of the Schrödinger equation both when $r \rightarrow 0$ and $r \rightarrow \infty$ can be useful : see, L. Bertocchi, S. Fubini and G. Furlan, CERN preprint, and E. Del Giudice and E. Galzenati, preprint.

$$\left\{ \begin{aligned}
 S(k) &= \lim_{p \rightarrow \infty} \frac{\sum_0^p \int_{E(p)}^{\infty} \sin kr V(r) R_q(k, r) dr}{\sum_0^p \int_{E(p)}^{\infty} \sin kr V(r) R_q(-k, r) dr} \quad E(p) \gg E_L(p) \\
 F(k, k) &= \lim_{p \rightarrow \infty} - \frac{\sum_0^p \int_{E(p)}^{\infty} k \sin kr V(r) T_q(k, r) dr}{\sum_0^p \int_{E(p)}^{\infty} \sin kr V(r) R_q(-k, r) dr} \quad E(p) \gg E_L(p) \\
 R(-k) &= \lim_{p \rightarrow \infty} \sum_0^p \left[\int_{E(p)}^{\infty} \sin kr V(r) R_q(-k, r) dr \right] \left[\int_{E(p)}^a r V(r) Y_{\text{sing}}(r) dr \right]^{-1} \quad E(p) \gg E_L(p) \\
 &\quad \text{a small const} > 0
 \end{aligned} \right. \quad (15)$$

where $E_L(p)$ has been investigated in Ref. ¹⁾ for many types of singular potentials. We recall that if the most singular part $\bar{V}(r)$ of $V(r)$ is

$$\begin{aligned}
 \text{a) } \bar{V}(r) &\simeq \frac{\lambda}{r^{2n}} \quad E_L(p) = \frac{\text{const}}{p^{(1/n-1)-\eta}} \quad n > 1, \eta \text{ small} > 0 \\
 \text{b) } \bar{V}(r) &\simeq \lambda \frac{(\log r)^\beta}{r^{2n}} \quad E_L(p) = \frac{\text{const}}{p^{(1/n-1)-\eta}} \quad n > 1 \\
 \text{c) } \bar{V}(r) &\simeq \frac{\lambda}{r^{2n}} \exp(\delta/r^\gamma), \delta > 0, \gamma > 0, E_L(p) = \frac{1}{(\text{const} \log p^{2(1-\gamma)/5})^{1/\gamma}} \\
 \text{d) } \bar{V}(r) &\simeq \lambda \frac{(\log r)^\beta}{r^2} \quad \beta \gg 0 \quad E_L(p) = \frac{1}{\exp[\text{const} p^{1/(1+\beta/2)-\eta}]} \quad (16) \\
 &\quad \beta < 0, |\beta| < 1, E_L(p) = \frac{1}{\exp[\text{const} p^{1/(1-|\beta|)-\eta}]}
 \end{aligned}$$

In the case d) for the waves $\ell \neq 0$ and $\beta < 0, |\beta| < 1$ we have ¹⁾

$$\epsilon_L(p) = \frac{1}{\exp[\text{const } p^{1-\gamma}]}$$

4) Fredholm's type of solution of the Equation (2)

In fact we know that there are only two different formulations for the solutions of integral equations like (2) ($\epsilon \neq 0$). The first is the Neumann-Liouville (Born series) method of successive substitutions and the second one is the Fredholm method which gives the solution as a ratio. Then we think that the formulations (10), (11), of (2) must be similar to the Fredholm one. Thus we write (10), (11) for $r > \epsilon$ like

$$\psi(r, k, \epsilon) = -2i \sin kr + \frac{N(r, k, \lambda, \epsilon)}{D(k, \lambda, \epsilon)} \quad (17)$$

where

$$D = 1 + \frac{\lambda}{k} \int_{\epsilon}^{\infty} \sin kx V(x) R(-k, x) dx = 1 + \sum_{q=1}^{\infty} \lambda^q D_q(k, \epsilon) \quad (18)$$

$$\begin{aligned} N &= R(k, r) - R(-k, r) + 2i \sin kr D + \\ &\quad + \frac{\lambda}{k} \int_{\epsilon}^{\infty} \sin kx V(x) [R(k, r) R(-k, x) - R(-k, r) R(k, x)] dx \\ N &= \sum_{q=0}^{\infty} \lambda^{q+1} N_q(r, k, \epsilon) \end{aligned} \quad (19)$$

We have seen above that D and N (for any r and k fixed) develop the same ϵ singularity when $\epsilon \rightarrow 0$; this singularity being $\int_{a=\text{const}}^{\infty} \sin kx V(x) Y_{\text{sing}}(x) dx$. Moreover $\lim_{\epsilon \rightarrow 0} \Psi(k, r, \epsilon)$ in (17) gives the right physical solution [see (14)] and then for any r and k finite we get

$$\Psi(r, k) = \lim_{\epsilon \rightarrow 0} \Psi(r, k, \epsilon) = -2i \sin kr + \lim_{p \rightarrow \infty} \frac{\sum_0^p \lambda^{q+1} N_q(r, k, \epsilon(p))}{1 + \sum_1^p D_q(k, \epsilon(p)) \lambda^q} \quad \epsilon(p) \gg \epsilon_L(p) \quad (20)$$

Now we consider the Fredholm solution of (2) for $\epsilon \neq 0$

$$\Psi(r, k, \epsilon) = -2i \sin kr + \frac{B(r, k, \lambda, \epsilon)}{\Delta(k, \lambda, \epsilon)} \quad (21)$$

$$B(r, k, \lambda, \epsilon) = \int_{\epsilon}^{\infty} \Delta(r, r', k, \lambda, \epsilon) (-2i \sin kr') dr' = \sum_0^{\infty} \lambda^{q+1} B_q(r, k, \epsilon)$$

$$\Delta(k, \lambda, \epsilon) = 1 + \sum_1^{\infty} \lambda^q A_q(k, \epsilon) \quad (22)$$

$\Delta(r, r', k, \lambda, \epsilon)$ and $\Delta(k, \lambda, \epsilon)$ are respectively the Fredholm first minor and the Fredholm determinant. From the Fredholm theory we get

$$A_q(k, \epsilon) = \frac{(-1)^q}{q!} \int_{\epsilon}^{\infty} dr_1 \cdots \int_{\epsilon}^{\infty} dr_q \begin{vmatrix} K(r_1, r_1, k) & \cdots & K(r_1, r_q, k) \\ \vdots & \ddots & \vdots \\ K(r_q, r_1, k) & \cdots & K(r_q, r_q, k) \end{vmatrix} \quad (23)$$

$$B_q(r, k, \epsilon) = \frac{(-1)^q}{q!} \int_{\epsilon}^{\infty} dr' \int_{\epsilon}^{\infty} dr_1 \dots \int_{\epsilon}^{\infty} dr_q (-2i \sin kr') \left| \begin{array}{c} K(r, r', k) K(r_1, r_1, k) \dots K(r_q, r_q, k) \\ K(r, r', k) K(r_1, r_1, k) \dots K(r_q, r_q, k) \\ \vdots \\ K(r_q, r', k) K(r_q, r_q, k) \dots K(r_q, r_q, k) \end{array} \right| \quad (24)$$

The conditions of validity of the Fredholm type of solution of the physical solution have been investigated for regular potentials ²⁾ and can be applied for $\epsilon \neq 0$. We assume $\Delta(k, \lambda, \epsilon) \neq 0$, from the uniqueness of Fredholm's theory of integral equations ($\epsilon \neq 0$), we get from (17) and (21)

$$N/D = B/\Delta \quad (25)$$

Further, we want to prove that

$$D(k, \lambda, \epsilon) \equiv \Delta(k, \lambda, \epsilon) \quad \text{or} \quad D_q(k, \epsilon) = A_q(k, \epsilon) \quad (26)$$

First, we consider

$$D(k, \lambda, \epsilon) = 1 + \frac{\lambda}{k} \int_{\epsilon}^{\infty} \sin(kr_1) V(r_1) \left(e^{ikr_1} + \sum_{q=1}^{\infty} \lambda^q R_q(-k, r_1) \right) dr_1 \quad (27)$$

where $R_q(-k, r_1)$ comes from successive substitutions in (9)

$$k^q R_q(-k, r_1) = \int_{\epsilon}^{\infty} dr_2 \dots \int_{\epsilon}^{\infty} dr_{q+1} e^{ikr_{q+1}} \prod_{j=1}^q V(r_{j+1}) \sin(k(r_{j+1} - r_j)) \quad (28)$$

with $r_1 \leq r_2 \leq \dots r_q \leq r_{q+1}$

$$\begin{aligned} D_q(k, \epsilon) &= k^{-1} \int_{\epsilon}^{\infty} \sin kr_1 V(r_1) R_{q-1}(-k, r_1) dr_1 \\ &= k^{-q} \int_{\epsilon}^{\infty} dr_1 \int_{r_1}^{\infty} dr_2 \dots \int_{r_{q-1}}^{\infty} dr_q \prod_{j=1}^q V(r_j) \sin(k(r_j - r_{j-1})) e^{ikr_q} \end{aligned} \quad (29)$$

where $r_0 = 0$ and $\epsilon \leq r_1 \leq \dots \leq r_q$.

Secondly, we study $A_q(k, \epsilon)$ given by (23), the integrand in (23) being symmetric in all the variables, we get

$$A_q(k, \epsilon) = k^{-q} \int_{\epsilon}^{\infty} dr_1 \dots \int_{r_{q-1}}^{\infty} dr_q \mathcal{D}_q(r_1, r_2, \dots, r_q, k) \prod_{j=1}^q V(r_j) \quad (30)$$

where $\epsilon \leq r_1 \dots \leq r_q$ and $\mathcal{D}_q$ is the determinant

$$\mathcal{D}_q(r_1, \dots, r_q, k) = \begin{vmatrix} g(r_1, r_1, k) & \dots & g(r_1, r_q, k) \\ \vdots & \ddots & \vdots \\ g(r_q, r_1, k) & \dots & g(r_q, r_q, k) \end{vmatrix} \quad (30')$$

But if $r_i > r_j$, $g(r_i, r_j, k) = g(r_j, r_i, k) = \sin kr_j e^{ikr_i}$ and $g(r_i, r_j, k) = \sin kr_i e^{ikr_j}$. We put $\sin kr_i = y_i$ and $e^{ikr_j} = x_j$ then $\mathcal{D}_q = y_1 x_q \mathcal{E}_q$ where $\mathcal{E}_1 = 1$ and

where $R_0(\pm k, z) = e^{\mp i k z}$. We write (32) like $N = \sum_{q=0}^{\infty} \lambda^{q+1} N_q$ and investigate the terms N_q .

In Appendix A we have verified for $q = 0$ and $q = 1$ the relation $N_q = B_q$. From the fact that $N \equiv B$ and $D \equiv \Delta$ it follows that all previous results obtained in the formulation N/D for the physical solution $\Psi(k, r, \epsilon)$ are still valid for the Fredholm type of solution B/Δ . Always for potentials when the most singular part is repulsive we get :

- i) the ϵ singularities when $\epsilon \rightarrow 0$ of the whole numerator and of the whole denominator in the Fredholm type of solution are the same and the limit of the ratio when $\epsilon \rightarrow 0$ has a meaning ;
- ii) this limit $\epsilon \rightarrow 0$ of the ratio is indeed the right physical solution

$$\lim_{\epsilon \rightarrow 0} \Psi(r, k, \epsilon) = \Psi(r, k) = -2i \sin k r + \lim_{\epsilon \rightarrow 0} \frac{-2i \int_0^{\infty} \Delta(r, r', k, \lambda, \epsilon) \sin k r' dr'}{\Delta(k, \lambda, \epsilon)} \quad (33)$$

- iii) for potentials such that the most singular repulsive part is of the type previously considered [Eq. (16), a, b, c, d] we have

$$\Psi(r, k) = -2i \sin k r + \lim_{p \rightarrow \infty} \frac{\sum_0^p \lambda^{q+1} B_q(r, k, \epsilon(p))}{\sum_0^p \lambda^q A_q(k, \epsilon(p))} \quad (34)$$

if $\epsilon(p) \geq \epsilon_L(p)$

where the limiting dependences $\epsilon_L(p)$ are the same as for the S matrix and have been given above for different families of repulsive singular potentials.

B. Waves $\ell \neq 0$

There are two different ways to extend the results obtained in A. to the $\ell \neq 0$ case. The first one is to simply substitute $V(r) + \ell(\ell+1)/r^2$ for $V(r)$ in the previous formulation A., in which case we can take over the previous results in the original form. For families of singular repulsive potentials quoted in A.3) a, b, c, more singular than $r^{-\varphi}$ ($\varphi > 2$) near the origin, the centrifugal potential does not enter into the singular part $Y_{\text{sing}}(r)$. The only little change is for

$$\bar{V}(r) \underset{r \rightarrow 0}{\simeq} \frac{\lambda (\log r)^\beta}{r^2} \quad \beta > -1$$

where the singular part $Y_{\text{sing}}(r)$ depends both on $\bar{V}(r)$ and on the centrifugal potential. The corresponding modification ($\beta < 0$, $|\beta| < 1$) for $\epsilon_L(p)$ is given in A.3, d. Note that for what we called "regular potentials", where $Y_{\text{sing}}(r) = r^{-\ell}$ is only given by the centrifugal potential, the corresponding $\epsilon_L(p)$ is similar to that of

$$\bar{V}(r) \underset{r \rightarrow 0}{\simeq} \frac{\lambda (\log r)^\beta}{r^2}$$

with $\beta = 0$.

The second method is to substitute into the integral equation (2) the corresponding spherical Bessel and Hankel $\ell \neq 0$ terms for the free term $(2-i \sin kr)$ and the Green function $k^{-1}g(r, r', k)$. Then all the previous formalism can be easily extended.

III. LIPPMAN-SCHWINGER EQUATION

We consider

$$F(p, k) = \lambda \int_0^{\infty} \sin pr V(r) \frac{\Psi(r, k)}{2ik} dr \quad (35)$$

where $\Psi(r, k)$ is the physical solution studied in II : $\Psi(r, k) = \lim_{\epsilon \rightarrow 0} \Psi(r, k, \epsilon)$ and such that $F(k, k)$ is the usual amplitude $e^{i\delta} \sin \delta$. The method we want to use in order to get $F(p, k)$ will be similar to the previous one for $\Psi(r, k)$. We introduce a cut-off $V \rightarrow V \theta(r - \epsilon)$ and define

$$F(p, k, \epsilon) = \lambda \int_{\epsilon}^{\infty} \sin pr V(r) \frac{\Psi(r, k, \epsilon)}{2ik} dr \quad (36)$$

where $\Psi(r, k, \epsilon)$ is the solution of (2). We shall obtain $F(p, k, \epsilon)$ in two different manners : $F(p, k, \epsilon) = \bar{N}(p, k, \lambda, \epsilon) / \bar{D}(k, \lambda, \epsilon) = \bar{N}(p, k, \lambda, \epsilon) / \bar{D}(k, \lambda, \epsilon)$. The first one (for instance $\bar{N}/\bar{D}$) will be the Fredholm solution of the Lippman-Schwinger (cut-off dependent) equation, the other one ($\bar{N}/\bar{D}$) will be obtained from the exact Jost solutions $R(\pm k, r)$. We shall show that $\bar{N}$ and $\bar{D}$ diverge in the same way when $\epsilon \rightarrow 0$ and we shall identify the two formulations : $\bar{N} \equiv \bar{N}$ and $\bar{D} \equiv \bar{D}$.

1) Lippman-Schwinger (cut-off dependent) solution

We define

$$V(p, q, \epsilon) = - \int_{\epsilon}^{\infty} \sin pr \frac{\sin qr}{q} V(r) dr \quad (37)$$

and use the relation

$$\int_0^{\infty} \frac{\sin qr \sin qr'}{q^2 - k^2 - i\eta} dq = \frac{\pi}{2k} g(r, r', k) \quad (38)$$

We start from the integral equation (2) of $\Psi(r, k, \epsilon)$ and with (36), (37), (38), we get for $F(p, k, \epsilon)$ an integral equation, the so-called Lippman-Schwinger equation

$$F(p, k, \epsilon) = \lambda V(p, k, \epsilon) + \lambda \int_0^{\infty} \frac{2q}{\pi} \frac{V(p, q, \epsilon)}{q^2 - k^2 - i\eta} F(q, k, \epsilon) dq \quad (39)$$

We call

$$G(p, q, k, \epsilon) = \frac{2q V(p, q, \epsilon)}{\pi(q^2 - k^2 - i\eta)}$$

the kernel of (39).

Now we consider the Fredholm solution of (39)

$$F(p, k, \epsilon) = \lambda V(p, k, \epsilon) + \frac{\sum_0^{\infty} \lambda^{m+2} C_m(p, k, \epsilon)}{1 + \sum_1^{\infty} \lambda^m \bar{D}_m(k, \epsilon)} \quad (40)$$

where

$$\bar{D}_m(k, \epsilon) = \frac{(-1)^m}{m!} \int_0^{\infty} dq_1 \cdots \int_0^{\infty} dq_m \begin{vmatrix} G(q_1, q_1, k, \epsilon) & \cdots & G(q_1, q_m, k, \epsilon) \\ \vdots & & \vdots \\ G(q_m, q_1, k, \epsilon) & \cdots & G(q_m, q_m, k, \epsilon) \end{vmatrix} \quad (41)$$

and

$$C_m(p, k, \epsilon) = \frac{(-1)^m}{m!} \int_0^\infty dq_1 \int_0^\infty dq_2 \dots \int_0^\infty dq_m V(q, k, \epsilon) \begin{vmatrix} G(p, q_1, k, \epsilon) G(p, q_2, k, \epsilon) \dots G(p, q_m, k, \epsilon) \\ G(q_1, q_1, k, \epsilon) G(q_1, q_2, k, \epsilon) \dots G(q_1, q_m, k, \epsilon) \\ \vdots \\ G(q_m, q_1, k, \epsilon) G(q_m, q_2, k, \epsilon) \dots G(q_m, q_m, k, \epsilon) \end{vmatrix}$$

(42)

We note that the difference from the Fredholm formulation of the physical solution $\Psi(r, k, \epsilon)$ is the fact that here the Born term $V(p, k, \epsilon)$ diverges when $\epsilon \rightarrow 0$ if $V(r)$ is more singular near the origin than $r^{-3+\epsilon'}$, (ϵ' arbitrarily small > 0). Then, because we do not want to restrict the singularities of the potential we rewrite the Fredholm solution (40) in the following form:

$$F(p, k, \epsilon) = \frac{\bar{N}(p, k, \lambda, \epsilon)}{\bar{D}(k, \lambda, \epsilon)} = \frac{\sum_0^\infty \lambda^{m+1} \bar{N}_m(p, k, \epsilon)}{1 + \sum_1^\infty \lambda^m \bar{D}_m(k, \epsilon)} \quad (43)$$

where $\bar{N}_0 = V(p, k, \epsilon)$, $\bar{N}_1 = C_0 + \bar{D}_1 V$, $\bar{N}_m = C_{m-1} + \bar{D}_m V$.

2) Formulation of $F(p, k, \epsilon)$ in terms of Jost solutions

We have shown above in Section II that $\Psi(r, k, \epsilon)$ can be obtained from the Jost solutions [see Eq. (14)]. If we substitute (14) in (36) we get

$$F(p, k, \epsilon) = \frac{\bar{N}(p, k, \lambda, \epsilon)}{\bar{D}(k, \lambda, \epsilon)} = \frac{\sum_0^\infty \lambda^{m+1} \bar{N}_m(p, k, \epsilon)}{1 + \sum_1^\infty \lambda^m \bar{D}_m(k, \epsilon)} \quad (44)$$

where

$$\bar{D}(k, \lambda, \epsilon) = 1 + \frac{\lambda}{k} \int_{\epsilon}^{\infty} \sin kr V(r) R(-k, r) dr = D(k, \lambda, \epsilon) = A(k, \lambda, \epsilon) \quad (45)$$

$$\begin{aligned} \bar{N}(p, k, \lambda, \epsilon) = & \frac{\lambda}{2ik} \int_{\epsilon}^{\infty} \sin pr V(r) [R(k, r) - R(-k, r)] dr \\ & + \frac{\lambda^2}{2ik^2} \int_{\epsilon}^{\infty} dr \int_{\epsilon}^{\infty} dx \sin pr \sin kx V(x) V(r) [R(k, r) R(-k, x) - R(-k, r) R(k, x)] \end{aligned} \quad (46)$$

First, we note that for $p = k$ the second term in the right-hand side of (46) is identically zero such that

$$\lim_{\epsilon \rightarrow 0} F(k, k, \epsilon) = \lim_{\epsilon \rightarrow 0} \frac{\bar{N}(k, k, \lambda, \epsilon)}{\bar{D}(k, \lambda, \epsilon)} = F(k, k)$$

Secondly, we want to show that for p and k finite and fixed $\bar{N}$ and $\bar{D}$ diverge in the same way when $\epsilon \rightarrow 0$. For this we define a regular solution

$$v(k, r) = \frac{R(k, r) - \frac{R(k)}{R(-k)} R(-k, r)}{R(-k)} = \text{const } U_{\text{reg}}(r, k)$$

Then the second term in the right-hand side of (46) can be written

$$\frac{\lambda^2}{2ik^2} \int_{\epsilon}^{\infty} dr \int_{\epsilon}^{\infty} dx \sin pr \sin kx V(r) V(x) [R(-k, x) v(k, r) - R(-k, r) v(k, x)] \quad (46')$$

We consider always the case where the most singular part of the potential is repulsive ; then ^{*)} $\bar{N}$ and $\bar{D}$ develop the same ϵ singularity when $\epsilon \rightarrow 0$, this singularity being $\int_{\epsilon}^a x V(x) Y_{\text{sing}}(x) dx$ (where a is a small constant). And from (14), (36), (44), (45), (46), we get

$$F(p, k) = \lim_{\epsilon \rightarrow 0} F(p, k, \epsilon) = \lim_{\epsilon \rightarrow 0} \frac{\bar{N}(p, k, \lambda, \epsilon)}{\bar{D}(k, \lambda, \epsilon)} \quad (47)$$

$$F(p, k) = \lim_{p \rightarrow \infty} \frac{\sum_{m=0}^p \lambda^{m+1} \bar{N}_m(p, k, \epsilon(p))}{\sum_{m=0}^p \lambda^m \bar{D}_m(k, \epsilon(p))} \quad (48)$$

if $\epsilon(p) \geq \epsilon_L(p)$

where $\epsilon_L(p)$ are the same as in Section II.

3) Equivalence between the two formulations

For $\epsilon \neq 0$, from the uniqueness of the Fredholm's type of solution we get

$$\bar{N}/\bar{D} = \bar{N}/\bar{D} \quad (49)$$

We want to prove now that

$$\bar{D}(k, \lambda, \epsilon) = \bar{D}(k, \lambda, \epsilon) \text{ or } \bar{D}_m(k, \epsilon) = \bar{D}_m(k, \epsilon) \quad (50)$$

Because of the above results (Section II) $\bar{D}_m(k, \epsilon) = A_m(k, \epsilon)$, it is equivalent to prove that

^{*)} This result can be seen directly, from (36) and (44) we get :

$$\lim_{\epsilon \rightarrow 0} \frac{\bar{N}}{\bar{D}} = \lim_{\epsilon \rightarrow 0} \lambda \int_{\epsilon}^{\infty} \sin pr V(r) \frac{\psi(r, k, \epsilon)}{2ik} dr = \lambda \int_0^{\infty} \sin pr \frac{\psi(r, k, 0)}{2ik} V(r) dr$$

and the last integral exists following (14) and is equal to $F(p, k)$.

$$\bar{D}_m(k, \epsilon) = A_m(k, \epsilon) \quad (51)$$

We recall [see Eq. (23)] that A_m is given by

$$A_m(k, \epsilon) = \frac{1}{k^m} \int_{\epsilon}^{\infty} dr_1 \cdots \int_{\epsilon}^{\infty} dr_m \prod_{j=1}^m V(r_j) \begin{vmatrix} g(r_1, r_1, k) & \cdots & g(r_1, r_m, k) \\ \vdots & & \vdots \\ g(r_m, r_1, k) & \cdots & g(r_m, r_m, k) \end{vmatrix} \quad (52)$$

and $\bar{D}_m$ can be written [see Eqs. (41), (39), (37)]

$$\bar{D}_m(k, \epsilon) = \left(\frac{2}{\pi}\right)^m \int_0^{\infty} dq_1 \cdots \int_0^{\infty} dq_m \prod_{j=1}^m \frac{1}{q_j^2 - k^2 - i\eta} \begin{vmatrix} \bar{V}_{\epsilon}(q_1, q_1) & \cdots & \bar{V}_{\epsilon}(q_1, q_m) \\ \vdots & & \vdots \\ \bar{V}_{\epsilon}(q_m, q_1) & \cdots & \bar{V}_{\epsilon}(q_m, q_m) \end{vmatrix} \quad (53)$$

$$\text{where } \bar{V}_{\epsilon}(q_i, q_j) = \int_{\epsilon}^{\infty} \sin q_i r \sin q_j r V(r) dr = \bar{V}_{\epsilon}(q_j, q_i).$$

We start from $\bar{D}_m$ and want to show that it is equal to A_m .

Case $m = 1$. We use the relation

$$\frac{2}{\pi} \int_0^{\infty} \frac{\bar{V}_{\epsilon}(q_i, q_i) dq_i}{q_i^2 - k^2 - i\eta} = \int_{\epsilon}^{\infty} \frac{g(r_i, r_i, k) V(r_i) dr_i}{k} \quad (54)$$

$$\text{and we get } \bar{D}_1(k, \epsilon) = A_1(k, \epsilon).$$

Case $m = 2$. The determinant in (53) is the sum of two terms. We use the relation (54) and

$$\left(\frac{2}{\pi}\right)^2 \int_0^{\infty} dq_i \int_0^{\infty} dq_j \frac{\bar{V}_{\epsilon}(q_i, q_j) \bar{V}_{\epsilon}(q_j, q_i)}{(q_i^2 - k^2 - i\eta)(q_j^2 - k^2 - i\eta)} = \frac{1}{k^2} \int_{\epsilon}^{\infty} dr_i \int_{\epsilon}^{\infty} dr_j g(r_i, r_j, k) g(r_j, r_i, k) V(r_i) V(r_j) \quad (55)$$

$$\text{and we get } \bar{D}_2 = A_2.$$

Case $m = 3$. The determinant in (53) is the sum of six terms. Each of these terms is the product of three $V_E(q_i, q_j)$, in each term q_i appears only two times either following $\bar{V}_E(q_i, q_i)$ and we apply (54), or following $\bar{V}_E(q_i, q_j) \bar{V}_E(q_j, q_i)$ and we apply (55) or in the form $\bar{V}_E(q_i, q_j) \bar{V}_E(q_j, q_l) \bar{V}_E(q_l, q_i)$ and we apply

$$\left(\frac{2}{\pi}\right)^3 \int_0^\infty dq_i \int_0^\infty dq_j \int_0^\infty dq_l \frac{\bar{V}_E(q_i, q_l) \bar{V}_E(q_l, q_j) \bar{V}_E(q_j, q_i)}{(q_i^2 - k^2 - i\eta)(q_l^2 - k^2 - i\eta)(q_j^2 - k^2 - i\eta)} \quad (56)$$

$$\equiv \frac{1}{k^3} \int_E^\infty dr_i \int_E^\infty dr_j \int_E^\infty dr_l g(r_i, r_l, k) g(r_l, r_j, k) g(r_j, r_i, k) V(r_i) V(r_j) V(r_l)$$

and we get $\bar{D}_3 = A_3$.

The proof can be generalized for $m = m$. The determinant in (53) is the sum of $m!$ terms. Each of these terms is the product of $m V_E$, in each term q_i appears only two times in the general form $\bar{V}_E(q_i, q_j) \bar{V}_E(q_j, q_l) \dots \bar{V}_E(q_k, q_n) \bar{V}_E(q_n, q_i)$ with all available permutations of the indices and we use (54), (55), (56), or the generalization of these relations and we get $\bar{D}_m = A_m$.

From $\bar{N}/\bar{D} = \bar{N}/\bar{D}$ and $\bar{D} = \bar{D}$ it follows that

$$\bar{N}(p, k, \lambda, \epsilon) = \bar{N}(p, k, \lambda, \epsilon) \text{ or } \bar{N}_m(p, k, \epsilon) = \bar{N}_m(p, k, \epsilon) \quad (57)$$

In Appendix B the result (57) is verified for $m = 0$ and $m = 1$. From the fact that $\bar{N} = \bar{N}$ and $\bar{D} = \bar{D}$ it follows that we get for the Fredholm solution of the Lippman-Schwinger equation

$$i) \quad F(p, k) = \lim_{\epsilon \rightarrow 0} \frac{\bar{N}(p, k, \epsilon)}{\bar{D}(p, k, \epsilon)}$$

$$\text{ii)} \quad F(p, k) = \lim_{q \rightarrow \infty} \frac{\sum_0^q \lambda^{m+1} \bar{N}_m(p, k, \epsilon(q))}{\sum_0^q \lambda^m \bar{D}_m(k, \epsilon(q))} \quad \epsilon(q) \gg \epsilon_L(q) \quad (58)$$

iii) Here also the Jost solution $R(-k)$ and the bound states can be obtained

$$R(-k) = \lim_{p \rightarrow \infty} \left[\sum_0^p \lambda^m \bar{D}_m(k, \epsilon(p)) \right] \left[\int_{\epsilon(p)}^a y_{\text{sing}}^{(x)} V(x) dx \right]^{-1} \\ \epsilon(p) \gg \epsilon_L(p)$$

If the potential is near the origin less or equally singular than $r^{-3+\epsilon'}$ we can also consider the Fredholm formulation (40) of the Lippman-Schwinger equation and we identify it with

$$F = \lambda V + \frac{\bar{N} - \lambda V \bar{D}}{\bar{D}}$$

Following (40), (43), (50), (57), it is obvious that these two formulations are equivalent. Then we have also in this case

$$F(p, k) = \lambda V(p, k, \epsilon=0) + \lim_{\epsilon \rightarrow 0} \frac{\sum_0^\infty \lambda^{m+2} C_m(p, k, \epsilon)}{1 + \sum_1^\infty \lambda^m \bar{D}_m(k, \epsilon)} \quad (59)$$

$$F(p, k) = \lambda V(p, k, \epsilon=0) + \lim_{q \rightarrow \infty} \frac{\sum_0^q \lambda^{m+2} C_m(p, k, \epsilon(q))}{\sum_1^q \lambda^m \bar{D}_m(k, \epsilon(q))} \\ \epsilon(q) \gg \epsilon_L(q) \quad (60)$$

For the case $\ell \neq 0$, we must substitute in (35), (36), (37) the corresponding $\ell \neq 0$ spherical Bessel terms and $\ell \neq 0$ physical solution instead of $\sin pr$, $\sin qr$ and ψ . Then the above formalism is easily extended. Then also for the corresponding Born

term (37) $V_\ell(p, q, \epsilon)$ we have always, if $V(r) \underset{r \rightarrow 0}{\sim} \frac{\text{const}}{r^{2n}}$, a particular value n_0 of n such that $V_\ell(p, q, \epsilon = 0)$ is finite ^{*)} for $n < n_0$. Following $n \geq n_0$ we can consider the two corresponding formulations (40) or (43) of the Fredholm solutions. (Note that this distinction is not possible if the potential is more singular than any inverse power near the origin, like an exponential for example.) In fact we note that if $V_\ell(p, q, \epsilon = 0)$ is finite we are in a situation similar to the co-ordinate case where the free term was cut-off independent and in our method (58)-(60), there is not a big difference between the two cases: Born term finite or infinite. We remark also that (58) contains the two cases.

4) Application

All above results are valid for any kind of singular potential where the most singular part is repulsive, we want now to verify explicitly in the particular case $k = 0$ that $\epsilon \lim_{\epsilon \rightarrow 0} F(p, k, \epsilon) = F(p, k)$. We call $T(r) = \lim_{k \rightarrow 0} T(k, r)$

$$R(r) = \lim_{k \rightarrow 0} R(\pm k, r)$$

$$\Psi(r) = \lim_{k \rightarrow 0 - 2ik} \frac{\Psi(r, k)}{k}$$

We get from (35), (43), (44), (45), (46), (50), (57),

$$F(p, k=0, \epsilon) = \frac{-\lambda \int_0^\infty \sin pr V(r) \left[T(r) + \int_0^\infty \lambda x V(x) (T(r)R(x) - T(x)R(r)) dx \right] dr}{1 + \lambda \int_0^\infty R(r) r V(r) dr}$$

$$F(p, k=0) = -\lambda \int_0^\infty \sin pr V(r) \Psi(r) dr$$

^{*)} We remark that this can be useful if we consider the analytic continuation with respect to ℓ of the solution $F_\ell(p, k)$ (see Arbuzov et al. ³⁾).

We consider the case $V = r^{-4}$. We recall that ¹⁾

$$\begin{aligned} T &= r \operatorname{ch}(\sqrt{\lambda}/r) \\ R &= \frac{r}{\sqrt{\lambda}} \operatorname{sh}(\sqrt{\lambda}/r) \end{aligned} \quad \Psi = r \exp(-\sqrt{\lambda}/r)$$

such that

$$F(p, k=0, \epsilon) = -\frac{\lambda \int_0^\infty \frac{\sin pr}{r^3} \operatorname{ch}\left(\sqrt{\lambda}\left(\frac{1}{r} - \frac{1}{\epsilon}\right)\right) dr}{\operatorname{ch}(\sqrt{\lambda}/\epsilon)} \xrightarrow{\epsilon \rightarrow 0} -\lambda \int_0^\infty \frac{\sin pr}{r^3} e^{-\sqrt{\lambda}/r} dr$$

$$\text{and } \lim_{\epsilon \rightarrow 0} F(p, k=0, \epsilon) = F(p, k=0).$$

CONCLUSION

In this paper we have considered two different examples of singular Fredholm integral equations (by singular we mean here in particular that the Fredholm determinants are infinite or that we cannot apply directly neither the method of successive substitutions nor the Fredholm's type of solution).

The first one in co-ordinate space is equivalent to a second order differential equation. The second one is the Lippman-Schwinger equation in p space. When the Lippman-Schwinger equation can be reduced to a differential equation, its order depends on the interaction (see Arbuzov et al. ³⁾ who have shown that for $V = \lambda / r^{2n}$ we have a differential equation of order $2n$). But these two different equations have been solved by the same method. The remarkable fact (which was also found for energy equal to zero in Ref. ¹⁾) is that although we consider Green functions (in r space) which are correct only for regular interactions, these Green functions with the introduction of a cut-off ϵ are able to reproduce in all cases the correct solution when $\epsilon \rightarrow 0$ even for singular interactions. The other result that we can construct the solution by connecting in a certain manner the order of the expansions (written as a ratio) and the cut-off going to zero, is only a practical and general tool in order to use explicitly the first result. We note that we have taken great advantage of the fact that the free Green functions are always valid (in r space) at one of the ends of the integration path (∞). We note also that we have always the restriction that the most singular part of the interaction is repulsive in order to have a finite ratio when $\epsilon \rightarrow 0$.

We hope that these results can be extended to other kinds of singular Fredholm equations.

ACKNOWLEDGEMENTS

I thank Dr. A. Martin for very useful discussions. I thank also Dr. A. Contogouris, Dr. M.B. Halpern and Dr. J. Høgasen.

A P P E N D I X A

We want to investigate the numerator of the Fredholm's type solution of the physical (cut-off dependent) solution of (2).

We want to verify for $q = 0$ and $q = 1$ the equality

$$N_q = B_q.$$

- 1) First we consider the case $q = 0$. From (24) for B_0 and (32) for N_0 we get

$$\begin{aligned} B_0(r, k, \epsilon) &= - \int_{\epsilon}^{\infty} K(r, r', k) 2i \sin(kr') dr' \\ N_0(r, k, \epsilon) &= \frac{2i}{k} \int_{\epsilon}^{\infty} \sin kr' V(r') \left[-\sin k(r'-r) \Theta(r'-r) + \sin kr' e^{ikr} + \sin k(r'-r) \right] dr' \quad (A.1) \\ &= \frac{2i}{k} \int_{\epsilon}^{\infty} \sin kr' V(r') \left[\sin kr' e^{ikr} \Theta(r-r') + \sin kr' e^{ikr} \Theta(r'-r) \right] dr' \\ &= B_0(r, k, \epsilon) \end{aligned}$$

- 2) Secondly, we consider the case $q = 1$. Still from (24) we get

$$\begin{aligned} B_1(r, k, \epsilon) &= \\ &= \int_{\epsilon}^{\infty} dr' \int_{\epsilon}^{\infty} dr'' 2i \sin(kr') \left[K(r, r', k) K(r'', r'', k) - K(r, r'', k) K(r'', r', k) \right] \end{aligned}$$

and, after some algebra,

$$B_1(r, k, \epsilon) = \frac{2i}{k} \int_{\epsilon}^{\infty} dr' \int_{\epsilon}^r dr'' \sin(kr') V(r'') K(r, r', k) \sin k(r'' - r') \quad (A.2)$$

From (32) we get

$$\begin{aligned} N_1(r, k, \epsilon) = & [R_2(k, r) - R_2(-k, r)] + \\ & + \left[\int_{\epsilon}^{\infty} \frac{\sin kx}{k} V(x) [R_0(k, r) R_1(-k, x) - R_0(-k, r) R_1(k, x) + \right. \\ & \left. + R_1(k, r) R_0(-k, x) - R_1(-k, r) R_0(k, x)] dx \right] + \left[2i \frac{\sin kr}{k} \int_{\epsilon}^{\infty} \sin kx V(x) R_1(-k, x) dx \right] \end{aligned} \quad (A.3)$$

By re-arrangement, the first part in the right-hand side of (A.3) can be written

$$\frac{2i}{k^2} \int_{\epsilon}^{\infty} dr' \int_{\epsilon}^r dr'' V(r') V(r'') \sin k(r'' - r') \sin kr'' [-\sin k(r' - r) \Theta(r'' - r') \Theta(r' - r)] \quad (A.4)$$

By re-arrangement, the second part in the right-hand side of (A.3) can also be written

$$\frac{2i}{k^2} \int_{\epsilon}^{\infty} dr' \int_{\epsilon}^r dr'' V(r') V(r'') \sin k(r'' - r') \sin kr'' [\sin k(r' - r) (\Theta(r' - r) - \Theta(r' - r''))] \quad (A.5)$$

and the last term of (A.3) can be written

$$\frac{2i}{k^2} \int_{\epsilon}^{\infty} dr' \int_{\epsilon}^r dr'' V(r') V(r'') \sin k(r'' - r') \sin kr'' [-\sin kr e^{ikr'} \Theta(r' - r'')] \quad (A.6)$$

Now we add (A.4), (A.5) and (A.6) and use the relation

$$\begin{aligned} \sqrt{\epsilon} V(r') K(r, r', k) \Theta(r' - r'') = & -\sin k(r' - r) [\Theta(r'' - r') \Theta(r' - r) - \Theta(r' - r) + \Theta(r' - r'')] - \\ & - \sin k r e^{i k r'} \Theta(r' - r'') \end{aligned}$$

Then (A.3) which is the sum of (A.4), (A.5) and (A.6) gives the result (A.2), then we get

$$B_1(r, k, \epsilon) = N_1(r, k, \epsilon)$$

The proof $B_q = N_q$ can be extended for $q = 2, \dots$

A P P E N D I X B

We want to investigate the numerator of the Fredholm solution (cut-off dependent) of the Lippman-Schwinger equation written following (43). We want to verify for $m = 0$ and $m = 1$ the equality $\bar{N}_m = \bar{N}_m$. We recall that [see (43)] $\bar{N}_0 = V$ and $\bar{N}_m = C_{m-1} + \bar{D}_m V$ where $\bar{D}_m = \bar{D}_m$ and C_m is given by (42).

We consider the λ expansion given by (46) $\bar{N} = \sum_{m=0}^{\infty} \lambda^{m+1} \bar{N}_m$

and

$$\bar{N}(p, k, \lambda, \epsilon) = \sum_{m=0}^{\infty} \lambda^{m+1} \int_{\epsilon}^{\infty} \frac{\sin pr}{2ik} V(r) [R_m(k, r) - R_m(-k, r)] dr +$$

$$+ \sum_{n=0}^{\infty} \lambda^{n+2} \sum_{\ell=0}^{\infty} \lambda^{\ell} \int_{\epsilon}^{\infty} dr \int_{\epsilon}^{\infty} dx \frac{\sin pr \sin kx}{2ik^2} V(r) V(x) [R_{\ell}(-k, x) R_n(k, r) - R_{\ell}(k, x) R_n(-k, r)] \quad (B.1)$$

- 1) First we consider the case $m = 0$. For $\bar{N}_0$ we have only to take the first term in the first part of the right-hand side of (B.1) and we get

$$\bar{N}_0 = - \int_{\epsilon}^{\infty} \sin pr V(r) \frac{\sin kr}{k} dr = V(p, k, \epsilon) = \bar{N}_0 \quad (B.2)$$

- 2) Secondly, we consider the case $m = 1$. For $\bar{N}_1$ we take the second term ($m = 1$) in the first part of the right of (B.1) and the first term ($n = 0, \ell = 0$) in the second part of the right of (B.1). We get

$$\bar{N}_1 = \frac{1}{k^2} \int_{\epsilon}^{\infty} \sin pr V(r) \int_{\epsilon}^{\infty} \sin k(r-r') V(r') \sin kr' dr' dr \quad (B.3)$$

For $\bar{N}_1$ we have $\bar{N}_1 = C_0 + \bar{D}_1 V$

$$C_0 = \frac{2}{\pi} \int_0^\infty q \frac{V(p, q, \epsilon) V(q, k, \epsilon) dq}{q^2 - k^2 - i\eta}$$

With the use of (37) and (38) we get

$$C_0 = \frac{1}{k^2} \int_{\epsilon}^{\infty} dr \int_{\epsilon}^{\infty} dr' \sin pr \sin kr' V(r) V(r') g(r, r', k) \quad (B.4)$$

But $\bar{D}_1 V = A_1(k, \epsilon) V(p, k, \epsilon)$ such that

$$\bar{D}_1 V = \frac{1}{k^2} \int_{\epsilon}^{\infty} dr \int_{\epsilon}^{\infty} dr' \sin pr \sin kr' V(r) V(r') (-\sin kre^{ikr'}) \quad (B.5)$$

We consider the sum of (B.4) and (B.5) and use the relation

$$g(r, r', k) - \sin kre^{ikr'} = \sin k(r' - r) \phi(r - r')$$

Finally, we get $\bar{N}_1 = \bar{N}_1$.

The proof $\bar{N}_m = \bar{N}_m$ can be extended for $m = 2, \dots$

R E F E R E N C E S

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