GAUGE INVARIANCE AND STRUCTURE OF THE CORRELATION FUNCTIONS OF A BOSE LIQUID

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ABSTRACT

In this paper we develop criteria to construct correlation functions complying with the continuity equation and the longitudinal sum rule, in the case of a homogeneous Bose liquid with condensation in the zero momentum state. The approach can be compared with methods previously used in the theory of superconductivity as the main idea consists in exploiting systematically a generalized Ward identity which follows from the invariance of the theory under infinitesimal gauge transformations. However, the structure of this identity, analysed in terms of diagrams, turns out to be rather different from the structure of similar relationships already studied in the case of Fermi systems. A general conclusion following from the identity alone is that the correlation functions must be singular at $k = 0$. As a rule then the spectrum of density fluctuations will have no gap. We also conclude that the theorem of Hugenholtz and Pines on the chemical potential and the spectrum of single-particle excitations is strictly connected with the generalized Ward identity and can actually be deduced from it. This leads to the identification of the $k = 0$ singularity of the propagator with the $k = 0$ singularity of the correlation functions.

The usefulness of the method is illustrated by discussing specific approximations such as the Bogoliubov approximation and the non-linear self-consistent scheme of Girardeau and Arnowitt.

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INTRODUCTION

The properties of a system of many interacting bosons have been intensely studied in recent years ¹⁾ and the main motivation is that such a system should in principle provide an adequate model of superfluid helium. However, the problem is a very difficult one and research has been concentrated on the properties at low density where an expansion of the quantities of physical interest in terms of a small parameter is possible ²⁾. At the density of liquid He this parameter is not small and in order that the whole question of giving a microscopic foundation to the behaviour of superfluid helium does not become academic, one has to find means to enlarge the schemes of calculation proposed until now. In this connection it has been stressed by various authors ³⁾ that an essential requirement to be imposed on any approximation scheme is that it satisfies conservation laws such as the conservation of the number of particles which is locally expressed by the continuity equation for the matter field. A global expression of the same conservation law is afforded, for example, by the so-called longitudinal sum rule. The quantities that one would like to calculate consistently are the one-particle propagator and the correlation functions describing density and current fluctuations (see below), in terms of which all the relevant physical properties of the system can be expressed.

An approach to the problem of superfluids making progress in this direction has been recently proposed by Martin ⁴⁾ and it has been applied by Hohenberg ⁵⁾ to the study of particular approximations. It is essentially an iterative scheme to solve the hierarchy of the Green's function integro-differential equations. In this paper we shall present an alternative and in a certain sense complementary method of calculation which allows, in some cases, a more detailed and intuitive picture of the physics involved. The problem that will be discussed can be formulated as follows : given an approximation for the one-particle Green's function construct correlation functions that satisfy the continuity equation and

the longitudinal sum rule. As we shall see the solution of this problem indicates in turn how to modify the one-particle propagator from which we started, if the initial choice for that function was not a good one. Our approach is entirely diagrammatic and is based on a generalized Ward identity (GWI) that follows from the invariance of the theory under infinitesimal gauge transformations. We shall limit our study to a system of interacting bosons with condensation in the zero momentum state and we shall use Beliaev's rules for the construction of graphs ⁶⁾. The underlying physical picture consists in looking at the particles with momentum different from zero as a system in equilibrium with a reservoir given by the condensate.

The method described in this paper should be compared with a technique previously used by Nambu ⁷⁾ to find the collective modes in superconductors. It can be viewed as a way of calculating the residual interaction of the quasiparticles until the conservation of the current is restored. The effect of this residual interaction gives rise to what Feynman ³⁾ calls the back-flow.

In the first Section we shall introduce the correlation functions and show that for the purpose of analysing their structure it is convenient to split them up in two parts. In the subsequent Section we introduce the GWI and establish the connection with the correlation functions. This leads immediately to the conclusion that the density fluctuation spectrum must have no gap. The GWI is then analysed in terms of graphs and its irreducible parts are determined. The theorem of Hugenholtz and Pines ⁸⁾ follows as a by-product and it is concluded that the propagator and the correlation functions have the same singularity at $K = 0$ ⁹⁾.

The remaining two Sections of the paper are devoted to the discussion of Bogoliubov and of the Girardeau-Arnoult ¹⁰⁾ approximations. The latter can be regarded as an iteration of the first but it does not comply with the theorem of Hugenholtz and Pines as the single-particle excitations show a gap. This circumstance gives us the opportunity of further illustrating the strict connection

existing between this theorem and the GWI. In the case of the Bogoliubov approximation the GWI gives particularly simple prescriptions for the calculation of the residual interaction of the quasiparticles and we are led to propose new expressions for the correlation functions which satisfy current conservation and longitudinal sum rule.

All our conclusions refer to systems at zero temperature. Work is in progress to extend the approach to finite temperatures including the cases of inhomogeneous and multiply connected systems.

1. THE CORRELATION FUNCTIONS

It is well known that when we deal with a homogeneous system of many interacting bosons in which a finite fraction N^0 of the particles occupies the zero momentum state, the number N' of particles with $k \neq 0$ becomes a good quantum number in the limit of very large volume⁸⁾. This circumstance has various consequences that can be roughly expressed by saying that the physical conservation law of the total number of particles N splits up in two separate conservation laws for N^0 and N' . It is the purpose of this Section to give a precise formulation to this idea in so far as the correlation functions are concerned.

We make the preliminary observation that in the limit of very large volume one can write after using the canonical commutation relations

$$\langle [\eta'(\underline{x}), \underline{j}(\underline{y})] \rangle = i \langle \eta' \rangle \nabla_{\underline{y}} \delta(\underline{x} - \underline{y}) \quad (1)$$

$$\begin{aligned} n_0^{\frac{1}{2}} \langle [\psi(\underline{x}), \underline{j}(\underline{y})] \rangle &= n_0^{\frac{1}{2}} \langle [\psi^\dagger(\underline{x}), \underline{j}(\underline{y})] \rangle = \\ &= \frac{i}{2} n_0 \nabla_{\underline{y}} \delta(\underline{x} - \underline{y}) \end{aligned} \quad (2)$$

where $\psi(\underline{x})$ is the quantized boson field and $\eta'(\underline{x})$ can be expressed as

$$\eta'(\underline{x}) = \varphi^\dagger(\underline{x}) \varphi(\underline{x}) \quad (3)$$

with $\varphi(\underline{x}) = \psi(\underline{x}) - \langle \psi(\underline{x}) \rangle = \psi(\underline{x}) - n_0^{\frac{1}{2}}$. $\langle \rangle$ denotes the ground state expectation value. $\underline{j}(\underline{x})$ is the current operator. We have set $\hbar = 1$ and the mass of the particles $m = 1$.

Let us introduce now the density current correlation functions

$$i D_{\mu\nu}(x, y) = \langle T(J_\mu(x) J_\nu(y)) \rangle - \langle J_\mu(x) \rangle \langle J_\nu(y) \rangle \quad (4)$$

where $J_i(x) = j_i(x)$ for $i = 1, 2, 3$ and $J_4(x) = n(x)$. We are mainly interested in the components $D_{\mu 4}$ which clearly satisfy the continuity equation

$$\frac{\partial}{\partial x_4} D_{44}(x, y) + \sum_i \frac{\partial}{\partial x_i} D_{i4}(x, y) = 0 \quad (5)$$

D_{44} is the usual density correlation function

$$i D_{44}(x, y) = \langle T(n(x) n(y)) \rangle - \langle n(x) \rangle \langle n(y) \rangle$$

It is convenient to write this function as the sum of two contributions

$$D_{44}(x, y) = D_{44}^0(x, y) + D'_{44}(x, y) \quad (6)$$

where

$$\begin{aligned} i D_{44}^0(x, y) = n_0 \{ & \langle T(\varphi(x) \varphi^+(y)) \rangle + \langle T(\varphi(y) \varphi^+(x)) \rangle \quad (7) \\ & + \langle T(\varphi(x) \varphi(y)) \rangle + \langle T(\varphi^+(y) \varphi^+(x)) \rangle \} \\ & + n_0^{\frac{1}{2}} \{ \langle T(n'(x) \varphi(y)) \rangle + \langle T(n'(x) \varphi^+(y)) \rangle \} \end{aligned}$$

$$\begin{aligned} i D'_{44}(x, y) = & \langle T(n'(x) n'(y)) \rangle - \langle n' \rangle^2 \quad (8) \\ & + n_0^{\frac{1}{2}} \{ \langle T(\varphi(x) n'(y)) \rangle + \langle T(\varphi^+(x) n'(y)) \rangle \} \end{aligned}$$

A similar decomposition can be introduced for the D_{i4}

$$i D_{i4}^0(x, y) = \frac{i}{2} n_0 \{ \langle T(\nabla_i \varphi^+(x) \varphi(y)) \rangle - \langle T(\nabla_i \varphi(x) \varphi^+(y)) \rangle + \langle T(\nabla_i \varphi^+(x) \varphi^+(y)) \rangle - \langle T(\nabla_i \varphi(x) \varphi(y)) \rangle \}^{(7')} \\ + n_0^{\frac{1}{2}} \{ \langle T(j_i'(x) \varphi(y)) \rangle + \langle T(j_i'(x) \varphi^+(y)) \rangle \}$$

$$i D_{i4}'(x, y) = \langle T(j_i'(x) n'(y)) \rangle \quad (8') \\ + \frac{i}{2} n_0^{\frac{1}{2}} \{ \langle T(\nabla_i \varphi^+(x) n'(y)) \rangle - \langle T(\nabla_i \varphi(x) n'(y)) \rangle \}$$

Using now (1) and (2) it is easily proved that the imaginary parts of the Fourier transforms of D_{44}^0 and D_{44}' , that will be denoted $-S^0(\underline{k}, \omega)$ and $-S'(\underline{k}, \omega)$, satisfy the sum rules

$$\frac{1}{\pi} \int_0^\infty d\omega \omega S^0(\underline{k}, \omega) = n_0 \frac{k^2}{2} \quad (9)$$

$$\frac{1}{\pi} \int_0^\infty d\omega \omega S'(\underline{k}, \omega) = n' \frac{k^2}{2} \quad (10)$$

The next step consists in showing that $D_{\mu 4}^0$ and $D_{\mu 4}'$ obey separate local continuity equations

$$\frac{\partial}{\partial x_4} D_{44}^0(x, y) + \sum_i \frac{\partial}{\partial x_i} D_{i4}^0(x, y) = 0 \quad (11)$$

$$\frac{\partial}{\partial x_4} D_{44}'(x, y) + \sum_i \frac{\partial}{\partial x_i} D_{i4}'(x, y) = 0 \quad (12)$$

This result follows from the continuity equation for J_μ .

2. THE GENERALIZED WARD IDENTITY (GWI)

We use a dichotomic notation and define the one particle Green's function as a matrix

$$iG(x-y) = \langle T(\Phi(x) \Phi^\dagger(y)) \rangle \quad (13)$$

where

$$\Phi(x) \equiv \begin{pmatrix} \varphi(x) \\ \varphi^\dagger(x) \end{pmatrix} \quad (14)$$

By performing now infinitesimal gauge transformations and using a standard procedure we obtain in the limit of infinite volume

$$i\tau_3 G(x-y) \delta(x'-x) - iG(x-y) \tau_3 \delta(x'-y) \\ = \iint d^4\xi d^4\eta G(x-\xi) \Gamma(\xi-x', x'-\eta) G(\eta-y) \quad (15)^*$$

where Γ is a matrix of the form

$$\Gamma(\xi-x', x'-\eta) = \frac{\partial}{\partial x'_4} \Gamma_4(\xi-x', x'-\eta) + \sum_i \frac{\partial}{\partial x'_i} \Gamma_i(\xi-x', x'-\eta) \quad (16)$$

Γ_4 and Γ_i describe the coupling of the four current to an external vector field and are defined through the equations

*) The τ_i are the Pauli matrices in the usual representation

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

8.

$$\begin{aligned}
 & - \langle T \left(\bar{\Phi}(x) \begin{bmatrix} \eta(x') - \eta \\ j_i(x') \end{bmatrix} \Phi^+(y) \right) \rangle = \\
 & = \iint d^4\xi d^4\eta G(x-\xi) \begin{bmatrix} \Gamma_4(\xi-x', x'-\eta) \\ \Gamma_i(\xi-x', x'-\eta) \end{bmatrix} G(\eta-y)
 \end{aligned} \tag{17}$$

By going over to momentum space we find

$$\begin{aligned}
 i\tau_3 G(p) - iG(p')\tau_3 &= G(p') \Gamma(p', p) G(p) \\
 &= iG(p') \{ (p'_4 - p_4) \Gamma_4(p', p) - (p' - p) \Gamma(p', p) \} G(p)
 \end{aligned} \tag{18}$$

This equation can be inverted to give

$$iG^{-1}(p')\tau_3 - i\tau_3 G^{-1}(p) = \Gamma(p', p) \tag{18'}$$

The various Γ_{ik} can now be expressed in terms of self-energy parts if we remember that

$$G^{-1}(p) = \begin{pmatrix} p_4 - \epsilon_p + \mu - \Sigma_{11}(p) & -\Sigma_{12}(p) \\ -\Sigma_{12}(p) & -p_4 - \epsilon_p + \mu - \Sigma_{11}(-p) \end{pmatrix} \tag{19}$$

The Σ_{ik} are defined as in Ref. 5). μ is the chemical potential. The result is

*) We introduce the notation $G_0^{-1}(p) = p_4 - \epsilon_p + \mu$ with $\epsilon_p = \frac{1}{2}p^2$.

$$\begin{aligned}
\Gamma_{11}(p', p) &= i \{ G_0^{-1}(p') - G_0^{-1}(p) + \Sigma_{11}(p) - \Sigma_{11}(p') \} \\
\Gamma_{12}(p', p) &= i \{ \Sigma_{12}(p') + \Sigma_{12}(p) \} \\
\Gamma_{21}(p', p) &= -\Gamma_{12}(p', p) \\
\Gamma_{22}(p', p) &= \Gamma_{11}(-p, -p')
\end{aligned} \tag{20}$$

To make the connection with the correlation functions we take the matrix element 11 of Eq. (18)

$$\begin{aligned}
i G_{11}(p) - i G_{11}(p') &= \sum_{i, k} G_{1i}(p') \Gamma_{ik}(p', p) G_{k1}(p) \\
&= G_{11}(p') \{ \Gamma_{11}(p', p) + \Gamma_{12}(p', p) g(-p) \Sigma_{12}(p) + \\
&+ g(-p') \Sigma_{12}(p') \Gamma_{21}(p', p) + g(-p') \Sigma_{12}(p') \Gamma_{22}(p', p) \cdot \\
&\cdot g(-p) \Sigma_{12}(p) \} G_{11}(p) = G_{11}(p') \chi(p', p) G_{11}(p)
\end{aligned} \tag{21}$$

where we have defined

$$g^{-1}(p) = G_0^{-1}(p) - \Sigma_{11}(p)$$

and used the relationship ⁶⁾

$$G_{12}(p) = g(-p) \Sigma_{12}(p) G_{11}(p)$$

γ is again of the form

$$\gamma(p', p) = i \left\{ \omega \gamma_4(p', p) - \underline{k} \gamma(p', p) \right\} \quad \begin{aligned} \underline{k} &= \underline{p}' - \underline{p} \\ \omega &= p_4' - p_4 \end{aligned} \quad (22)$$

From the comparison of (8) with (17), (21) and (22) it follows

$$D'_{44}(\underline{k}, \omega) = \frac{i}{(2\pi)^4} \int d^4 p \, G_{11}(p + \frac{\underline{k}}{2}) \gamma_4(p + \frac{\underline{k}}{2}, p - \frac{\underline{k}}{2}) G_{11}(p - \frac{\underline{k}}{2}) \quad (23)^*)$$

Similarly one has

$$D'_{i4}(\underline{k}, \omega) = \frac{i}{(2\pi)^4} \int d^4 p \, G_{11}(p + \frac{\underline{k}}{2}) \gamma_i(p + \frac{\underline{k}}{2}, p - \frac{\underline{k}}{2}) G_{11}(p - \frac{\underline{k}}{2}) \quad (23')$$

The continuity equation (12) can be obtained also from (23), (23') and (21). (11) then follows from (12) and (5). It is instructive to check the validity of (11) and (12) in the long wavelength limit using the results contained in the articles of Gavoret and Gavoret and Nozières⁹⁾.

There is an important conclusion that emerges at this stage. From (20) we see that

$$\lim_{k \rightarrow 0} \Gamma(p', p) = \lim_{k \rightarrow 0} i k_\mu \Gamma_\mu(p', p) = -2 \chi_2 \sum_{12} (p) \quad (24)$$

*) The p_4 integration must be performed over a path closed in the upper half-plane.

This means that $\bar{\Gamma}_{\mu}(p', p)$ cannot be regular at $k = 0$. According to (23) and (23') then the low-lying excitations corresponding to density fluctuations will have no gap. This argument does not depend on whether the single-particle spectrum has a gap or not. However, it will be seen in the next Section that the GWI contains enough information to show that the one-particle propagator must be singular at $k = 0$ and that this is the singularity appearing in the correlation functions.

From the computational point of view the above discussion indicates that in order to comply with the continuity equation (12) and the sum rule (10), it is sufficient to make sure that the GWI holds. The conservation law for $D_{\mu 4}^0$ has, on the other hand, a more complicated structure. However, once the approximation for $D'_{\mu 4}$ has been selected also the contributions to be included in $D_{\mu 4}^0$ are uniquely determined.

3. DIAGRAMMATIC STRUCTURE OF THE GWI

We want now to describe the graphical structure of the identities (20). A little reflection shows that the diagrams contributing to anyone of the Γ_{ik} can be quite generally divided in two classes. In order to make our discussion definite we consider first $\Gamma_{11}(p', p)$. Let us introduce for the elementary vertex $G_0^{-1}(p') - G_0^{-1}(p) = p_4' - p_4 - \frac{1}{2}(p'^2 - p^2)$ the notation

$$\text{Diagram: a circle with an incoming arrow from below labeled } p \text{ and an outgoing arrow from top labeled } p' = i \{ G_0^{-1}(p') - G_0^{-1}(p) \} \quad (25)$$

This represents the modification of the free-particle propagator due to the gauge field. A first class of diagrams is then obtained by substituting successively in all possible ways the free propagators appearing as internal lines in the irreducible ¹¹⁾ self-energy part $\Sigma_{11}(p)$, with the vertex (25). If we add the result of this operation to (25) we obtain all the irreducible graphs contributing to Γ_{11} and we shall indicate them

$$\text{Diagram: a shaded circle with an incoming arrow from below labeled } p \text{ and an outgoing arrow from top labeled } p' = \tilde{\Gamma}_{11}(p', p) \quad (26)$$

The second class of contributions is given by the terms describing the scattering of a particle of the condensate with the gauge field and the subsequent interaction of this particle with a $k \neq 0$ line. They have, for example, the structure

$$\text{Diagram: a shaded circle with an incoming arrow from bottom-left labeled } p \text{ and an outgoing arrow from top labeled } p'. \text{ It is connected to another shaded circle below by a line labeled } k = p' - p. \quad (27)$$

where the hatched areas represent respectively an irreducible three particle kernel and the sum of all irreducible diagrams describing the scattering of a particle of the condensed phase with the gauge field. $\Gamma_{11}(p', p)$ has, therefore, the following decomposition in terms of irreducible contributions

$$\Gamma_{11}(p', p) = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \text{[Diagram 4]} + \text{[Diagram 5]} \quad (28)$$

Similarly we find for Γ_{12}

$$\Gamma_{12}(p', p) = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \text{[Diagram 4]} + \text{[Diagram 5]} \quad (29)$$

The above equations can be used in various ways according to the particular problem we are discussing. We notice that all information necessary to reconstruct the different contributions on the right-hand side are contained in the self-energies $\sum_{ik}$.

We are now in a position to prove that the one-particle propagator is singular at $k = 0$ in agreement with the theorem of Hugenholtz and Pines. If we define

$$\text{diagram of a shaded circle with an arrow labeled } k \text{ pointing out of it} = F(k, 0) \quad (30)$$

we have by construction

$$\begin{aligned} F(0, 0) &= 0 \\ F(k, 0) &= -F(0, k) \end{aligned} \quad (31)$$

Let us take now the limit of (29) for $k \rightarrow 0$. In this limit the left-hand side has the value $2i \sum_{12}(p)$ while the right-hand side can be written ^{*)}

$$\left[P_-^{++}(p, p) - P^{+++}(p, p) \right] \lim_{k \rightarrow 0} \left[G_{11}(k) F(k, 0) + G_{12}(k) F(0, -k) \right] \quad (32)$$

where P_-^{++} and P^{+++} ¹²⁾ are the three-particle kernels. A very simple calculation ¹³⁾ shows that

$$P_-^{++}(p, p) - P^{+++}(p, p) = \frac{2}{m_0^2} \sum_{12}(p) \quad (33)$$

This yields

$$\lim_{k \rightarrow 0} \frac{1}{im_0^2} \left[G_{11}(k) F(k, 0) + G_{12}(k) F(0, -k) \right] = 1 \quad (34)$$

^{*)} The function $G_{11}(k)F(k, 0) + G_{12}(k)F(0, -k)$ is assumed to be continuous at $k = 0$.

from which the singular behaviour of $G(k)$ at $k = 0$ follows. Of course, this is not a very direct way to derive the theorem of Hugenholtz and Pines. Here we are interested in pointing out how the connection between this theorem and the GWI can be established. This discussion completes the argument given in the previous Section.

Equation (34) is a particular case of the more general relationship

$$\frac{1}{i\mathcal{M}_0^{\frac{1}{2}}} [G_{11}(k) F(k, 0) + G_{12}(k) F(0, -k)] = 1 \quad (35)$$

There are several ways of deriving (35). It could be proved directly by exhibiting order by order the cancellation mechanism which operates between the diagrams contributing to $F(k, 0)$ and the diagrams contributing to the propagator. This method, however, becomes rather involved when applied to arbitrarily complicated graphs. A simpler even if more indirect way consists in taking the limit of (28) and (29) for p or $p' \rightarrow 0$. In this limit the right-hand side of (28) and (29) can be expressed in terms of $F(k, 0)$ and the $\sum_{ik}$ and one obtains an equation for $F(k, 0)$ which can be solved explicitly. This is done in the Appendix. If we take into account (35), (28) and (29) can be written in the form

$$\begin{aligned} i \left[G_0^{-1}(p') - G_0^{-1}(p) + \sum_{11}(p) - \sum_{11}(p') \right] &= \quad (28') \\ &= \tilde{\Gamma}_{11}(p', p) + i\mathcal{M}_0^{\frac{1}{2}} \left[P_-^{++}(p', p) - P_-^{++}(p', p) \right] \end{aligned}$$

$$\begin{aligned} i \left[\sum_{12}(p') + \sum_{12}(p) \right] &= \quad (29') \\ &= \tilde{\Gamma}_{12}(p', p) + i\mathcal{M}_0^{\frac{1}{2}} \left[P_-^{++}(p', p) - P_-^{++}(p', p) \right] \end{aligned}$$

where $\tilde{\Gamma}_{11}$ and $\tilde{\Gamma}_{12}$ represent the irreducible parts of Γ_{11} and Γ_{12} . (28') and (29') are equations involving irreducible parts only.

4.

THE BOGOLIUBOV APPROXIMATION

This approximation is obtained by taking

$$\begin{aligned}\Sigma_{11}(p) &= n_0 V(0) + n_0 V(p) \\ \Sigma_{12}(p) &= n_0 V(p) \\ \mu &= n_0 V(0)\end{aligned}\tag{36}$$

where $v(p)$ is the Fourier transform of the interaction potential. Sometimes in the following it will be convenient to leave the chemical potential μ undetermined.

We calculate first $D_{\mu 4}$. If we specialize Eq. (20) in the present approximation, we obtain for the relevant quantities

$$\begin{aligned}\Gamma_{11}(p', p) &= i \{ G_0^{-1}(p') - G_0^{-1}(p) + n_0 V(p) - n_0 V(p') \} \\ \Gamma_{12}(p', p) &= i n_0 \{ V(p') + V(p) \}\end{aligned}\tag{37}$$

Let us find the graphical structures underlying these equations. According to the analysis of the previous paragraph the following three-particle vertices have to be considered

where $\sim$ stands for the interaction potential. On the other hand, it is clear that because of the absence of self-energy closed loops in the approximation, there is only one irreducible diagram which can contribute to the Γ_{11}

and this is given by (25). We shall immediately verify that Eqs. (28) and (29) are satisfied by this choice. We have

$$\begin{aligned}
 -i\Gamma_{11}(p', p) = & G_0^{-1}(p') - G_0^{-1}(p) \\
 & + n_0 V(k) \{ G_{11}(k) [G_0^{-1}(k) - \mu] + G_{12}(k) [\mu - G_0^{-1}(-k)] \} \\
 & + n_0 V(k) \{ G_{11}(-k) [\mu - G_0^{-1}(-k)] + G_{12}(k) [G_0^{-1}(k) - \mu] \} \\
 & + n_0 V(p) \{ G_{11}(k) [G_0^{-1}(k) - \mu] + G_{12}(k) [\mu - G_0^{-1}(-k)] \} \\
 & + n_0 V(p') \{ G_{11}(-k) [\mu - G_0^{-1}(-k)] + G_{12}(k) [G_0^{-1}(k) - \mu] \}
 \end{aligned} \tag{39}$$

$G_{11}(k)$ and $G_{12}(k)$ are calculated in the Bogoliubov approximation. If we recall now that $G_{12}(k) = \sum_{12}(k) g(-k) G_{11}(k)$ we find in accordance with (35)

$$\begin{aligned}
 G_{11}(k) [G_0^{-1}(k) - \mu] + G_{12}(k) [\mu - G_0^{-1}(-k)] = \\
 = G_{11}(k) \left\{ \omega - \epsilon_k + \frac{n_0 V(k) (\omega + \epsilon_k)}{-\omega - \epsilon_k - n_0 V(k)} \right\} = 1
 \end{aligned} \tag{40}$$

where we have used for μ the value (36). (39) then reduces to

$$-i\Gamma_{11}(p', p) = G_0^{-1}(p') - G_0^{-1}(p) + n_0 V(p) - n_0 V(p')$$

as required by (37). An entirely similar verification can be carried out for

Γ_{12} .

At this stage it is a rather simple matter to construct χ_4 and $\underline{\chi}$ which we need to calculate $D_{\mu 4}$. The analysis we have made indicates explicitly the contributions to be considered. We find

$$\begin{aligned}
 \chi_4(p', p) = & 1 + n_0^2 V(p) V(p') g(-p) g(-p') \\
 & + n_0 \{ V(k) [G_{11}(k) + G_{11}(-k) + 2 G_{12}(k)] \\
 & + V(p) [G_{11}(k) + G_{12}(k)] + V(p') [G_{11}(-k) + G_{12}(k)] \} \\
 & + n_0^2 (V(p) + V(p')) [G_{11}(k) + G_{12}(k)] g(-p) V(p) \\
 & + n_0^2 (V(p) + V(p')) [G_{11}(-k) + G_{12}(k)] g(-p') V(p') \\
 & + n_0^3 V(p) V(p') g(-p) g(-p') \{ V(k) [G_{11}(k) + G_{11}(-k) + 2 G_{12}(k)] \\
 & + V(p) [G_{11}(-k) + G_{12}(k)] + V(p') [G_{11}(k) + G_{12}(k)] \}
 \end{aligned} \tag{41}$$

$$\begin{aligned}
 \underline{\chi}(p', p) = & \frac{1}{2} (\underline{p} + \underline{p}') \{ 1 - n_0^2 V(p) V(p') g(-p) g(-p') \} \\
 & + \frac{1}{2} \underline{k} n_0 \{ V(k) [G_{11}(k) - G_{11}(-k)] \\
 & + V(p) [G_{11}(k) - G_{12}(k)] + V(p') [G_{12}(k) - G_{11}(-k)] \} \\
 & + \frac{1}{2} \underline{k} n_0^2 (V(p) + V(p')) [G_{11}(k) - G_{12}(k)] g(-p) V(p) \\
 & + \frac{1}{2} \underline{k} n_0^2 (V(p) + V(p')) [G_{12}(k) - G_{11}(-k)] g(-p') V(p') \\
 & + \frac{1}{2} \underline{k} n_0^3 V(p) V(p') g(-p) g(-p') \{ V(k) [G_{11}(k) - G_{11}(-k)] \\
 & + V(p) [G_{12}(k) - G_{11}(-k)] + V(p') [G_{11}(k) - G_{12}(k)] \}
 \end{aligned} \tag{42}$$

(41) and (42) satisfy (21) as it is seen by direct computation.

Finally, we show that D'_{44} as obtained from (41) and (23) does satisfy the sum rule (10). For simplicity we consider a δ function interaction which corresponds to a constant potential v in momentum space. From the analyticity properties of $D'_{44}(\underline{k}, \omega)$ as a function of ω it follows that (10) is equivalent to

$$\lim_{\omega \rightarrow \infty} \omega^2 D'_{44}(\underline{k}, \omega) = n' K^2 \quad (43)$$

From (41) we obtain asymptotically

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \omega^2 D'_{44}(\underline{k}, \omega) = \lim_{\omega \rightarrow \infty} \left\{ \omega^2 [A(\underline{k}, \omega) + C(\underline{k}, \omega)] \right. \\ \left. + 2n_0 V \omega [B(\underline{k}, \omega) - B(-\underline{k}, -\omega)] \right\} \quad (44) \end{aligned}$$

where we have defined

$$\begin{aligned} A(\underline{k}, \omega) &= \frac{i}{(2\pi)^4} \int d^4 p \, G_{11}(p + \frac{\underline{k}}{2}) G_{11}(p - \frac{\underline{k}}{2}) \\ B(\underline{k}, \omega) &= \frac{i}{(2\pi)^4} \int d^4 p \, G_{11}(p + \frac{\underline{k}}{2}) G_{12}(p - \frac{\underline{k}}{2}) \quad (45) \\ C(\underline{k}, \omega) &= \frac{i}{(2\pi)^4} \int d^4 p \, G_{12}(p + \frac{\underline{k}}{2}) G_{12}(p - \frac{\underline{k}}{2}) \end{aligned}$$

The limit (44) is evaluated without difficulty and it is found that (43) holds with

$$\pi' = \frac{i}{(2\pi)^4} \int d^4 p G_{11}(p) \quad (46)$$

The calculation has been repeated for a p dependent potential and the answer is, of course, the same. In the latter case the verification is considerably more involved. It is interesting to note that in this approximation the sum rule (10) holds whatever the value of μ may be.

We now turn our attention to the D^0 part of the density correlation function. Here we have the peculiar situation that the part of (7) proportional to n_0 and containing only one-particle Green's functions satisfies by itself both (9) and (11) provided we take for μ the value (36). If we add to D_{44}^0 contributions containing three-particle kernels such as those included in D_{44}' we spoil the conservation law and the sum rule as well. We see then that we can construct a density correlation function complying with the continuity equation and the longitudinal sum rule, provided we discard arbitrarily a certain number of diagrams. This indicates that the Bogoliubov approximation is not self-consistent as it has been noticed by several authors.

5. THE GIRARDEAU-ARNOWITT APPROXIMATION

The approximation that will be described in the following can be considered as an iteration of the Bogoliubov approximation. It is a self-consistent non-linear scheme but contrary to the Bogoliubov approximation it does not satisfy the theorem of Hugenholtz and Pines. The defining equations can be written diagrammatically

$$\begin{array}{c} \uparrow P \\ \circlearrowleft \Sigma_{11} \\ \uparrow P \end{array} = \begin{array}{c} \uparrow P \\ | \\ \uparrow P \end{array} + \begin{array}{c} \uparrow P \\ | \\ \uparrow P \end{array} + \begin{array}{c} \uparrow P \\ | \\ \uparrow P \end{array} \text{ (loop) } + \begin{array}{c} \uparrow P \\ | \\ \uparrow P \end{array} \text{ (loop) } \quad (47)$$

$$\begin{array}{c} \uparrow P \quad \uparrow -P \\ \circlearrowleft \Sigma_{12} \end{array} = \begin{array}{c} \uparrow P \quad \uparrow -P \\ | \\ \uparrow P \quad \uparrow -P \end{array} + \begin{array}{c} \uparrow P \quad \uparrow -P \\ | \\ \uparrow P \quad \uparrow -P \end{array} \text{ (loop) }$$

$$n_0 \mu = \text{wavy line} + \text{wavy line (loop)} + \text{wavy line (loop)} + \text{wavy line (loop)}$$

or analytically

$$\Sigma_{11}(p) = n_0 (V(0) + V(p)) + \frac{i}{(2\pi)^4} \int d^4 q (V(0) + V(p-q)) G_{11}(q)$$

$$\Sigma_{12}(p) = n_0 V(p) + \frac{i}{(2\pi)^4} \int d^4 q V(p-q) G_{12}(q) \quad (47')$$

$$\begin{aligned} \mu = n_0 V(0) &+ \frac{i}{(2\pi)^4} \int d^4 q (V(0) + V(q)) G_{11}(q) + \\ &+ \frac{i}{(2\pi)^4} \int d^4 q V(q) G_{12}(q) \end{aligned}$$

The expression of the chemical potential is such that it gives a minimum for the ground-state energy. Clearly the single-particle spectrum has a gap as we have for example

$$G_{11}^{-1}(0) = \sum_{12}(0) - 2n_0 V(0) - \frac{\sum_{12}^2(0)}{\sum_{12}(0) - 2n_0 V(0)} \neq 0 \quad (48)$$

We shall now show that it is not possible to satisfy the GWI in this approximation. It is not difficult to apply the method of analysis of Section 3 and conclude that in this case the following coupled equations for Γ_{11} and Γ_{12} should be satisfied

$$\begin{aligned} \Gamma_{11}(p', p) = & i \left[G_0^{-1}(p') - G_0^{-1}(p) \right] + i n_0 V(k) \left\{ G_{11}(k) f(k, 0) + G_{12}(k) f(0, -k) \right\} \\ & + i n_0 V(k) \left\{ G_{11}(-k) f(0, -k) + G_{12}(k) f(k, 0) \right\} \\ & + i n_0 V(p) \left\{ G_{11}(k) f(k, 0) + G_{12}(k) f(0, -k) \right\} \\ & + i n_0 V(p') \left\{ G_{11}(-k) f(0, -k) + G_{12}(k) f(k, 0) \right\} \\ & + \sum_{i, k} \frac{i}{(2\pi)^4} \int d^4 q \left(V(k) + V(q - p - \frac{k}{2}) \right) G_{1i}(q + \frac{k}{2}) \Gamma_{ik}(q + \frac{k}{2}, q - \frac{k}{2}) G_{k1}(q - \frac{k}{2}) \end{aligned} \quad (49)$$

$$\begin{aligned} \Gamma_{12}(p', p) = & i n_0 \left(V(p) + V(p') \right) \left\{ G_{11}(k) f(k, 0) + G_{12}(k) f(0, -k) \right\} \\ & + \sum_{i, k} \frac{i}{(2\pi)^4} \int d^4 q V(q - p - \frac{k}{2}) G_{1i}(q + \frac{k}{2}) \Gamma_{ik}(q + \frac{k}{2}, q - \frac{k}{2}) G_{k2}(q - \frac{k}{2}) \end{aligned}$$

A straightforward substitution shows that these equations are solved by (20) when (47') are taken into account, provided (35) holds, i.e.,

$$G_{11}(k)f(k,0) + G_{12}(k)f(0,-k) = 1 \quad (50)$$

However, it is immediately seen that difficulties arise in connection with this equation. If we include in $f(k,0)$ only irreducible diagrams as we should - to agree with the rules of Section 3 - (50) does not hold for any value of k . In order to make our discussion as clear as possible, we first write down an explicit expression for $f(k,0)$ which solves (50) and then try to understand it. We have

$$\begin{aligned} f(k,0) = & G_0^{-1}(k) - G_0^{-1}(0) \\ & + \frac{i}{(2\pi)^4} \sum_{i,k} \int d^4q (V(k) + V(q - \frac{k}{2})) G_{1i}(q + \frac{k}{2}) \Pi_{ik}(q + \frac{k}{2}, q - \frac{k}{2}) G_{k1}(q - \frac{k}{2}) \\ & + \frac{i}{(2\pi)^4} \sum_{i,k} \int d^4q V(q - \frac{k}{2}) G_{1i}(q + \frac{k}{2}) \Pi_{ik}(q + \frac{k}{2}, q - \frac{k}{2}) G_{k2}(q - \frac{k}{2}) \end{aligned} \quad (51)$$

If we insert (20) in (51) and make use of (47') we find

$$\begin{aligned} f(k,0) = & G_0^{-1}(k) - \Sigma_{11}(k) + \Sigma_{12}(k) \\ f(0,-k) = & -G_0^{-1}(-k) + \Sigma_{11}(-k) - \Sigma_{12}(k) \end{aligned} \quad (52)$$

which indeed satisfies the required condition. However, this self-consistent

choice of $f(k,0)$ does not satisfy the condition $f(0,0) = 0$ nor the exact equation (A.3) of the Appendix. Furthermore, a number of self-energy insertions not included in the original approximation for the propagator are introduced by (51). To see the reason for this, let us consider as an example the second order self-energy closed loops:

The first two diagrams are not included in our approximation for Σ_{11} . The other two, on the other hand, are summed by the integral equations for Σ_{12} or Σ_{21} . Now the third graph gives rise to the following irreducible contribution to the vertex $F(k,0)$

$$F^{(2)}(k,0) = i n_0^{-1} f^{(2)}(k,0) = \text{diagram 1} + \text{diagram 2} =$$

$$= i \left\{ \frac{1}{n_0} \left[\text{diagram 3} \right] + \text{diagram 4} - \text{diagram 5} - \text{diagram 6} \right\} \quad (54)$$

the second line following from a simple direct calculation. The first two diagrams of the second line are cancelled in (50) by the corresponding terms of the propagator, but there is no contribution in our approximation for G to eliminate the other two. So in order to satisfy (50) one has to insert self-energy contributions which cancel the terms originating from the irreducible parts of $F(k,0)$.

Notice that, in the example considered the new insertions represent just the terms necessary to restore the validity of the theorem of Hugenholtz and Pines for that type of diagrams. In fact

$$\mu^{(2)} = \frac{1}{n_0} \left[\text{diagram} \right] = \sum_{11}^{(2)}(0) - \sum_{12}^{(2)}(0) \quad (55)$$

The argument can be extended without difficulty to the more complicated diagrams. The result is that besides the first two graphs (53), the following $\sum_{11}$ type insertions are contained in (51)

The new insertions of $\sum_{12}$ type can be obtained similarly. The next step would be to set up a new approximation scheme in which the diagrams (53) and (56) are included from the beginning in the approximation for G^*). It is easily seen that the new scheme includes the diagrams of the second order Beliaev's approximation ⁶⁾ and it is well known that this approximation has no gap in the single particle excitation spectrum.

The conclusions of this Section can be compared with those reached by Hohenberg ⁵⁾ who has also carried out an extensive study of the second order approximation.

Finally, we want to point out that even if it is not possible to construct within the limits of the Girardeau-Arnouitt approximation current conserving correlation functions, it is still possible to have a density correlation function that satisfies consistently the sum rules (9) and (10). The verification of this statement comes out after a long and tedious but not difficult calculation and will not be reported here.

^{*)} Of course, (51) contains other (actually infinite) self-energy diagrams, besides the contributions (53) and (56) where the interaction kernels appear in the lowest order. They also must be taken into account when setting up a new scheme beyond the approximation discussed in this Section.

CONCLUDING REMARKS

In this paper we have shown how a connection between the single particle Green's functions and the correlation functions can be established without explicit use of the equations of motion. The attractive features of the generalized Ward identity are first represented by its simple meaning in terms of diagrams which are perhaps closer to our intuition than the hierarchy of the Green's function equations. In general, the GWI will provide a complementary tool of analysis to be used in conjunction with the equations of motion. As a further interesting feature we may mention the fact that the GWI allows a unified picture of the problems connected with the long wavelength behaviour of the single-particle propagators and the correlation functions.

The preliminary analysis of the Bogoliubov and of the Girardeau-Arnouitt approximations presented in this paper indicate that the method deserves further study.

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A P P E N D I X

Proof of Eq. (35)

We subtract (29) from (28) and take the limit for $p \rightarrow 0$.

$$\begin{aligned} \Gamma_{11}(p, 0) - \Gamma_{12}(p, 0) &= \tilde{\Gamma}_{11}(p, 0) - \tilde{\Gamma}_{12}(p, 0) \\ &- [P_-^{++}(p, p) - P_+^{++}(p, p)] [G_{11}(-p)F(-p, 0) - G_{12}(p)F(p, 0)] \end{aligned} \quad (\text{A.1})$$

where $\tilde{\Gamma}_{11}(p', p)$ and $\tilde{\Gamma}_{12}(p', p)$ represent the irreducible parts of Γ_{11} and Γ_{12} .

Using the same argument which led to (33), we find

$$\tilde{\Gamma}_{11}(p, 0) - \tilde{\Gamma}_{12}(p, 0) = \frac{1}{m_0^{\frac{1}{2}}} F(p, 0) \quad (\text{A.2})$$

From (33) and (A.2) we then have the functional equation for $F(p, 0)$

$$\begin{aligned} m_0^{\frac{1}{2}} [\Gamma_{11}(p, 0) - \Gamma_{12}(p, 0)] &= \\ &= F(p, 0) - 2 \Sigma_{12}(p) [G_{11}(-p)F(-p, 0) - G_{12}(p)F(p, 0)] \end{aligned} \quad (\text{A.3})$$

The solution of (A.3) which satisfies $F(0, 0) = 0$ is given by

$$F(p, 0) = i m_0^{\frac{1}{2}} [G_0^{-1}(p) - \Sigma_{11}(p) + \Sigma_{12}(p)] \quad (\text{A.4})$$

with

$$\mu = \sum_{11} (0) - \sum_{12} (0) \quad (\text{A.5})$$

The latter equation is the expression of the chemical potential first obtained by Hugenholtz and Pines⁸⁾. (A.4) makes (35) identically satisfied.

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- 11) We use here the word "irreducible" as equivalent to "connected proper".

- 12) In this notation a + sign stands for an outgoing line and a - sign for an ingoing line. It is only slightly different from the notation used in Ref. ⁹⁾.

- 13) Equation (33) can be found in Ref. ⁹⁾. We repeat the derivation here for completeness. Consider a $\sum_{12}$ diagram of order s in n_0 . A direct counting of the incomplete vertices leads to

$$P_-^{++}(P, P) = \frac{(s+1)}{n_0^{\frac{1}{2}}} \sum_{12} (P)$$

$$P_+^{++}(P, P) = \frac{(s-1)}{n_0^{\frac{1}{2}}} \sum_{12} (P)$$

from which (33) follows.