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NEUTRINO PRODUCTION OF LEPTON PAIRS THROUGH THE
POINT FOUR-FERMION INTERACTION *)

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ABSTRACT

The total cross-sections for production of lepton pairs through the point four-fermion interaction by high-energy neutrinos and antineutrinos in the nuclear electromagnetic field are calculated numerically from the threshold up to very high energies where the asymptotic formulae are applicable. The asymptotic cross-sections are calculated analytically and found to be (for high enough energies) in excellent agreement with the exact numerical calculations. The importance of such detailed calculations, especially in the search for the intermediate vector boson, is stressed. Appendix B contains many details of the invariant phase space calculations.

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1. INTRODUCTION

The advent of high energy neutrino beams as useful experimental tools has opened many new avenues for studies of the nature of weak interactions. Already, for example, the Brookhaven ¹⁾ and CERN ²⁾ experiments have shown the existence of two neutrinos, one associated with the muon and one with the electron. The study of the inverse μ capture process

$$\nu_{\mu} + n \longrightarrow \mu^{-} + p \quad (1.1)$$

as a function of both energy and momentum transfer, yields a great deal of information about the nature of the weak coupling ^{1),2)}. One of the most interesting possibilities for the experiments is to look for the intermediate vector boson as suggested by Lee and Yang ³⁾. If the weak interactions take place through the emission and absorption of a vector boson, then the coupling of the boson to the lepton is only semi-weak as it would be proportional to the square root of the weak interaction coupling constant and one should be able to produce the vector boson as a real particle with quite reasonable cross-sections, using high energy neutrino beams ^{3),4)}. This, of course, depends on the mass of the boson and the spectrum of the incident neutrino beam. Experimentally, it is observed ^{1),2)} that the dominant process for the present neutrino beams, which come from the decay of high-energy pions and peak in energy around 1 or 2 GeV, is the inverse μ capture process (1.1). Since the boson, if it exists, has a mass greater than 1 GeV ²⁾, it will only be produced by the high-energy tail of the neutrino spectrum and thus the experimental identification of this smaller fraction of the events becomes a problem. The boson (W^{+}) must be charged to account for the charged weak currents and will have the leptonic decay modes

$$\begin{aligned} W^{+} &\longrightarrow e^{+} + \nu_e \\ &\longrightarrow \mu^{+} + \nu_{\mu} \end{aligned} \quad (1.2)$$

Thus, one method of identification is to look for the production of lepton pairs coming from the chain

$$\nu_\mu + Z \longrightarrow \mu^- + W^+ + Z \quad (1.3)$$

$$\quad \quad \quad \searrow \left\{ \begin{array}{l} e^+ + \nu_e \\ \mu^+ + \nu_\mu \end{array} \right.$$

The difficulty with the observation of this process so far has been the unambiguous identification of lepton pair events among all the others and also the fact that since two neutrinos of unknown energy are involved, it is difficult to directly relate the observations to the properties of a W . The W can be expected to have other decay modes. One such is the decay into two pions

$$W^\pm \longrightarrow \pi^\pm + \pi^0 \quad (1.4)$$

with a branching ratio to leptonic decays given by (neglecting $\frac{m_\pi^2}{m_W^2} \ll 1$)

$$\frac{\omega(W \rightarrow 2\pi)}{\omega(W \rightarrow \text{leptons})} = \frac{1}{8} \left| F_\pi(-m_W^2) \right|^2 \quad (1.5)$$

where $F_\pi(-m_W^2)$ is the electromagnetic form factor of the pion evaluated at a time-like momentum transfer of m_W^2 ⁵⁾. Thus it can be expected to compete with the leptonic modes. The two pion decay mode offers the possibility of directly observing the W by looking at the invariant mass of the two pions. This has not yet proved possible, one difficulty being the large number of pions coming from inelastic processes associated with (1.1). The present indications are that lepton pairs are produced with a frequency which is less than, or of the order of, one percent of the elastic events (1.1)²⁾.

Now even if the W does not exist, or has a very large mass, there is still a process of lepton pair production by neutrinos proceeding through the point four-fermion interaction and then a subsequent coherent (or incoherent) scattering of the charged particles in the Coulomb field of the nucleus, as in ordinary electromagnetic pair production. The processes we have in mind are

- (a) $\nu_{\mu} + Z \rightarrow \mu^{-} + e^{+} + \nu_e + Z$
 (b) $\nu_{\mu} + Z \rightarrow \mu^{-} + \mu^{+} + \nu_{\mu} + Z$
 (c) $\nu_e + Z \rightarrow e^{-} + \mu^{+} + \nu_{\mu} + Z$
 (d) $\nu_e + Z \rightarrow e^{-} + e^{+} + \nu_e + Z$

It is shown in Appendix A that the cross-sections for the processes obtained by replacing leptons by antileptons in (a)-(d) are the same.

The two Feynman diagrams for this general process are shown in Fig. 1. The weak vertex is just the contact V-A interaction

$$H_W = \frac{G}{\sqrt{2}} \left[\bar{\psi}_{(l_1)} \gamma_{\lambda} (1 + \gamma_5) \psi_{(\nu_1)} \right] \left[\bar{\psi}_{(\nu_2)} \gamma_{\lambda} (1 + \gamma_5) \psi_{(l_2)} \right] \quad (1.6)$$

where

$$G = \frac{1.01 \times 10^{-5}}{M_p^2} \quad (1.7)$$

(M_p = proton mass). The existence of such an interaction between muons and electrons is implied by the muon decay. There is no direct evidence for such a term involving only muons, but any lepton symmetry scheme or attempt to write weak interactions as the product of currents would imply it *). The electromagnetic interactions of the particles in Fig. 1 are the usual ones. One can roughly estimate the contribution of this lepton pair production process relative to the elastic process (1.1) as being down by a factor of $((\alpha^2 Z^2)/N)$ per nucleus, where N is the number of neutrons. Since this number is a fraction of a percent for heavy nuclei, it is crucial to have an accurate calculation of this process before

*) We have assumed that there are no terms $(\bar{\nu}\nu)(\bar{N}N)$, where N is a nucleon, present in the Hamiltonian. There is experimental evidence supporting this assumption ²⁾.

concluding anything about the existence of intermediate bosons from the presence of lepton pairs. The calculation is, however, also interesting in its own right since it is a process which is exactly calculable in terms of what we know about the point four-fermion interaction and electromagnetic interactions. Thus an exact prediction can be made for it. If the experiments disagree with the predictions then there must be some profound modification of the weak interactions, such as the existence of a vector boson or the coming into play of higher orders in the weak interactions as suggested by Pais and Feinberg ⁶⁾. It is also easily seen (because of the extra powers of the boson mass in the propagators) that even if an intermediate boson exists, the result calculated from the diagrams of Fig. 1 will be correct in the limit of a fixed incident energy with the boson mass going to infinity.

There have been several attempts to estimate process (b) on the basis of the Weizsäcker-Williams approximation ^{7),8),9),10)}, the most extensive being the calculations of Kozhushner and Shabalin ⁷⁾ and Shabalin ¹⁰⁾. It was argued in Ref. ⁷⁾ that the nuclear form factor limits the four momentum transfer to be less than the reciprocal of the size of the nucleus.

$$q < \frac{1}{R} = \frac{1}{1.2 A^{1/3} f} \quad (\text{roughly}) \quad (1.8)$$

whereas, kinematically, the momentum transfer can be of the order of the incident neutrino energy. Thus it is the nuclear form factor which limits the size of the cross-section for high-energy neutrinos. Furthermore,

$$\frac{q}{m_\mu} < \frac{1.6}{A^{1/3}} \quad (\text{roughly}) \quad (1.9)$$

Therefore, for heavy nuclei, one can hope to get a good approximation by letting $q^2 \rightarrow 0$ and, replacing the upper parts of the graphs of Fig. 1 by photoproduction of muon pairs off neutrinos, the Weizsäcker-Williams approximation. This procedure simplifies the γ algebra and phase space integrals immensely. In Refs. ^{7),10)} the photoproduction cross-section was

only computed in the limits $\mathcal{E}_1 \rightarrow \infty$ and $\mathcal{E}_1 \rightarrow \text{threshold}$. One noticeable feature of the high energy asymptotic cross-section of Refs. ^{7),10)} is the almost complete cancellation of the first two terms in the asymptotic cross-section for (b), resulting in the conclusion that the cross-section for (b) is very small. It is of the same interest, therefore, to see if this cancellation holds up in the exact calculation. The approximations discussed above are completely inadequate if there is an electron involved in the process [as pointed out in Ref. ¹⁰⁾], and, as far as we know, there are no reliable estimates of these processes in the literature.

Since the nuclear form factor limits the high-energy dependence of the cross-sections for the coherent processes (a)..(b) to be of the form $\sigma \sim \mathcal{E}_1 \ln \mathcal{E}_1$, one can ask what about the incoherent processes where the individual nucleons contribute independently to the cross-section (each being ejected from the nucleus). This gives an incoherent contribution which must be added to the coherent contribution (where the nucleus stays in its ground state), and the cross-section here is limited by the nucleon form factors. Thus the form factors die off slower but one loses the coherence. The cross-section here again goes asymptotically as $\sigma \sim \mathcal{E}_1 \ln \mathcal{E}_1$. One can also ask what about production off atomic electrons? Since the electron is (assumed to be) a point particle, there is no form factor and one can expect that at very high energies these processes will actually dominate since the cross-section can be expected to go as $\sigma \sim \mathcal{E}_1 \ln^2 \mathcal{E}_1$. With incident ν_μ , however, the thresholds are very high

$$\nu_\mu + e^- \longrightarrow \mu^- + \mu^+ + \nu_\mu + e^- \quad \mathcal{E}_{1\text{thresh}} = \frac{(m_e + 2m_\mu)^2 - m_e^2}{2m_e} \cong 412m_\mu \quad (1.10)$$

$$\nu_\mu + e^- \longrightarrow \mu^- + e^+ + \nu_e + e^- \quad \mathcal{E}_{1\text{thresh}} = \frac{(m_\mu + 2m_e)^2 - m_e^2}{2m_e} \cong 103m_\mu \quad (1.11)$$

The threshold with incident ν_e for the process

$$\nu_e + e^- \longrightarrow e^- + e^+ + \nu_e + e^- \quad \mathcal{E}_{1\text{thresh}} = 4m_e \quad (1.12)$$

is lower. It takes place, however, only through the ν_e and the present neutrino beams are mainly ν_μ . Also, because of the extra α^2 , it is expected to be much less than

$$\nu_e + e^- \longrightarrow \nu_e + e^- \quad (1.13)$$

until extremely high energies are reached. Only a few candidates for (1.13) are known to exist ²⁾. We have not succeeded in getting the asymptotic form of the cross-section for (1.12). [There are other diagrams beside Fig. 1, those where the incident neutrino interacts with the target electron, which contribute to processes (1.10) - (1.12).]

In this paper we shall be concerned with calculating exactly to order $G^2 \alpha^2$ [see Fig. 1] the total cross-sections for the processes (a)..^(d). The two main problems are the γ matrix algebra and the phase space integrals (four particle final state). We shall use invariance arguments and the fact that we are calculating the total cross-section to simplify the γ matrix algebra, and then shall write the phase space integrals as integrals over the invariants appearing in the matrix element. We have carried out a numerical integration of the resulting sixfold integrals and have obtained the total coherent cross-sections as a function of incident energy. We have also extracted analytically the first two terms in the high energy asymptotic forms of these cross-sections. The leading order terms in the asymptotic expansions are given in a previous paper by two of the authors ¹¹⁾. We have computed numerically the incoherent cross-sections for processes (a) and (b) by just summing the contributions of the individual protons and neutrons, and found analytically the first term in the asymptotic form of the incoherent cross-section for process (b).

The nuclear physics in the coherent processes can be completely taken into account in these calculations through the elastic form factors [this result can be generalized to include all inelastic nuclear processes ¹²⁾]. We shall write our answers in a form such that the last integral is over the squared momentum transfer and thus over the nuclear form factor. To evaluate these integrals we have used the following two nuclear form factors

$$F_I(q^2) = \frac{1}{\left(1 + \frac{q^2}{\beta_I^2}\right)^2} \quad \beta_I = \frac{\sqrt{20}}{R_0} \quad (1.14)$$

$$F_{II}(q^2) = e^{-\frac{1}{2} \frac{q^2}{\beta_{II}^2}} \quad \beta_{II} = \frac{\sqrt{5}}{R_0} \quad (1.15)$$

with

$$R_0 = 1.2 A^{1/3} f \quad (1.16)$$

being chosen so that both form factors reduce to

$$F = 1 - \frac{1}{10} q^2 R_0^2 \quad (1.17)$$

at low momentum transfer in order to fit electron scattering data ¹³⁾. Since F_I falls off as a power with high q^2 and F_{II} falls off exponentially, these two form factors should bracket the behaviour of the real nuclear form factor. We emphasize that this is just a matter of convenience and one can insert the real nuclear form factor into the last step in our calculation, but the above two form factors give essentially the same results in the region of energies we are interested in.

For the individual nucleon form factors in the incoherent production we use the simple form ¹⁴⁾

$$f_{1p} = f_{2p} = f_{2n} = \frac{1}{\left(1 + 1.21 \frac{q^2}{M_p^2}\right)^2} \quad (1.18)$$

$$f_{1n} = 0$$

We note that if the nucleon form factors were to have a hard core, the incoherent cross-sections could become much larger. We consider this unlikely however.

2. CALCULATIONS

The S matrix element corresponding to the two Feynman diagrams in Fig. 1 can be immediately written down according to the usual rules but first it is convenient to carry out a Fierz transformation on H_W and write

$$H_W = \frac{G}{\sqrt{2}} [\bar{\psi}_{(l_1)} \gamma_\lambda (1 + \gamma_5) \psi_{(l_2)}] [\bar{\psi}_{(l_2)} \gamma_\lambda (1 + \gamma_5) \psi_{(l_1)}] \quad (2.1)$$

(assuming the lepton fields anticommute). With this one finds (with $q \equiv P - P'$)

$$(f|S|i) = -\frac{(2\pi)^4}{\Omega^2} \delta^{(4)}(P + p_1 - P' - p_2 - p_3 - p_4) \frac{Ze^2 G}{\sqrt{2}} \frac{1}{q^2} (P' | \frac{J_\mu(0)}{Z} | P) \cdot \quad (2.2)$$

$$\cdot [\bar{u}(\vec{p}_2) \gamma_\lambda (1 + \gamma_5) u(\vec{p}_1)] \bar{u}(\vec{p}_4) \left\{ \gamma_\mu \frac{1}{i(\not{p}_4 - \not{q}) + m_1} \gamma_\lambda (1 + \gamma_5) + \gamma_\lambda (1 + \gamma_5) \frac{1}{i(\not{q} - \not{p}_3) + m_2} \gamma_\mu \right\} u(\vec{p}_3)$$

where $(P' | J_\mu(0) | P)$ is the transition matrix element between exact nuclear states $(P'^2 = P^2 = M_T^2, \text{ the mass of the target})$ of the electromagnetic current operator of the nucleus taken at the origin. When multiplied by $1/q^2$ this is just the familiar Møller potential provided by the nucleus. Ω is our normalization volume and we use a metric with $a_\mu = (\vec{a}, ia_0)$, and Hermitian γ matrices satisfying $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$. The fine structure constant is $\alpha = \frac{1}{137} = \frac{e^2}{4\pi}$ and we have set $\hbar = c = 1$.

The total cross-section can be immediately obtained from this in the standard fashion and we find, with $q \equiv P - P'$,

$$\sigma = \frac{2Z^2 \alpha^2 G^2}{(2\pi)^6} \frac{1}{4M_T E_1} \int \frac{d\vec{P}'}{2E'} F_{\mu\nu}(P, q) M_{\mu\nu}(P', q) \frac{1}{q^4} \quad (2.3)$$

$$F_{\mu\nu}(P, q) = (2E\Omega)(2E'\Omega) (P | \frac{J_\nu(0)}{Z} | P') (P' | \frac{J_\mu(0)}{Z} | P) \quad (2.4)$$

$$\begin{aligned}
M_{\mu\nu}(p,q) &= \int \frac{d\vec{p}_2}{2\varepsilon_2} \int \frac{d\vec{p}_3}{2\varepsilon_3} \int \frac{d\vec{p}_4}{2\varepsilon_4} \delta^{(4)}(q+p_1-p_2-p_3-p_4) \text{Tr} \left[\gamma_\lambda (1+\gamma_5) \not{p}_1 \gamma_\sigma (1+\gamma_5) \not{p}_2 \right] \cdot \\
&\cdot \text{Tr} \left[\gamma_\mu \frac{1}{i(\not{p}_4 - \not{q}) + m_1} \gamma_\lambda (1+\gamma_5) + \gamma_\lambda (1+\gamma_5) \frac{1}{i(\not{q} - \not{p}_3) + m_2} \gamma_\mu \right] \left[-(m_2 + i\not{p}_3) \right] \cdot \\
&\cdot \left[\gamma_\nu \frac{1}{i(\not{q} - \not{p}_3) + m_2} \gamma_\sigma (1+\gamma_5) + \gamma_\sigma (1+\gamma_5) \frac{1}{i(\not{p}_4 - \not{q}) + m_1} \gamma_\nu \right] \left[m_1 - i\not{p}_4 \right]
\end{aligned} \tag{2.5}$$

where we have carried out a sum over lepton spins. We can just sum over initial neutrino spins if the initial neutrino beam is completely polarized as is the experimental case, since the $(1+\gamma_5)$ in the neutrino interaction is a spin projection operator and therefore the wrong value of the spin gives zero in the sum. There is no factor of $\frac{1}{2}$ as there would be if one were to just average over incident neutrino spins. σ is evidently a Lorentz invariant (note $M_T \epsilon_1 = -P \cdot p_1$) while $M_{\mu\nu}$ and $F_{\mu\nu}$ transform as second rank tensors. One can see immediately, by making use of the projection operators standing in the traces, that

$$q_\mu M_{\mu\nu} = M_{\mu\nu} q_\nu = 0 \tag{2.6}$$

This is a statement of electromagnetic current conservation for the leptonic parts of the process.

If we have elastic scattering from a spin zero target, then we can write from invariance arguments and electromagnetic current conservation for the nucleus (note $P \cdot q = \frac{1}{2} q^2$)

$$\langle P' | \frac{J_\mu(0)}{Z} | P \rangle = \frac{1}{\sqrt{EE'\Omega^2}} F(q^2) \left(P_\mu - \frac{1}{2} q_\mu \right) \tag{2.7}$$

$$F(0) = 1$$

where the normalization of F comes from looking at $\mu = 4$ and $q^2 \rightarrow 0$. This leads to

$$F_{\mu\nu}(P, q) = 4 |F(q^2)|^2 (P_\mu - \frac{1}{2} q_\mu)(P_\nu - \frac{1}{2} q_\nu) = F_{\nu\mu}(P, q) \quad (2.8)$$

In the case of elastic scattering from a spin $\frac{1}{2}$ target (incoherent process off nucleons) then ¹²⁾

$$\begin{aligned} F_{\mu\nu}(P, q) &= q^2 |F_1(q^2) + 2M_T F_2(q^2)|^2 \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \\ &+ (|F_1(q^2)|^2 + q^2 |F_2(q^2)|^2) 4 (P_\mu - \frac{1}{2} q_\mu)(P_\nu - \frac{1}{2} q_\nu) = \quad (2.9) \\ &= F_{\nu\mu}(P, q) \end{aligned}$$

where $F_1(0) = 1$, $F_2(0) = \frac{\lambda}{2M_T}$, and λ is the anomalous magnetic moment of the target. We note that in both (2.8) and (2.9) $F^{\mu\nu} = F_{\nu\mu}$ is symmetric. This means that only the symmetric part of $M^{\mu\nu}$ will contribute in our answer. Now one can explicitly separate $M^{\mu\nu}$ into

$$M_{\mu\nu} = M_{\mu\nu}^{(SYM)} + M_{\mu\nu}^{(ANTISYM)} \quad (2.10)$$

Equation (2.6) (or explicit evaluation) then implies

$$q_\mu M_{\mu\nu}^{(SYM)} = M_{\mu\nu}^{(SYM)} q_\nu = 0 \quad (2.11)$$

Therefore, since $M_{\mu\nu}^{(SYM)}$ can only depend on p_1 and q and must satisfy (2.11), it must have the form ¹⁵⁾

$$M_{\mu\nu}^{SYM}(q, p) = f_1(q^2, q \cdot p) \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + f_2(q^2, q \cdot p) \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \quad (2.12)$$

Once we have the explicit form (2.12), however, we can simplify the γ algebra considerably. Since the only unknowns are now f_1 and f_2 , all we need are two linearly independent equations to solve for these. The first can be obtained by taking

$$M_{\mu\mu}^{(\text{SYM})} = M_{\mu\mu} = 3f_1(q^2, q \cdot p_1) - \frac{(p_1 \cdot q)^2}{q^2} f_2(q^2, q \cdot p_1) \quad (2.13)$$

and computing $M_{\mu\mu}$ reduces the number of γ matrices by two in the well-known fashion (this is the expression one gets in the Weizsäcker-Williams approximation where one replaces the virtual photon in Fig. 1 by a real photon and then performs a covariant polarization sum). One other equation can be obtained by dotting both sides of (2.12) into p_1

$$p_{1\mu} M_{\mu\nu}^{(\text{SYM})} p_{1\nu} = p_{1\mu} M_{\mu\nu} p_{1\nu} = - \frac{(p_1 \cdot q)^2}{q^2} \left[f_1(q^2, q \cdot p_1) - \frac{(p_1 \cdot q)^2}{q^2} f_2(q^2, q \cdot p_1) \right] \quad (2.14)$$

This again reduces the γ algebra by virtue of the fact that $\not{p}_1 \not{p}_1 = p_1^2 = 0$. We can therefore solve [noting $(q \cdot p_1)^2 > 0$ always]

$$f_1(q^2, q \cdot p_1) = \frac{1}{2} \left[M_{\mu\mu} + \frac{q^2}{(q \cdot p_1)^2} p_{1\mu} M_{\mu\nu} p_{1\nu} \right] \quad (2.15)$$

$$f_2(q^2, q \cdot p_1) = \frac{1}{2} \left[M_{\mu\mu} + \frac{3q^2}{(q \cdot p_1)^2} p_{1\mu} M_{\mu\nu} p_{1\nu} \right] \frac{q^2}{(q \cdot p_1)^2} \quad (2.16)$$

We can now join (2.12) and (2.8) or (2.9) to get the integrand in (2.3)

At this point it is convenient to introduce a complete, linearly independent set of invariants describing our process. As far as the integrand in $M_{\mu\nu}$ is concerned, there are four independent four vectors which yield ten scalar invariants and four mass shell restrictions reduce this number to six. We shall take these invariants to be :

$$\begin{aligned}
x_1 &= q^2 \\
x_2 &= -q \cdot p_1 \\
x_3 &= -q \cdot p_3 \\
x_4 &= -q \cdot p_4 \\
x_5 &= -p_1 \cdot p_3 \\
x_6 &= -p_1 \cdot p_4
\end{aligned} \tag{2.17}$$

The choice is somewhat arbitrary but we will find that our phase space integrals are fairly simple in terms of these invariants (we have not been able to find any other set which makes things simpler). In joining (2.14) to (2.8) or (2.9) one gets additional terms in x_1, x_2 , or $p_1 \cdot p = -\epsilon_1 m_T$. Thus for fixed incident energy ϵ_1 , one must be able to write all phase space integrals and integrands in terms of $x_1 \dots x_6$. The other invariants can be given in terms of these by :

$$\begin{aligned}
-p_1 \cdot p_2 &= x_2 - x_5 - x_6 \\
-p_2 \cdot p_3 &= x_2 - x_4 - x_6 - \frac{1}{2} [m_2^2 + x_1 - m_1^2] \\
-p_2 \cdot p_4 &= x_2 - x_3 - x_5 - \frac{1}{2} [m_1^2 + x_1 - m_2^2] \\
+p_3 \cdot p_4 &= x_2 - x_3 - x_4 - x_5 - x_6 + \frac{1}{2} [m_1^2 + m_2^2 - x_1] \\
-q \cdot p_2 &= x_2 - x_3 - x_4 - x_1
\end{aligned} \tag{2.18}$$

The reduction of the integrals in (2.3) and (2.5) to integrals over the invariants $x_1 \dots x_6$ is discussed in detail in Appendix B. The main idea is that since our expression is in covariant form, we can work in any frame we want. One first goes to the frame defined by $\vec{q} + \vec{p}_1 - \vec{p}_3 = 0$ (the centre of momentum of ν_2 and l_1). In this frame one can carry out

$$\int \frac{d\vec{p}_4}{2\epsilon_4} \int \frac{d\vec{p}_2}{2\epsilon_2} \delta^{(4)}(q + p_1 - p_2 - p_3 - p_4)$$

The $\delta^{(4)}$ takes care of four integrations ($d\vec{p}_2 d\epsilon_4$) and one is left with two angular integrations. These can be related to integrals over $\int dx_4 \int dx_6$.

It turns out that it is simpler to do instead the following : one finds that in this frame the only dependence on θ_{p_4} occurs in x_6 since one can take $\hat{q}$ as the direction of the $\hat{z}$ axis. This integration can be carried out explicitly and is trivial since x_6 appears only to the first and second power in the numerators of the integrand (this is the reason for our statement above that this is a convenient set of invariants).

After carrying it out, one can then re-express the result in terms of the invariants $x_1 \dots x_5$. The remaining integral over $d \cos \theta_{p_4}$ can then be converted to an integral over dx_4 with a simple Jacobian. The limits on this angle ($\cos \theta_{p_4} = \pm 1$) are easy to compute but turn out to be rather complicated. Since the expression is now again in explicitly covariant form, one can go to another convenient frame $\vec{q} + \vec{p}_1 = 0$ (the centre of momentum system of ν_2 , l_1 , and l_2). In this frame it is easy to show

$$\frac{d\vec{p}_3}{2\varepsilon_3} = \frac{\pi}{x_2} dx_3 dx_5 \quad (2.19)$$

the limits now depending on the various maxima and minima of the variables involved for fixed x_1 and x_2 . Finally, one can go to the lab. frame to show

$$\frac{d\vec{p}'}{2E'} = \frac{\pi}{2m_T \varepsilon_1} dx_1 dx_2 \quad (2.20)$$

the limits again being straightforward but complicated. As a result of these operations we can write

$$\sigma = \frac{Z^2 \alpha^2 G^2}{\pi^3} \frac{1}{\varepsilon_1^2} \int_{\underline{x}_1}^{\bar{x}_1} \frac{dx_1}{x_1^2} \int_{\underline{x}_2}^{\bar{x}_2} \frac{dx_2}{x_2} \int_{\underline{x}_5}^{\bar{x}_5} dx_5 \int_{\underline{x}_3}^{\bar{x}_3} dx_3 \frac{1}{[(x_2 - x_3)^2 - 2x_1 x_5 + x_1 m_2^2]^{1/2}} \int_{\underline{x}_4}^{\bar{x}_4} dx_4 \quad (2.21)$$

$$\cdot \int \frac{d\phi}{2\pi} \left\{ A(x_1, x_2) \mathcal{M}_1(x_1, \dots, x_6) + B(x_1, x_2) \mathcal{M}_2(x_1, \dots, x_6) \right\}$$

The functions $\mathcal{M}_1$ and $\mathcal{M}_2$ come from the traces in the integrands of $\mathcal{M}_{\mu\mu}$ and $p_{1\mu}\mathcal{M}_{\mu\nu}p_{1\nu}$ and are given by

$$\begin{aligned} \mathcal{M}_1(x_1 \dots x_6) = & \frac{1}{(x_3 + \frac{x_1}{2})^2} \left\{ x_2(x_3 + m_2^2) \left(x_2 - x_3 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) + x_5 \left(\frac{x_1}{2} - m_2^2 \right) \left(x_2 - x_3 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) \right\} \\ & + \frac{1}{(x_4 + \frac{x_1}{2})^2} \left\{ x_5(x_4 + m_1^2) (x_2 - x_3 - x_4 - x_1) + x_5 \left(\frac{x_1}{2} - m_1^2 \right) \left(x_2 - x_3 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) \right\} \end{aligned} \quad (2.22)$$

$$\begin{aligned} & - \frac{1}{(x_3 + \frac{x_1}{2})(x_4 + \frac{x_1}{2})} \left\{ x_3(x_5 - x_6) \left(x_2 - x_3 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) + x_4 x_5 (x_4 + x_6 - x_3 - x_5 - m_1^2 + m_2^2) + \right. \\ & + (x_2 - x_3 - x_4 - x_5 - x_6 + \frac{1}{2}[m_1^2 + m_2^2 - x_1]) \left[2x_5 \left(x_2 - x_3 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) - x_5 (x_2 - x_3 - x_4 - x_1) - x_2 \left(x_2 - x_3 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) \right] \\ & \left. + \frac{x_1}{2} \left[x_5 \left(x_2 - x_3 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) - (x_2 - x_5 - x_6) \left(x_2 - x_3 - x_4 - x_5 - x_6 + \frac{1}{2}[m_1^2 + m_2^2 - x_1] \right) - x_6 \left(x_2 - x_4 - x_6 - \frac{1}{2}[m_2^2 + x_1 - m_1^2] \right) \right] \right\} \end{aligned}$$

$$\mathcal{M}_2(x_1 \dots x_6) = \frac{1}{(x_3 + \frac{x_1}{2})^2} \left\{ x_5 \left(x_2 - x_3 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) (x_2 - x_5)^2 \right\} \quad (2.23)$$

$$\begin{aligned} & + \frac{1}{(x_4 + \frac{x_1}{2})^2} \left\{ x_5 x_6 (x_2 - x_6) \left(x_5 - x_4 - x_1 + \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) + \left(x_4 + \frac{x_1}{2} \right) (x_5 x_6) (x_2 - x_5 - x_6) \right\} \\ & + \frac{1}{(x_3 + \frac{x_1}{2})(x_4 + \frac{x_1}{2})} \left\{ -x_5 \left(x_2 - x_3 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) (x_2 - x_5) (x_2 - x_6) + \right. \\ & \left. + x_5 x_6 (x_2 - x_5) \left(x_1 + x_4 - x_5 - \frac{1}{2}[x_1 + m_1^2 - m_2^2] \right) + x_4 x_5 (x_2 - x_5 - x_6) (x_2 - x_5) \right\} \end{aligned}$$

The limits on the phase space integrals are given by

$$\begin{aligned}\bar{x}_1 &= \frac{2\varepsilon_1^2}{1 + \frac{2\varepsilon_1}{m_T}} \left\{ \left(1 - \frac{(m_1+m_2)^2}{2\varepsilon_1^2} \left[1 + \frac{\varepsilon_1}{m_T} \right] \right) \pm \left[\left(1 - \frac{(m_1+m_2)^2}{2\varepsilon_1^2} \left[1 + \frac{\varepsilon_1}{m_T} \right] \right)^2 - \frac{(m_1+m_2)^4}{4\varepsilon_1^4} \left(1 + \frac{2\varepsilon_1}{m_T} \right) \right]^{1/2} \right\} \\ \bar{x}_2 &= \varepsilon_1 \left\{ \left(x_1 + \frac{x_1^2}{4m_T^2} \right)^{1/2} - \frac{x_1}{2m_T} \right\} \quad x_2 = \frac{1}{2} \left[x_1 + (m_1+m_2)^2 \right] \\ \bar{x}_5 &= \frac{x_2}{2(2x_2-x_1)} \left\{ (2x_2-x_1+m_2^2-m_1^2) \pm \left[(2x_2-x_1-m_1^2-m_2^2)^2 - 4m_1^2m_2^2 \right]^{1/2} \right\} \\ \bar{x}_3 &= \frac{1}{2} (2x_2-x_1+m_2^2-m_1^2) - x_5 \quad x_3 = \frac{m_2^2}{2} \frac{x_2}{x_5} - \frac{x_1}{2} \frac{x_5}{x_2} \\ \bar{x}_4 &= \frac{(x_2-x_1-x_3)(2[x_2x_3-x_5]-x_1+m_2^2+m_1^2) \pm (2[x_2x_3-x_5]-x_1+m_2^2-m_1^2)[(x_2-x_3)^2-2x_1x_5+x_1m_2^2]^{1/2}}{2[2(x_2-x_3-x_5)-x_1+m_2^2]}\end{aligned}\tag{2.24}$$

The functions $A(x_1x_2)$ and $B(x_1x_2)$ come from joining (2.12) with (2.8) and (2.9) and thus depend on the type of target. We find :

Case 1 : spin 0 target

$$\begin{aligned}A(x_1x_2) &= |F(x_1)|^2 \left[\frac{\varepsilon_1^2 x_1}{x_2^2} \left(1 - \frac{x_2}{m_T \varepsilon_1} \right) - 1 \right] \\ B(x_1x_2) &= |F(x_1)|^2 \frac{x_1}{x_2^2} \left[3 \frac{\varepsilon_1^2 x_1}{x_2^2} \left(1 - \frac{x_2}{m_T \varepsilon_1} \right) - \left(1 - \frac{x_1}{2m_T^2} \right) \right]\end{aligned}\tag{2.25}$$

Case 2 : spin $\frac{1}{2}$ target

$$\begin{aligned}A(x_1x_2) &= \frac{x_1}{2m_T^2} |F_1(x_1) + 2m_T F_2(x_1)|^2 + (|F_1(x_1)|^2 + x_1 |F_2(x_1)|^2) \left[\frac{\varepsilon_1^2 x_1}{x_2^2} \left(1 - \frac{x_2}{m_T \varepsilon_1} \right) - 1 \right] \\ B(x_1x_2) &= \frac{x_1}{x_2^2} (|F_1(x_1)|^2 + x_1 |F_2(x_1)|^2) \left[3 \frac{\varepsilon_1^2 x_1}{x_2^2} \left(1 - \frac{x_2}{m_T \varepsilon_1} \right) - \left(1 - \frac{x_1}{2m_T^2} \right) \right]\end{aligned}\tag{2.26}$$

The $\int \frac{d\phi}{2\pi}$ can be carried out as discussed above and results in :

i) where there is no x_6 :

$$\int \frac{d\phi}{2\pi} f(x_1 \dots x_5) = f(x_1 \dots x_5) \quad (2.27)$$

ii) where x_6 appears linearly :

$$\begin{aligned} \int \frac{d\phi}{2\pi} x_6 = & \frac{1}{(x_2-x_3)^2 - 2x_1x_5 + x_1m_2^2} \left\{ x_1(x_2-x_5)(x_2-x_3-x_5 + \frac{1}{2}[m_1^2 + m_2^2 - x_1]) + \right. \\ & \left. + x_2(x_2-x_3-x_1)(x_2-x_3-x_5 + \frac{1}{2}[m_1^2 + m_2^2 - x_1]) + x_4(x_2-x_3-x_1)(x_2-x_5) - x_2x_4(2[x_2-x_3-x_5] + m_2^2 - x_1) \right\} \end{aligned}$$

iii) where x_6 appears quadratically :

$$\begin{aligned} \int \frac{d\phi}{2\pi} x_6^2 = & \frac{1}{[(x_2-x_3)^2 - 2x_1x_5 + x_1m_2^2]^2} \left\{ \frac{1}{2} \left[x_1(x_2-x_5)^2 + 2x_2(x_2-x_5)(x_2-x_3-x_1) - x_2^2(2[x_2-x_3-x_5] + m_2^2 - x_1) \right] \right. \\ & \cdot \left[x_1(x_2-x_3-x_5 + \frac{1}{2}[m_1^2 + m_2^2 - x_1])^2 - m_1^2((x_2-x_3)^2 - 2x_1x_5 + x_1m_2^2) + 2x_4(x_2-x_3-x_1)(x_2-x_3-x_5 + \frac{1}{2}[m_1^2 + m_2^2 - x_1]) \right. \\ & \quad \left. \left. - x_4^2(2[x_2-x_3-x_5] + m_2^2 - x_1) \right] + \right. \\ & \left. + \left[x_1(x_2-x_5)(x_2-x_3-x_5 + \frac{1}{2}[m_1^2 + m_2^2 - x_1]) + x_2(x_2-x_3-x_1)(x_2-x_3-x_5 + \frac{1}{2}[m_1^2 + m_2^2 - x_1]) + x_4(x_2-x_3-x_1)(x_2-x_5) \right. \right. \\ & \quad \left. \left. - x_2x_4(2[x_2-x_3-x_5] + m_2^2 - x_1) \right]^2 \right\} . \end{aligned}$$

We note that x_6 only appears linearly in the interference term in $\mathcal{M}_1(x_1 \dots x_6)$ and both linearly and quadratically in $\mathcal{M}_2(x_1 \dots x_6)$.

Although the problem has been reduced to a sixfold integral over a rational function of $(x_1 \dots x_6)$, exact analytic evaluation appears to the authors to be extremely tedious, the main problem being the square roots which enter into the limits, and into the Jacobian. (It should also be noted that the main complication is the presence of x_1 in these square roots.) It is possible, however, to extract an analytic expression

expression for the first two terms in the asymptotic, $E_1 \rightarrow \infty$, form of the cross-sections. These terms go as $E_1 \ln E_1$ and E_1 . We find that the following simplification of Eqs. (2.21) to (2.27) correctly yields these first two terms :

$$\sigma_{E_1 \rightarrow \infty} = \frac{Z^2 \alpha^2 G^2}{\pi^3} \frac{1}{E_1^2} \int_{\underline{x}_1}^{\overline{x}_1} \frac{dx_1}{x_1^2} \int_{\underline{x}_2}^{\overline{x}_2} \frac{dx_2}{x_2} \left[A(x_1 x_2) G_1(x_1 x_2) + B(x_1 x_2) G_2(x_1 x_2) \right] \quad (2.28)$$

$$G_1(x_1 x_2) = \int_{a_5}^{b_5} \int_{a_3}^{b_3} \frac{dx_3}{x_2 - x_3} \int_{a_4}^{b_4} dx_4 \left\{ \left[1 - \frac{2x_5}{x_2 - x_3} + \frac{2x_5^2}{(x_2 - x_3)^2} \right] \left[\frac{x_5(x_2 - x_3)}{x_4 + \frac{x_1}{2}} - (x_2 + x_5) + \frac{x_2(x_2 - x_5)}{x_3 + \frac{x_1}{2}} \right] \right. \\ \left. + \left(m_2^2 - \frac{x_1}{2} \right) \frac{(x_2 - x_5)^2}{\left(x_3 + \frac{x_1}{2} \right)^2} + \left(m_1^2 - \frac{x_1}{2} \right) \frac{x_5^2}{\left(x_4 + \frac{x_1}{2} \right)^2} \right\} \quad (2.29)$$

$$G_2(x_1 x_2) = \int_{a_5}^{b_5} \int_{a_3}^{b_3} \frac{dx_3}{x_2 - x_3} \int_{a_4}^{b_4} dx_4 \left\{ \frac{x_5(x_2 - x_5)^2(x_2 - x_3 - x_5)}{\left(x_3 + \frac{x_1}{2} \right)^2} + \frac{x_5^3 x_2^2 (x_2 - x_3 - x_5)}{\left(x_4 + \frac{x_1}{2} \right)^2 (x_2 - x_3)^2} \right\}$$

where the limits have been expanded and are given by

$$\begin{aligned} b_5 &= x_2 - \frac{1}{2} m_1^2 \\ a_5 &= \frac{1}{2} m_2^2 \\ b_3 &= x_2 - x_5 - \frac{1}{2} (m_1^2 - m_2^2) - \frac{1}{2} x_1 \\ a_3 &= \frac{m_2^2}{2} \left(\frac{x_2}{x_5} \right) + \frac{x_1}{2} \left(\frac{x_2 - x_5}{x_2} \right) - \frac{1}{2} x_1 \\ b_4 &= x_2 - x_3 - \frac{x_1}{2} \left(\frac{x_5}{x_2 - x_3} \right) - \frac{1}{2} x_1 \\ a_4 &= \frac{m_1^2}{2} \left(\frac{x_2 - x_3}{x_2 - x_3 - x_5} \right) + \frac{x_1}{2} \left(\frac{x_5}{x_2 - x_3} \right) - \frac{1}{2} x_1 \end{aligned} \quad (2.30)$$

The basic observation which goes into the reduction leading to (2.28) is to note that since x_1 is limited by the nuclear form factor, the only place ε_1 effectively appears in the limits of integration is in the upper x_2 limit. It also appears in the lower x_1 limit, making this limit go to zero as $((m_1+m_2)^4)/(4\varepsilon_1^2)$. One might expect to pick up a contribution from the $1/x_1^2$ appearing in the integrand (as one does in electromagnetic pair production for example). One can see, however, that

$$\frac{1}{\varepsilon_1^2} \int_{\frac{(m_1+m_2)^4}{4\varepsilon_1^2}}^{\varepsilon_1^2} \frac{dx_1}{x_1^2}$$

cannot contribute terms in either $\varepsilon_1 \ln \varepsilon_1$ or ε_1 . Since x_2 gets as big as ε_1 and since $x_3 \dots x_6$ can get as big as x_2 , one expects to be able to start from the idea that $x_2 \dots x_6 \gg x_1, m_1^2, m_2^2$. Consistently throwing away these mass terms in numerators, in the expressions for $\frac{d\sigma}{2\pi}$, in the Jacobian and in the corrections to the limits, one arrives at the above. One can check at each step that no contribution to $\varepsilon_1 \ln \varepsilon_1$ or ε_1 has been lost. This can be done by using the bounds $|x_i| \leq 2x_2$ $i=1 \dots 6$ in the integrand and in processes (a), (b) and (d), for example, the bounds

$$\begin{aligned} \frac{1}{2}m_2^2 &\leq x_5 \leq x_2 - \frac{1}{2}m_1^2 \\ \frac{1}{2}(m_1^2 - x_1) &\leq x_4 \leq x_2 - x_3 - \frac{1}{2}x_1 \\ \frac{1}{2}(m_2^2 - x_1) &\leq x_3 \leq x_2 - x_5 - \frac{1}{2}x_1 \end{aligned} \quad (2.31)$$

on the region of integration in order to limit the discarded terms. The remaining calculations and extractions of the terms in $\varepsilon_1 \ln \varepsilon_1$ and ε_1 are easy. The results for the coherent processes (a).. (d) from spin zero nuclei, neglecting $\sqrt{x_1}/m_T$, can be written

$$\sigma = \frac{2Z^2\alpha^2G^2}{9\pi^3} \varepsilon_1 \int_0^\infty \frac{|F(x_1)|^2}{\sqrt{x_1}} dx_1 \left\{ \ln \frac{\varepsilon_1 \sqrt{x_1}}{\frac{1}{2}\lambda_1} + \frac{1}{3} \ln \frac{\varepsilon_1 \sqrt{x_1}}{\frac{1}{2}\lambda_2} - \Lambda \right\} \quad (2.32)$$

T A B L E 1

PROCESS	λ_1	λ_2	Λ
(a) $\mu^- e^+$	x_1	m_μ^2	$\frac{19}{6} + \frac{1}{30} \left(\frac{x_1}{m_\mu^2} \right)$
(b) $\mu^- \mu^+$	m_μ^2	m_μ^2	$\frac{16}{3} + \frac{2}{15} \left(\frac{x_1}{m_\mu^2} \right)$
(c) $e^- \mu^+$	m_μ^2	x_1	$\frac{83}{18} + \frac{1}{10} \left(\frac{x_1}{m_\mu^2} \right)$
(d) $e^- e^+$	x_1	x_1	$\frac{22}{9}$

Quantities appearing in the asymptotic cross-sections for processes (a)..(d) as given by Eq. (2.32).

where λ_1 , λ_2 , and Λ are given in Table 1. In computing Λ we have neglected m_e^2/x_1 and expanded to first order in x_1/m_μ^2 .

Inserting the form factor (1.13)

$$F_I(x_1) = \frac{1}{\left(1 + \frac{x_1}{\beta^2}\right)^2} \quad \beta = \frac{\sqrt{20}}{R_0} \quad (1.14)$$

$$(R_0 = 1.2 A^{1/3} f)$$

$$\sigma_{I, \varepsilon_1 \rightarrow \infty} = \frac{4Z^2 \alpha^2 G^2}{9\pi^3} (\varepsilon_1 \beta) \frac{5\pi}{32} \left\{ \ln \frac{\varepsilon_1 \beta}{\frac{1}{2} p_1} + \frac{1}{3} \ln \frac{\varepsilon_1 \beta}{\frac{1}{2} p_2} - R_I \right\} \quad (2.33)$$

and inserting the form factor (1.14)

$$F_{II}(x_1) = e^{-\frac{1}{2} \frac{x_1}{\beta^2}} \quad \beta = \frac{\sqrt{5}}{R_0} \quad (1.15)$$

$$\sigma_{II, \varepsilon_1 \rightarrow \infty} = \frac{4Z^2 \alpha^2 G^2}{9\pi^3} (\varepsilon_1 \beta) \frac{\sqrt{\pi}}{2} \left\{ \ln \frac{\varepsilon_1 \beta}{\frac{1}{2} p_1} + \frac{1}{3} \ln \frac{\varepsilon_1 \beta}{\frac{1}{2} p_2} - R_{II} \right\} \quad (2.34)$$

We give p_1 , p_2 , R_I and R_{II} in Table 2. The leading terms in (2.32)-(2.34) were given in Reference ¹²).

We see that in all cases the second term in the asymptotic expansion, $\sigma \sim \varepsilon_1$, tends to cancel the leading term, $\sigma \sim \varepsilon_1 \ln \varepsilon_1$ as was suggested in Reference ¹²). This cancellation is most severe for the case (b) ($\mu^- \mu^+$) and least severe for the case (c) ($e^- e^+$) as one might expect. In the truly asymptotic region $\ln \varepsilon_1 \gg 1$, all the cross-sections for (a)..^(d) become equal

T A B L E 2

PROCESS	ρ_1	ρ_2	R_I	R_{II}
(a) $\mu^- e^+$	β^2	m_μ^2	$\frac{193}{90} + \frac{1}{150} \frac{\beta^2}{m_\mu^2}$	$-\frac{1}{3}(C+2\ell n 2) + \frac{19}{6} + \frac{1}{60} \frac{\beta^2}{m_\mu^2}$
(b) $\mu^- \mu^+$	m_μ^2	m_μ^2	$\frac{332}{45} + \frac{2}{75} \frac{\beta^2}{m_\mu^2}$	$+\frac{2}{3}(C+2\ell n 2) + \frac{16}{3} + \frac{1}{15} \frac{\beta^2}{m_\mu^2}$
(c) $e^- \mu^+$	m_μ^2	β^2	$\frac{507}{90} + \frac{1}{50} \frac{\beta^2}{m_\mu^2}$	$\frac{1}{3}(C+2\ell n 2) + \frac{83}{18} + \frac{1}{20} \frac{\beta^2}{m_\mu^2}$
(d) $e^- e^+$	β^2	β^2	$\frac{2}{5}$	$-\frac{2}{3}(C+2\ell n 2) + \frac{22}{9}$

Quantities appearing in the asymptotic cross-sections for processes (a)..(d) as given by Eqs. (2.33) and (2.34) which are obtained from the form factors (1.13) and (1.14). Note for the first form factor $\beta = \beta_I = \frac{\sqrt{20}}{R_0}$ and for the second $\beta = \beta_{II} = \frac{\sqrt{5}}{R_0}$. $C = 0.577$ (Euler's constant).

$$\sigma_I = \frac{5Z^2\alpha^2G^2}{54\pi^2}(\varepsilon, \beta_I) \ln\left(\frac{\varepsilon_1}{m_\mu}\right)$$

$$\sigma_{II} = \frac{8}{5\sqrt{\pi}}\sigma_I = .90\sigma_I \quad (2.35)$$

In practice, as one can see from the above, this equality is not reached until extremely high energies, much higher than energies where one might have expected lepton mass differences to have stopped playing any role. We also see that the two different nuclear form factors give rise to only a 10% difference (the exponential giving the smaller result) in the truly asymptotic region.

We have also calculated the leading term in the asymptotic cross-section for process (b) taking place incoherently off a nucleus by just adding the contributions of the protons and neutrons individually. One starts from (2.28)-(2.30) using (2.9). An equation very similar to (2.32) is obtained, which gives, when the nucleon form factors (1.17) are inserted

$$\sigma_{\text{INCOH}}(\bar{\mu}\mu^+)_{\varepsilon_1 \rightarrow \infty} = \frac{4\alpha^2G^2}{9\pi^3}(\varepsilon, M_p) \left[\ln\left(\frac{\varepsilon_1 M_p}{\frac{1}{2}m_p^2}\right) \right]. \quad (2.36)$$

$$\cdot \left\{ Z(.486 + .015\lambda_p^2 + .009(1+\lambda_p)^2) + N(.024\lambda_n^2) \right\}$$

where $\lambda_p = +1.78$ and $\lambda_n = -1.91$ are the anomalous moments of the proton and neutron. Because of the large cancellation for the $(\bar{\mu}\mu^+)$ process coming from the second term in the asymptotic expansion, as discussed above, this formula is not useful until quite high energies.

3. NUMERICAL CALCULATIONS

In order to have the exact cross-sections, we have carried out a numerical integration of equation (2.21), using (2.22)-(2.27), on the CERN I.B.M. 7090 computer. The method used was a modification of the Monte-Carlo method. The volume of integration, V , is transformed into the unit hypercube, H , by means of the transformation

$$X_i = \underline{X}_i + (\overline{X}_i - \underline{X}_i) y_i, \quad i = 1, \dots, 5 \quad (3.1)$$

where $\overline{x}_i$ and $\underline{x}_i$ are the upper and lower limits respectively, of the range of integration with respect to x_i . This unit hypercube is then subdivided into smaller hypercubes H_n by the subdivision of the range of integration (0-1) in each variable y_i , and two values of the integrand are calculated at random in each of the hypercubes H_n . From these, an estimate is made of the integral over H_n and of the variance of this estimate, and then, by summation, an estimate is obtained of the integral over the hypercube H and of the variance of this estimate. The mean of this and any previous estimates is taken as the over-all estimate of the value of the integral, and if the estimated accuracy is not less than 5% of its value, the process is repeated with the subdivisions of the ranges of integration of the variables amended in an attempt to minimize the total variance. The accuracy of the estimate is taken to be $\neq 2.58\sigma$ where σ^2 is the variance, since the probability that a random normal deviate lies in the range $\mu \neq 2.58\sigma$ is 99% if μ is the mean and σ^2 the variance of the distribution.

For the present problem it was also found helpful to make the transformation $Z_1 = \frac{1}{x_1}$ before the transformation (3.1) in order to reduce the variation of the integrand. However, for increasing values of $\varepsilon_{1/m\mu}$ this variation became worse until it became impossible to achieve an accuracy of 5% in reasonable computing time.

The results have been compared with the asymptotic calculations of the cross-section carried out in the previous Section and have shown good agreement. In Fig. 2 we give the comparison between the calculated values

and the asymptotic form in some typical cases [we have plotted the processes (a), (b) and (d) for $A = 56$ and $F_I(q^2)$]. The solid lines represent computed values and the dotted lines are the asymptotic values computed from Table 2. Because of the magnitude of the correction term R_I in Table 2 for the process (b) $\mu^- \mu^+$, the asymptotic form does not become applicable until $\epsilon_1/m_\mu \approx 10^2$. After this, however, it immediately becomes very good. For the other processes, the asymptotic form becomes good at much lower energies. Both processes (a) and (b) lie below the asymptotic form for process (a) at energies $\epsilon_1/m_\mu \approx 15$, as was suggested in Reference ¹²).

The computation of the integrand has also been checked by a hand calculation for a random set of values of the variables and been shown to be correct. We consider these checks to be sufficient to establish the validity of the calculations.

In Figs. 3 - 6, we give the computed cross-sections for the processes (a).. (d) . We have done the calculations for $A = 12, 56$ and 208 and one can easily interpolate between these values from the graphs. We have also carried out the calculations for both form factors F_I and F_{II} and except near threshold [especially for process (b) in heavy nuclei] the difference is always small. And as we pointed out in the introduction, any more realistic nuclear form factor should give results lying between these two. By the time one has reached the right-hand side of all the graphs, the asymptotic forms given in Table 2 are indistinguishable from the calculated values, so the analytical expressions of Table 2 can be used to go to any higher energy desired.

In Table 3 we give the calculated values of the cross-section for processes (a) and (b).

We have also carried out similar calculations for the incoherent cross-sections from individual neutrons and protons using (2.9) and (1.18). The results are shown in Fig. 7. Because the variations in the integrand prevented us from getting reliable cross-sections for processes (a) and (b)

T A B L E 3

$\frac{\epsilon_1/m_\mu}{A}$		1.1	1.5	2	3	4	6	8	12	15	20	35	50	100	500	1000	2000	5000	10,000
208	F_I	0.46×10^{-9}	0.22×10^{-5}	0.43×10^{-4}	0.53×10^{-3}	0.19×10^{-2}	0.82×10^{-2}	0.19×10^{-1}	0.54×10^{-1}	0.93×10^{-1}	0.17	0.50	0.94	2.8	25	60			
	F_{II}	0.22×10^{-9}	0.12×10^{-5}	0.26×10^{-4}	0.36×10^{-3}	0.14×10^{-2}	0.63×10^{-2}	0.15×10^{-1}	0.46×10^{-1}	0.79×10^{-1}	0.15	0.44	0.83	2.5	23	51			
56	F_I	0.14×10^{-8}	0.67×10^{-5}	0.12×10^{-3}	0.13×10^{-2}	0.45×10^{-2}	0.17×10^{-1}	0.38×10^{-1}	0.10	0.16	0.30	0.82	1.5	4.4	38	91	207	609	1320
	F_{II}	0.11×10^{-8}	0.55×10^{-5}	0.94×10^{-4}	0.10×10^{-2}	0.35×10^{-2}	0.14×10^{-1}	0.32×10^{-1}	0.87×10^{-1}	0.14	0.27	0.76	1.4	4.0	35	82			
12	F_I	0.21×10^{-8}	0.14×10^{-4}	0.25×10^{-3}	0.27×10^{-2}	0.84×10^{-2}	0.31×10^{-1}	0.69×10^{-1}	0.18	0.29	0.51	1.4	2.5	7.2	62	130			
	F_{II}	0.20×10^{-8}	0.13×10^{-4}	0.23×10^{-3}	0.24×10^{-2}	0.77×10^{-2}	0.28×10^{-1}	0.61×10^{-1}	0.17	0.26	0.46	1.2	2.3	6.7	57	130			

(a) $\nu_\mu + Z \rightarrow \nu_e + e^+ + \mu^- + Z$

$\frac{\epsilon_1/m_\mu}{A}$		3	4	6	8	12	15	20	35	50	100	500	1000	2000	5000	10,000
208	F_I	0.83×10^{-6}	0.13×10^{-4}	0.16×10^{-3}	0.67×10^{-3}	0.34×10^{-2}	0.68×10^{-2}	0.18×10^{-1}	0.73×10^{-1}	0.17	0.68	9.6	26	65	220	490
	F_{II}	0.73×10^{-7}	0.25×10^{-5}	0.57×10^{-4}	0.29×10^{-3}	0.18×10^{-2}	0.43×10^{-2}	0.11×10^{-1}	0.55×10^{-1}	0.13	0.55	8.1	22	57		
56	F_I	0.59×10^{-5}	0.73×10^{-4}	0.75×10^{-3}	0.26×10^{-2}	0.11×10^{-1}	0.21×10^{-1}	0.47×10^{-1}	0.18	0.40	1.4	18	45	110	340	780
	F_{II}	0.19×10^{-5}	0.32×10^{-4}	0.40×10^{-3}	0.16×10^{-2}	0.72×10^{-2}	0.15×10^{-1}	0.35×10^{-1}	0.14	0.31	1.1	15	40			
12	F_I	0.27×10^{-4}	0.31×10^{-3}	0.27×10^{-2}	0.84×10^{-2}	0.32×10^{-1}	0.59×10^{-1}	0.13	0.44	0.89	3.0	34	86			
	F_{II}	0.19×10^{-4}	0.22×10^{-3}	0.20×10^{-2}	0.64×10^{-2}	0.25×10^{-1}	0.40×10^{-1}	0.10	0.37	0.74	2.6	30	76	150		

(b) $\nu_\mu + Z \rightarrow \nu_\mu + \mu^+ + \mu^- + Z$

Numerically computed values of σ/σ_0 where $\sigma_0 \equiv Z^2 \alpha^2 G_\mu^2 = 3.03 \times 10^{-44} \text{ cm}^2 Z^2$ for processes (a) and (b) using both form factors F_I and F_{II} and for $A = 12, 56$ and 208 . The cross-sections are given as functions of the incident neutrino energy.

much above $\epsilon_1/m_\mu = 10^3$, we were not able to get to a region where the asymptotic form (2.36) was applicable. It is roughly a factor of 10 too large in the region of $\epsilon_1/m_\mu = 15$. This is consistent with the fact that the next term in the asymptotic expansion decreases the cross-section as we have shown.

We have also carried out a calculation of process (b) to compare with the results obtained by Shabalin¹⁰⁾ using the Weizsäcker-Williams approximation and both a threshold and asymptotic expansion of the necessary cross-section for photoproduction of lepton pairs from neutrinos. For this comparison, we have used the same form factor used by Shabalin (which, however, does not fit low momentum transfer electron scattering data). The results are given in Table 4.

We see that the exact calculation is just a factor of two larger than Shabalin's results for the low value of ϵ_1 . For $\epsilon_1 = 20$ GeV the disagreement is greater^{*)}. The cancellation present in Shabalin's calculation and discussed previously is seen, however, to hold up in the exact calculation and the present results for process (b), though somewhat larger, are never more than one order of magnitude larger than those of References⁷⁾ and ¹⁰⁾.

*) Since the details of the calculations are not given in Refs. 7), 10) we are unable to trace the origin of the factor of two. We believe, however, that it may come from an average rather than a sum over incident neutrino spins as discussed in the text. Our equation (2.32) for the asymptotic cross-section gives a result which is a factor of $8/3$ larger than the result of Reference⁷⁾. Part of this is the factor of two mentioned above. Besides, the Weizsäcker-Williams approximation used in Reference⁷⁾ differs from our asymptotic forms by another factor of $1/2$ and the absence of the second term in $A(x_1, x_2)$ in Eq. (2.25) which also contributes to the asymptotic cross-section.

TABLE 4

ϵ_1 (GeV)	A = 208 SHABALIN ¹⁰⁾	A = 208 PRESENT CALC.	A = 56 SHABALIN ¹⁰⁾	A = 208 PRESENT CALC.
0.3	3.4×10^{-47}	7.69×10^{-47}	1.6×10^{-47}	3.69×10^{-47}
0.4	6.6×10^{-46}	1.30×10^{-45}	2.8×10^{-46}	5.89×10^{-46}
0.5	3×10^{-45}	5.75×10^{-45}	1.3×10^{-45}	2.50×10^{-45}
1.0	7.4×10^{-44}	1.59×10^{-43}	2.6×10^{-44}	5.69×10^{-44}
2.0	7×10^{-43}	1.71×10^{-42}	2×10^{-43}	5.16×10^{-43}
20.0	9×10^{-41}	2.79×10^{-40}	1.8×10^{-41}	5.85×10^{-41}

A comparison of the present numerical calculation with the approximate calculation of Shabalin ¹⁰⁾ for process (b), $\nu_\mu + Z \rightarrow \nu_\mu + \mu^+ + \mu^- + Z$. The form factor used was the same as in Ref. ¹⁰⁾. The cross-sections are in cm^2 and are computed for A = 208 and 56.

4. CONCLUSIONS

We now have exact calculations from threshold to infinite incident energy of the total cross-sections for the processes

$$\begin{aligned}
 (a) \quad \nu_{\mu} + Z &\rightarrow \nu_e + e^+ + \mu^- + Z \\
 (b) \quad \nu_{\mu} + Z &\rightarrow \nu_{\mu} + \mu^- + \mu^+ + Z \\
 (c) \quad \nu_e + Z &\rightarrow \nu_{\mu} + \mu^+ + e^- + Z \\
 (d) \quad \nu_e + Z &\rightarrow \nu_e + e^+ + e^- + Z
 \end{aligned} \tag{4.1}$$

based on the mechanisms illustrated in Fig. 1. We have calculated with two nuclear form factors

$$\begin{aligned}
 F_I(q^2) &= \frac{1}{\left(1 + \frac{q^2 R_0^2}{20}\right)^2} \\
 F_{II}(q^2) &= e^{-\frac{1}{10} q^2 R_0^2}
 \end{aligned} \quad R_0 = 1.2 A^{1/3} \text{ f} \tag{4.2}$$

which fit low momentum transfer electron scattering data and should represent the two extremes as far as high momentum transfer behaviour is concerned. Figures 3 - 6 show that over almost all the range of interest, the difference is a minor one. If one finds experimental results in contradiction to these calculations it means that some profound changes in the point four-fermion interaction theory of weak interactions have to be introduced. One possibility is the currently discussed theory of the vector boson. Another is the possibility that higher order weak interactions may begin to play an important role *).

*) There is no simple unitarity limit, as in elastic scattering through a four point interaction which takes place in a single angular momentum state, which we can think of which says that higher order effects must become important at ultra-high energies for the processes considered here. If one says roughly that $\sigma_{\text{tot}} \leq \pi R^2$ one finds, with (2.35)

$$\left(\frac{\Sigma I}{m_{\mu}} \ln \frac{\Sigma I}{m_{\mu}} \right)_{\text{MAX}} \approx \frac{25A}{(Z\alpha)^2} \frac{1}{G^2 m_{\mu}^4} \approx \frac{25A}{(Z\alpha)^2} \times 10^{14}$$

which is an enormous energy.

TABLE 5

		COHERENT (a) $\mu^- e^+$	COHERENT (b) $\mu^- \mu^+$	INCOHERENT (a) $\mu^- e^+$	INCOHERENT (b) $\mu^- \mu^+$	$N\sigma_{el}$
$^{12}_6\text{C}$	F _I	3.2×10^{-43}	6.4×10^{-44}	1.0×10^{-43}	3.7×10^{-44}	3.8×10^{-38}
	F _{II}	2.8×10^{-43}	5.4×10^{-44}			
$^{56}_{26}\text{Fe}$	F _I	3.3×10^{-42}	4.3×10^{-43}	4.4×10^{-43}	1.7×10^{-43}	1.9×10^{-37}
	F _{II}	2.9×10^{-42}	3.1×10^{-43}			
$^{208}_{82}\text{Pb}$	F _I	1.9×10^{-41}	1.4×10^{-42}	1.4×10^{-42}	5.5×10^{-43}	7.9×10^{-37}
	F _{II}	1.6×10^{-41}	0.9×10^{-42}			

Numerical values of the cross-sections in cm^2 for processes (a) and (b) using an incident neutrino energy $\epsilon_1/m_\mu = 15^{12}$. The last column is $N\sigma_{el}$ where σ_{el} is the cross-section (4.3) for the process (4.4) and N is the number of neutrons in the nucleus.

If one takes

$$\frac{\epsilon_1}{m_\mu} \approx 15$$

which represents the average energy of the CERN neutrino beam ¹²⁾, we have not carried out a detailed average of our cross-sections over the computed neutrino spectrum at CERN because this is continually being improved⁷ then at this energy we have the results given in Table 5. We have used

$$\sigma_{EL} = .63 \times 10^{-38} \text{ CM}^2/\text{NEUTRON} \quad (4.3)$$

which represents the calculated cross-section for the process

$$\nu_\mu + n \longrightarrow \mu^- + p \quad (4.4)$$

averaged over the CERN neutrino spectrum and corrected for the Pauli principle ¹⁶⁾. We conclude that if one sees lepton pairs produced at rates of about 1% of the elastic events with a 1.6 GeV μ -neutrino beam it would not be possible to attribute them to this process. The cross-sections computed here go essentially as $\epsilon_1 \ln \epsilon_1$ while the elastic cross-section flattens off with incident energy above 1 GeV ³⁾, so an increase in the high energy neutrino tail would enhance the effect of the processes computed here. Thus, taking $\epsilon_1/m_\mu = 100$ ($\epsilon_1 = 10.5$ GeV)

$$\begin{aligned} \sigma_I(\mu^- e^+) &= 1.2 \times 10^{-40} \text{ CM}^2 && \text{for } {}^{56}_{26}\text{Fe} \\ &= 5.7 \times 10^{-40} \text{ CM}^2 && \text{for } {}^{208}_{28}\text{Pb} \end{aligned} \quad (4.5)$$

We also see that the incoherent cross-section is less than the coherent cross-section and relatively more important for process (b) than for process (a).

Finally, we note that the computed asymptotic cross-sections given in Table 2 which contain the first two terms in the asymptotic expansion become very good as soon as the final term ($\sim \varepsilon_1 \ln \varepsilon_1$) becomes bigger than the negative second term ($\sim \varepsilon_1$). For the process (b), however, the second term is so large that this does not happen until $\varepsilon_1 \approx 100 m_\mu$. In the far asymptotic region, the cross-sections for processes (a)..(d) all become equal.

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A P P E N D I X A

In this Appendix we show that the cross-sections obtained by replacing particles by antiparticles in processes (a).. $\cdot$ (d) are the same. If we replace all the leptons by antileptons in Fig. 1, the integrand in Eq. (2.5) becomes

$$\begin{aligned} \overline{M}_{\mu\nu} &\propto \text{Tr} \left[\gamma_\lambda (1 + \gamma_5) \not{p}_2 \gamma_\sigma (1 + \gamma_5) \not{p}_1 \right] \cdot \\ &\cdot \text{Tr} \left(\gamma_\mu \frac{1}{i(\not{p}_3 - \not{q}) + m_2} \gamma_\lambda (1 + \gamma_5) + \gamma_\lambda (1 + \gamma_5) \frac{1}{i(\not{q} - \not{p}_4) + m_1} \gamma_\mu \right) (-m_1 - i\not{p}_4) \cdot (\text{A.1}) \\ &\cdot \left(\gamma_\nu \frac{1}{i(\not{q} - \not{p}_4) + m_1} \gamma_\sigma (1 + \gamma_5) + \gamma_\sigma (1 + \gamma_5) \frac{1}{i(\not{p}_3 - \not{q}) + m_2} \gamma_\nu \right) (m_2 - i\not{p}_3). \end{aligned}$$

Since $\overline{M}_{\mu\nu}$ is a second rank tensor, it either contains $\delta_{\mu\nu}$ or is bilinear in any two of the four vectors. It is thus unchanged under

$$\begin{aligned} p_i &\rightarrow -p_i & i = 1, \dots, 4 \\ q &\rightarrow -q \end{aligned} \quad (\text{A.2})$$

This transformation, and a change of dummy indices $\mu \rightleftharpoons \nu$ and $\lambda \rightleftharpoons \sigma$ carries it directly into the integrand in (2.5). Thus the cross-section is the same as that given for the leptons. J.S. Bell ¹⁷⁾ has pointed out that this is a simple consequence of CP invariance, since if we look at

$$\nu + Z \longrightarrow \text{leptons} + Z$$

and make a CP transformation, we get

$$\overline{\nu} + \overline{Z} \longrightarrow \overline{\text{leptons}} + \overline{Z}$$

with the helicities and momenta correspondingly reversed. Since we are computing a total cross-section, this change of helicities and momenta does not affect anything and since the cross-section goes as $(Ze)^2$ we can change the nuclear particle $\bar{Z}$ back to Z without changing the result.

APPENDIX B

In this Appendix we discuss the reduction of the phase space integrals in (2.3) and (2.5) to integrals over the invariants $x_1 \dots x_6$ given in (2.17). Thus we have a phase space integral of the form

$$\int \frac{d\vec{P}'}{2E'} \left[\int \frac{d\vec{P}_3}{2E_3} \int \frac{d\vec{P}_4}{2E_4} \int \frac{d\vec{P}_2}{2E_2} \delta^{(4)}(q + \vec{P}_1 - \vec{P}_2 - \vec{P}_3 - \vec{P}_4) I(x_1 \dots x_6) \right] \quad (\text{B.1})$$

$$q \equiv P - P'$$

$$P_1^2 = P_2^2 = 0 \quad ; \quad P_3^2 = -m_2^2 \quad P_4^2 = -m_1^2$$

We first go to the frame determined by $\vec{q} + \vec{p}_1 - \vec{p}_3 = 0$ (this frame is just the c.m. system of particles ν_2 and l_1). In this frame we take $\hat{q}$ to be the $\hat{z}$ axis and let $\vec{p}_1$ and $\vec{p}_3$ lie in the $\hat{x}\hat{z}$ plane. Then

$$\int d\vec{P}_2 \delta^{(3)}(\vec{q} + \vec{p}_1 - \vec{P}_2 - \vec{p}_3 - \vec{p}_4) = 1 \quad (\text{B.2})$$

and we are left with

$$\int \frac{p_4^2 dp_4 d\cos\Theta_4 d\phi_4}{4E_4E_2} \delta(q_0 + E_1 - E_2 - E_3 - E_4) \quad (\text{B.3})$$

where $E_2 = |\vec{q} + \vec{p}_1 - \vec{p}_3 - \vec{p}_4| = |\vec{p}_4|$, that is the neutrino (p_2) and lepton 1 (p_4) come off back-to-back. The remaining δ function can be used to do the dp_4 integration and one finds

$$\int \frac{|\vec{p}_4|}{4(q_0 + E_1 - E_3)} d\cos\Theta_4 d\phi_4 \quad (\text{B.4})$$

Now the only dependence on ϕ_4 is in the variable x_6 which only occurs in the numerators of the integrand, so one can explicitly carry out the integral on ϕ_4 and then re-express the result in terms of the invariants $x_1 \dots x_5$. For example

$$x_6 = -p_1 p_4 = -\varepsilon_1 [|\vec{p}_4| \sin \Theta_1 \sin \Theta_4 \cos \phi_4 + |\vec{p}_4| \cos \Theta_4 \cos \Theta_1 - \varepsilon_4] \quad (\text{B.5})$$

$$\int \frac{d\phi}{2\pi} x_6 = -\varepsilon_1 [|\vec{p}_4| \cos \Theta_4 \cos \Theta_1 - \varepsilon_4] \quad (\text{B.6})$$

Now use

$$\begin{aligned} x_2 = -q \cdot p_1 &= \varepsilon_1 [q_0 - |\vec{q}| \cos \Theta_1] \\ x_4 = -q \cdot p_4 &= q_0 \varepsilon_4 - |\vec{q}| |\vec{p}_4| \cos \Theta_4 \end{aligned} \quad (\text{B.7})$$

to get rid of the angles. Then to re-express the various energy and momenta in terms of invariants, one uses the fact that the four vector $q+p_1-p_3$ only has a fourth component in this frame. Thus

$$\begin{aligned} -(q+p_1-p_3)^2 &= (q_0 + \varepsilon_1 - \varepsilon_3)^2 = 2(x_2 - x_3 - x_5) + m_2^2 - x_1 \\ q \cdot (q+p_1-p_3) &= -q_0(q_0 + \varepsilon_1 - \varepsilon_3) = x_1 - x_2 + x_3 \end{aligned} \quad (\text{B.8})$$

$$p_1 \cdot (q+p_1-p_3) = -\varepsilon_1(q_0 + \varepsilon_1 - \varepsilon_3) = -x_2 + x_5$$

$$p_4 \cdot (q+p_1-p_3) = -\varepsilon_4(q_0 + \varepsilon_1 - \varepsilon_3) = x_3 + x_5 - x_2 - \frac{1}{2}(m_1^2 + m_2^2 - x_1)$$

Using these equations one obtains

$$\begin{aligned} \int \frac{d\phi}{2\pi} x_6 &= \frac{1}{(x_2 - x_3)^2 - 2x_1 x_5 + x_1 m_2^2} \left\{ x_1(x_2 - x_5)(x_2 - x_3 - x_5 + \frac{1}{2}[m_1^2 + m_2^2 - x_1]) + \right. \\ &+ x_2(x_2 - x_3 - x_1)(x_2 - x_3 - x_5 + \frac{1}{2}[m_1^2 + m_2^2 - x_1]) + x_4(x_2 - x_3 - x_1)(x_2 - x_5) \\ &\left. - x_2 x_4(2[x_2 - x_3 - x_5] + m_2^2 - x_1) \right\} \end{aligned} \quad (\text{B.9})$$

and a similar equation for $\int \frac{d\phi}{2\pi} x_6^2$ which is given in (2.27). The remaining integral is over $\cos\theta_4$ which now only appears in the variable x_4 thus

$$dx_4 = |\vec{q}| |\vec{p}_4| d(\cos\theta_4)$$

and using (B.8) again, one finds

$$\begin{aligned} & \int \frac{d\vec{p}_4}{2\varepsilon_4} \int \frac{d\vec{p}_2}{2\varepsilon_2} \delta^{(4)}(q + p_1 - p_2 - p_3 - p_4) I(x_1, \dots, x_6) = \\ & = \frac{\pi}{2} \frac{1}{[(x_2 - x_3)^2 - 2x_1x_5 + m_2^2x_1]^{1/2}} \int_{\underline{x}_4}^{\overline{x}_4} dx_4 \left[\int \frac{d\phi}{2\pi} I(x_1, \dots, x_6) \right] \end{aligned} \quad (\text{B.10})$$

The limits $\overline{x}_4$ and $\underline{x}_4$ are easily found since once we have satisfied energy-momentum conservation in this frame, the angle θ_4 can take all values between 0 and π . Thus, using (A.8)

$$\begin{aligned} \overline{x}_4 &= q_0 \varepsilon_4 \pm |\vec{q}| |\vec{p}_4| \\ &= \frac{(x_2 - x_3 - x_1)(2[x_2 - x_3 - x_5] + m_2^2 - x_1 + m_1^2) \pm [(x_2 - x_3)^2 - 2x_1x_5 + x_1m_2^2]^{1/2}(2[x_2 - x_3 - x_5] + m_2^2 - x_1 - m_1^2)}{2(2[x_2 - x_3 - x_5] + m_2^2 - x_1)} \end{aligned} \quad (\text{B.11})$$

Since our expression is again in invariant form, we can go to another convenient frame, $\vec{q} + \vec{p}_1 = 0$, to do the integral $(d\vec{p}_3)/(2\varepsilon_3)$. In this frame we take $\hat{q}$ as the $\hat{z}$ axis (this frame is just the c.m. system of particles ν_1, l_1 , and l_2). We have

$$\int \frac{d\vec{p}_3}{2\varepsilon_3} = \int \frac{p_3 d\varepsilon_3 d\cos\theta_3 d\phi_3}{2} \quad (\text{B.12})$$

There is no ϕ_3 dependence in any invariants so one can immediately carry this out. In this frame we have

$$\begin{aligned}
 X_3 &= -\vec{g} \cdot \vec{p}_3 = g_0 \varepsilon_3 - |\vec{g}| |\vec{p}_3| \cos \Theta_3 \\
 X_5 &= -\vec{p}_1 \cdot \vec{p}_3 = \varepsilon_1 \varepsilon_3 + \varepsilon_1 |\vec{p}_3| \cos \Theta_3
 \end{aligned}
 \tag{B.13}$$

One has

$$\begin{aligned}
 dX_3 dX_5 &= J \begin{vmatrix} X_3 & X_5 \\ \varepsilon_3 & \cos \Theta_3 \end{vmatrix} d\varepsilon_3 d\cos \Theta_3 \\
 &= \varepsilon_1 |\vec{p}_3| (g_0 + \varepsilon_1) d\varepsilon_3 d\cos \Theta_3
 \end{aligned}
 \tag{B.14}$$

and using the fact that in this frame

$$X_2 = -\vec{g} \cdot \vec{p}_1 = \varepsilon_1 (\varepsilon_1 + g_0) \tag{B.15}$$

one finds the very simple relation

$$\frac{d\vec{p}_3}{2\varepsilon_3} = \frac{\pi}{X_2} dX_5 dX_3 \tag{B.16}$$

Finding the limits is only slightly more complicated. Again there are no restrictions on $\cos \Theta_3$ since energy and momentum conservation have been satisfied. Thus $\cos \Theta_3 = \pm 1$ gives one set of boundary curves of the region of integration. In both cases, from (B.13) and the fact that in this frame

$$X_1 = g^2 = \varepsilon_1^2 - g_0^2 \tag{B.17}$$

one finds

$$\cos \Theta_3 = \pm 1 \implies X_5 (2X_2 X_3 + X_1 X_5) = M_2^2 X_2^2$$

The other boundary curves come from the maximum and minimum possible values of Σ_3 in this frame, which are

$$(\varepsilon_3)_{\min} = m_2 \implies X_3 + X_5 = \sqrt{2X_2 - X_1} m_2$$

$$(\varepsilon_3)_{\max} = \frac{(q_0 + \varepsilon_1)^2 - m_1^2 + m_2^2}{2(q_0 + \varepsilon_1)} \implies X_3 + X_5 = \frac{1}{2} [(2X_2 - X_1) + m_2^2 - m_1^2] \quad (\text{B.18})$$

Actually, the first of Eq. (B.18) does not further limit the integration region. Thus, from (B.17) and (B.16) one arrives at the limits

$$\begin{aligned} \bar{X}_3 &= \frac{1}{2} [(2X_2 - X_1) + m_2^2 - m_1^2] - X_5 \\ \underline{X}_3 &= \frac{m_2^2 X_2}{2X_5} - \frac{X_1 X_5}{2X_2} \end{aligned} \quad (\text{B.19})$$

$$\bar{X}_5 = \frac{X_2}{2(2X_2 - X_1)} \left\{ (2X_2 - X_1) + m_2^2 - m_1^2 \pm \left[((2X_2 - X_1)^2 + m_2^2 - m_1^2)^2 - 4m_2^2(2X_2 - X_1) \right]^{1/2} \right\} \quad (\text{B.20})$$

Finally, to do the final $\frac{d\vec{P}'}{2E'}$ one can go to the lab. system (choosing $\hat{p}_1$ as the $\hat{z}$ axis) and use the relation

$$\vec{P}' + \vec{q} = \vec{P} \quad (\text{B.21})$$

to get

$$\frac{d\vec{P}'}{2E'} = \frac{d\vec{q}}{2E'} = \frac{\bar{q}^2 d|\vec{q}| d\cos\Theta_q d\phi_q}{2E'} \quad (\text{B.22})$$

In this frame

$$\begin{aligned} q_0 &= m_T - [m_T^2 + \bar{q}^2]^{1/2} \\ X_1 &= \bar{q}^2 - q_0^2 \\ X_2 &= \varepsilon_1 [q_0 - |\vec{q}| \cos\Theta_q] \end{aligned}$$

Therefore we find (again there is no θ_q dependence in the invariants)

$$\frac{d\vec{p}'}{2E'} = \frac{\pi}{2m_T \varepsilon_1} dx_1 dx_2 \quad (\text{B.23})$$

The upper limit on x_2 for fixed x_1 (or $|\vec{q}|$) is easily found by letting $\theta_q = \pi$ ($\vec{p}'$ in forward direction)

$$\bar{x}_2 = \varepsilon_1(q_0 + |\vec{q}|) = \varepsilon_1 \left\{ \left[x_1 + \frac{x_1^2}{4m_T^2} \right]^{1/2} - \frac{x_1}{2m_T} \right\} \quad (\text{B.24})$$

θ_q cannot take all values because the nucleus cannot recoil in the backward direction. A lower bound can be obtained from the inequality

$$-(p_2 + p_3 + p_4)^2 = -(p_1 + q)^2 = 2x_2 x_1 \geq (m_1 + m_2)^2 \quad (\text{B.25})$$

$$x_2 \geq \frac{1}{2} [x_1 + (m_1 + m_2)^2]$$

and it is easy to see that you can always attain this limit. Therefore

$$\underline{x}_2 = \frac{1}{2} [x_1 + (m_1 + m_2)^2] \quad (\text{B.26})$$

The final limits on x_1 can be obtained in a straightforward fashion (it is done most easily in the frame $\vec{p} + \vec{p}_1 = 0$) and one finds

$$\bar{x}_1 = \frac{2\varepsilon_1^2}{1 + \frac{2\varepsilon_1}{m_T}} \left\{ 1 - \frac{(m_1 + m_2)^2}{2\varepsilon_1^2} \left(1 + \frac{\varepsilon_1}{m_T} \right) \pm \left[\left(1 - \frac{(m_1 + m_2)^2}{2\varepsilon_1^2} \left(1 + \frac{\varepsilon_1}{m_T} \right) \right)^2 - \frac{(m_1 + m_2)^4}{4\varepsilon_1^4} \left(1 + \frac{2\varepsilon_1}{m_T} \right) \right]^{1/2} \right\} \quad (\text{B.27})$$

Putting the results together

$$\begin{aligned} & \int \frac{d\vec{p}'}{2E'} \int \frac{d\vec{p}_3}{2\varepsilon_3} \int \frac{d\vec{p}_4}{2\varepsilon_4} \int \frac{d\vec{p}_2}{2\varepsilon_2} \delta^4(q + p_1 - p_2 - p_3 - p_4) I(x_1, \dots, x_6) = \\ & = \frac{\pi^3}{4m_T \varepsilon_1} \int_{\underline{x}_1}^{\bar{x}_1} dx_1 \int_{\underline{x}_2}^{\bar{x}_2} \frac{dx_2}{x_2} \int_{\underline{x}_5}^{\bar{x}_5} dx_5 \int_{\underline{x}_3}^{\bar{x}_3} dx_3 \int_{\underline{x}_4}^{\bar{x}_4} dx_4 \frac{1}{[(x_2 x_3)^2 - 2x_5 x_1 + x_1 m_2^2]^{1/2}} \left[\int \frac{d\phi}{2\pi} I(x_1, \dots, x_6) \right] \end{aligned} \quad (\text{B.28})$$

where the limits are given above.

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FIGURE CAPTIONS

Fig. 1 The relevant Feynman graphs. The Z-vertex represents the nucleus as a whole (coherent production) or the nucleons (incoherent production).

Fig. 2 The exact (to order $G^2\alpha^2$ - see Fig. 1) numerical and the asymptotic cross-sections for the processes (a), (b), (d) for the target nucleus $A = 56$ and coherent production. The dotted lines represent the computed asymptotic cross-sections (see Table 2) and the solid lines are those computed numerically.

Fig. 3 The exact numerical (coherent) cross-sections of the process $\nu_\mu + Z \rightarrow \nu_e + e^+ + \mu^- + Z$ for $A = 12, 56, 208$ and two form factors:

$$F(q^2) = \frac{1}{\left(1 + \frac{q^2 R_0^2}{120}\right)^2} \quad (\text{full line}),$$

$$F(q^2) = e^{-\frac{1}{10} q^2 R_0^2} \quad (\text{dotted line}),$$

where $R_0 = 1.2A^{1/3} \text{f.}$

Fig. 4 The same results as shown in Fig. 3 for the coherent process

$$\nu_\mu + Z \rightarrow \nu_\mu + \mu^+ + \mu^- + Z.$$

Fig. 5 The same results as shown in Fig. 3 for the coherent process

$$\nu_e + Z \rightarrow \nu_\mu + \mu^+ + e^- + Z.$$

Fig. 6 The same results as shown in Fig. 3 for the coherent process

$$\nu_e + Z \rightarrow \nu_e + e^+ + e^- + Z \quad (\text{note the different abscissa scale}).$$

Fig. 7 The incoherent production processes:

$$\nu_{\mu} + N \longrightarrow \nu_{\mu} + \mu^{-} + \mu^{+} + N \quad (\text{full line})$$

$$\nu_{\mu} + N \longrightarrow \nu_{e} + \mu^{-} + e^{+} + N \quad (\text{dotted line})$$

for protons and neutrons. The nucleon form factor taken is the form:

$$F(q^2) = \frac{1}{(1 + 1.21 \frac{q^2}{M_p^2})^2}$$

FIG. 1

$$P - P' = q = P_2 + P_3 + P_4 - P_1$$

FIG. 2

FIG. 5

$$\sigma_0 = G^2 \alpha^2 Z^2 m_\mu^2$$

$$= 3.03 \times 10^{-44} Z^2 \text{ cm}^2$$

FIG. 4

FIG. 5

$$\sigma_0 = G^2 \alpha^2 Z^2 m_\mu^2$$

$$= 3.03 \times 10^{-44} Z^2 \text{ cm}^2$$

FIG. 6

FIG. 7

$$\begin{aligned}\sigma_0 &= G^2 \alpha^2 m_\mu^2 Z \quad (\text{PROTONS}) \\ &= G^2 \alpha^2 m_\mu^2 N \quad (\text{NEUTRONS})\end{aligned}$$

$$G^2 \alpha^2 m_\mu^2 = 3.03 \times 10^{-44} \text{ cm}^2$$