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K-MATRIX MODEL OF PERIPHERAL INTERACTIONS

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ABSTRACT

A unitary S matrix for peripheral interactions at a few GeV including two- and three-particle channels is constructed by means of the K matrix. For quasi-two-particle channels its elements are approximated by Born terms while uncorrelated three-particle channels are described by a statistical K matrix. Both assumptions are discussed and the tools for practical applications are given.

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INTRODUCTION

Collisions of elementary particles at a few GeV have two features in common: abundance of quasi-two-particle reactions and the dominance of small momentum transfer (= peripheral) interactions. Both features are described by the one-meson-exchange model, which takes as amplitude for a quasi-two-particle reaction simply the sum of the possible Born terms, multiplied by a form factor $F(t)$, where t is the square of the four-momentum transfer. This form factor proves to be necessary for the description of the extremely steep t -dependence of peripheral interactions¹⁻³⁾.

Recently, several models have been made which avoid the ad hoc form factor. Sopkovich⁴⁾ treated this problem by a distorted wave Born approximation, the distortion coming from absorption in the initial and final states. Since the absorption is largest in the lowest partial waves, the cross-section drops faster at large momentum transfer. Calculations by Gottfried and Jackson on π^-p collisions⁵⁾, and by Gottfried, Jackson, Keyser and Svensson on K^+p collisions^{6,7)} gave good results not only for the differential cross-sections, but also, due to a proper treatment of spin, for the angular decay distributions of the produced resonances (K^* or N^*), except for the $K^* + N^*$ channel.

In these calculations, the optical potential is assumed to be purely imaginary, spin and isospin independent and identical in all initial and final quasi-two-particle states^{*)}. These assumptions may hold true in the asymptotic energy region, but not below 10 GeV, as has been demonstrated by recent experiments⁸⁾. Here we propose a model in which the deviations from the "asymptotic model" come from the one-meson exchange graphs themselves, and therefore can be calculated. In this paper we shall describe

*) In Ref. 7), the scattering in the final state has been taken as an additional parameter of the model.

the procedure, leaving the application to the K^+p case for a subsequent publication.

As our starting point we choose the K matrix (also called resistance matrix), defined by the matrix equations^{9,10)}

$$S = \frac{1 + iK}{1 - iK}.$$

The requirement that S be unitary is fulfilled if K is hermitian. Moreover, in a time-reversal invariant theory S can be chosen symmetric, which makes K symmetric and hence real. For those K-matrix elements which refer to quasi-two-particle reactions, we insert $\frac{1}{2}T^{\text{Born}}$. (In the weak coupling limit, this reduces to $S = 1 + iT^{\text{Born}}$.) Since the Born terms are real and symmetric, unitarity of the S-matrix follows automatically. Moreover, since the Born terms without form factors exceed the unitarity limit in the low partial waves by enormous amounts (typical factors are 10-100), imposing the unitarity condition means a reduction of low partial waves by correspondingly large factors. This results in a fast decrease of cross-sections with momentum transfer, i.e., it has the same effect as a form factor.

For the remaining K-matrix elements, we assume a statistical distribution with zero mean value. The underlying idea is that the total number of states with three or more uncorrelated particles is very large and that their matrix elements are all small. The width of the distribution is conveniently obtained from the differential elastic scattering. Our statistical treatment replaces the optical model of Sopkovich. The important difference is, however, that in our model the elastic scattering is the combined shadow of the two- and more-particle reactions. In the same way, the damping of one reaction is due to both two- and more-particle reactions. The contributions of the two-particle channels to both items are calculated explicitly. Since experimentally

the quasi-two-particle reactions contribute (in the case of K^+p and πp collisions) ca. 50% to the total cross-section, the effects of the unknown more-particle reactions are largely reduced.

It has been mentioned^{3,5)} that the more-particle reactions may be especially effective "dampers" since they occur with higher momentum transfers. Experimentally, this is true only for a small portion of the more-particle reactions. The double-peripheral reactions for example, which do belong to the more-particle reactions, are as peripheral as the two-particle reactions¹¹⁾. In fact, from our model we could calculate the damping of double-peripheral reactions, a problem which cannot even be formulated in the absorption model. Of course, we do not expect the K-matrix method proposed here to be the ultimate solution. Generally the identification $K = \frac{1}{2}T^{\text{Born}}$ does not give the correct analyticity properties of the T matrix. A method to construct a unitary S matrix which does not have this drawback is to unitarize the Born terms by N/D equations. For reactions involving high spins, it has its own difficulties due to the bad high-energy behaviour of the Born terms. Finally, it is not yet known, except for special cases¹²⁾, how to include three-particle states in this method.

The plan of the paper is as follows: in chapter 1 we introduce notations and kinematical quantities; chapter 2 contains the solution of Heitler's integral equation for two- and three-particle states; in chapter 3 this is compared to the N/D method and the key relation (3.6) between T and K is written down after the introduction of the statistical assumption. This assumption is discussed in more detail in chapter 4. The partial wave decomposition of K is made in chapter 5, while specific examples are postponed to the appendix. In chapter 6 we discuss, finally, a possible modification of the Born terms due to the instability of external particles.

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1. KINEMATICS

In this chapter we will introduce some necessary kinematical formulae. We follow the helicity formalism of Jacob and Wick¹³⁾ (J.W.) and, for three-particle states we reproduce some formulae derived by Cook and Lee¹²⁾ (C.L.). The notations are those of C.L.

We denote two- and three-particle states by:

$$|\vec{k}_a \lambda_a; \vec{p}_b \lambda_b; \Omega\rangle \quad (1.1)$$

and

$$|\vec{k}_a \lambda_a; \vec{k}'_a \lambda'_a; \vec{p}_b \lambda_b; \Phi \Xi\rangle \quad (1.2)$$

where $\vec{k}_a$, $\vec{k}'_a$, and $\vec{p}_b$ are the momenta of the mesons and nucleons in the over-all c.m.s.; λ_a , λ'_a and λ_b are the corresponding helicities; $\Omega = (\theta, \phi)$ is the solid angle of $\vec{k}_a$ in a coordinate system in the c.m.s.; Φ is the solid angle of $\vec{k}_a + \vec{k}'_a$ in a coordinate system in the over-all c.m.s.; Ξ is the solid angle in a two-meson c.m.s. coordinate system. Some of these variables are redundant - they are kept for better reference to (C.L.). The state normalization is

$$\begin{aligned} &\langle \vec{k}'_a \lambda'_a; \vec{p}'_b \lambda'_b; \Omega' | \vec{k}_a \lambda_a; \vec{p}_b \lambda_b; \Omega \rangle \\ &= (2\pi)^6 \delta^{(3)}(\vec{k}'_a - \vec{k}_a) \cdot \delta^{(3)}(\vec{p}'_b - \vec{p}_b) \cdot \delta_{\lambda_a \lambda'_a} \cdot \delta_{\lambda_b \lambda'_b}, \end{aligned} \quad (1.3)$$

and the same for three-particle states.

We introduce the Lorentz invariant scattering amplitudes for the following processes (a, c, c' refer to mesons, b and d to nucleons)

i) $a + b \rightarrow c + d$:

$$\begin{aligned} & \langle \vec{k}_c \lambda_c; \vec{p}_d \lambda_d; \Omega' | T | \vec{k}_a \lambda_a; \vec{p}_b \lambda_b; \Omega \rangle = \\ & = (2\pi)^4 \delta^{(4)}(P'-P) \prod_{\alpha} (2E_{\alpha})^{-\frac{1}{2}} \langle \lambda_c \lambda_d; \theta' \phi' | T_{22}(s) | \lambda_a \lambda_b; \theta \phi \rangle \quad (1.4) \end{aligned}$$

The $\delta^{(4)}$ -function gives the conservation of total energy-momentum;
 E_{α} are the energies of the particles a, b, c, d (see Fig. 1).

For $2 \rightarrow 3$ and $3 \rightarrow 3$ particle processes similar formulae hold.

Using rotational invariance to introduce helicity amplitudes, one finds:

$$\begin{aligned} & \langle \lambda_c \lambda_d; \theta' \phi' | T_{22}(s) | \lambda_a \lambda_b; \theta \phi \rangle = \\ & = \frac{1}{4\pi} \sum_{J, M} (2J+1) \langle \lambda_c \lambda_d | T_{22}^J(s) | \lambda_a \lambda_b \rangle \times \\ & \times D_{\lambda M}^{J*}(\phi', \theta', -\phi') \cdot D_{\lambda' \mu}^J(\phi, \theta, -\phi); \quad (1.5) \\ & \lambda = \lambda_a - \lambda_b, \mu = \lambda_c - \lambda_d; D_{\lambda \lambda'}^J \text{ is defined in J.W.} \end{aligned}$$

Similarly, a suitable choice of the relative orientation of Ξ and Ξ' allows one to introduce helicity amplitudes for processes involving three-particle states (see C.L. chapter 3).

ii) For $a + b \rightarrow c + c' + d$ one gets the helicity amplitudes

$$\langle \lambda_c \lambda_{c'} \lambda_d | T_{23}^{J\ell\Lambda}(s, \sigma) | \lambda_a \lambda_b \rangle, \quad (1.6)$$

where

J is the total angular momentum;

ℓ is the angular momentum of the two-meson system;

Λ its helicity;

σ is the invariant mass of the two pions.

That means that the two mesons manifest themselves in this coupling scheme as a 'particle' with spin ℓ , helicity Λ and the variable mass σ .

iii) For $c + c' + d \rightarrow a + b$ one finds in the same way

$$\langle \lambda_a \lambda_b | T_{32}^{J\ell\Lambda}(s, \sigma) | \lambda_c \lambda_{c'} \lambda_d \rangle \quad (1.7)$$

iv) For $c + c' + d \rightarrow c + c' + d$ finally,

$$\langle \lambda_c' \lambda_{c'}' \lambda_d' | T_{33}^{J\ell\Lambda, \ell'\Lambda'}(s, \sigma, \sigma') | \lambda_c \lambda_{c'} \lambda_d \rangle \quad (1.8)$$

are the helicity amplitudes.

For later convenience we introduce the following matrices as a shorthand notation for these helicity amplitudes,

$$T_{22} = \langle \lambda_c \lambda_d | T_{22}^J(s) | \lambda_a \lambda_b \rangle, \quad (1.9)$$

or, of additional channels i , $i = 1, \dots, n$, (with e.g. different thresholds) are present

$$T_{22} = (\langle \lambda_c \lambda_d | T_{22}^{\vec{f}}(s) | \lambda_a \lambda_b \rangle)_{ik}. \quad (1.10)$$

Similarly, the expressions (1.6), (1.7) and (1.8) are considered as matrices $T_{23}(\sigma)$, $T_{32}(\sigma)$, $T_{33}(\sigma, \sigma')$ with indices $\lambda_a, \lambda_b, \lambda_c, \lambda_d$ etc., and $\ell, \Lambda; \ell', \Lambda'$. Additional two- or three-particle channels enlarge only these matrices.

2. HEITLER'S INTEGRAL EQUATION

In this chapter we give solutions to Heitler's integral equation which connects the K matrix with the scattering matrix T. For later reference, we first treat the simple and well-known¹⁰⁾ case of coupled two-particle channels only. Next we consider the case of coupled two- and three-particle channels. As already pointed out in the introduction, our problem is then to give an explicit solution for the two-particle scattering matrix T_{22} .

a) Two-particle channels

The K matrix is defined by:

$$S = \frac{1 + iK}{1 - iK}, \quad (2.1)$$

from which we get Heitler's integral equation for $T = (S - 1)/i$

$$(1 - iK)T = 2K. \quad (2.2)$$

We write this equation in terms of the amplitudes $\langle \lambda_c \lambda_d; \theta' \phi' | T(s) | \lambda_a \lambda_b; \theta \phi \rangle$ and $\langle \lambda_c \lambda_d; \theta' \phi' | K(s) | \lambda_a \lambda_b; \theta \phi \rangle$ defined in the first chapter and insert their partial wave decompositions (1.5). Then we can perform the integration over intermediate states and get in the elastic case

$$\begin{aligned} \langle \lambda_c \lambda_d | T_{22}^J(s) | \lambda_a \lambda_b \rangle - i g_2(s) \sum_{\lambda \lambda'} \langle \lambda_c \lambda_d | K_{22}^J(s) | \lambda \lambda' \rangle \langle \lambda \lambda' | T_{22}^J(s) | \lambda_a \lambda_b \rangle \\ = 2 \langle \lambda_c \lambda_d | K_{22}^J(s) | \lambda_a \lambda_b \rangle, \end{aligned} \quad (2.3)$$

with

$$\rho_2(s) = \frac{1}{16\pi^2} \cdot \frac{q}{\sqrt{s}} \theta(s - s_{th}),$$

q = c.m.s. momentum,

and

$$s_{th} = (\text{threshold energy})^2.$$

This is a system of linear equations which can be solved for

$$\langle \lambda_c \lambda_d | T \partial(s) | \lambda_a \lambda_b \rangle \quad \text{once the K-matrix elements are given.}$$

If there are n other channels labelled by different thresholds s_i , eq. (2.3) is easily rewritten and reads in terms of the shorthand notation (1.10)

$$(1 - i K_{22} \hat{\rho}_2) T_{22} = 2 K_{22}, \quad (2.4)$$

where

$$\hat{\rho}_2 = \delta_{\lambda_a \lambda_c} \delta_{\lambda_b \lambda_d} \delta_{ik} \theta(s - s_{th}(i)) \cdot \frac{q_i}{\sqrt{s}} \frac{1}{16\pi^2}. \quad (2.5)$$

The unitarity condition $SS^+ = S^+S = 1$ is fulfilled if K_{22} is hermitian.

For a particular choice of phases time reversal invariance allows one to make T_{22} symmetric; K_{22} is then real and symmetric.

b) Two- and three-particle channels

We now want to include three-particle channels. To be definite let us consider spinless and isospinless nucleons of mass m and pions of mass μ and look at the following reactions:

Introducing the helicity decompositions for $2 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 3$ processes in eq. (2.2), one finds in the same manner as under 2.a) the following integral equations connecting the T-matrix to the K-matrix elements:

$$T_{22} = 2 K_{22} + i p_2 K_{22} T_{22} + i \int d\sigma'' p_3(\sigma'') K_{23}(\sigma'') T_{32}(\sigma''), \quad (2.7)$$

$$T_{32}(\sigma) = 2 K_{32}(\sigma) + i p_2 K_{32}(\sigma) T_{22} + i \int d\sigma'' p_3(\sigma'') K_{33}(\sigma, \sigma'') T_{32}(\sigma''), \quad (2.8)$$

$$T_{23}(\sigma) = 2 K_{23}(\sigma) + i p_2 K_{22} T_{23}(\sigma) + i \int d\sigma'' p_3(\sigma'') K_{23}(\sigma'') T_{33}(\sigma'', \sigma), \quad (2.9)$$

$$\begin{aligned}T_{33}(\sigma, \sigma') &= 2 K_{33}(\sigma, \sigma') + i p_2 K_{32}(\sigma) T_{23}(\sigma') \\ &\quad + i \int d\sigma'' p_3(\sigma'') K_{33}(\sigma, \sigma'') T_{33}(\sigma'', \sigma'),\end{aligned}\quad (2.10)$$

where

$$f_3(\sigma'') = \frac{1}{32(2\pi)^5} \cdot \frac{\tilde{q}(s, \sigma)}{\sqrt{s}} \theta(\sigma - 4\mu^2) \theta((\sqrt{s} - m)^2 - \sigma) \sqrt{1 - 4\mu^2/\sigma}. \quad (2.11)$$

The T_{ik} and K_{ik} ($i, k = 2, 3$) are the matrices introduced in (1.10) etc. Unitarity and symmetry of the S matrix is guaranteed if (the symbol T means the transposed matrix)

$$\begin{aligned} K_{23}(\sigma) &= K_{32}^T(\sigma) \\ K_{33}(\sigma, \sigma') &= K_{33}^T(\sigma, \sigma') \\ K_{22} &= K_{22}^T \end{aligned} \quad (2.12)$$

To write our solutions in a compact way we abbreviate the convolution:

$$\begin{aligned} \int d\sigma'' f_3(\sigma'') K_{23}(\sigma'') T_{32}(\sigma'') &= K_{23} * T_{32}, \\ \int d\sigma'' f_3(\sigma'') K_{33}(\sigma, \sigma'') K_{33}(\sigma'', \sigma') &= K_{33} * K_{33} = (\times K_{33})^2 \end{aligned} \quad (2.13)$$

etc.

We observe that only (2.7) and (2.8) as well as (2.9) and (2.10) are mutually coupled together. We are interested in T_{22} and therefore look only at the first two equations which we now write as (the σ -dependence is not indicated)

$$T_{22} = 2 K_{22} + i g_2 K_{22} T_{22} + i K_{23} \times T_{32}, \quad (2.14)$$

$$T_{32} = 2 K_{32} + i g_2 K_{32} T_{22} + i K_{33} \times T_{32}. \quad (2.15)$$

We introduce the resolvent kernel of (2.15) which can formally be written as $(1 - i K_{33})^{-1}$ and find

$$T_{32} = (1 - i K_{33})^{-1} \times K_{32} (2 + i g_2 T_{22}). \quad (2.16)$$

We insert this expression in (2.14),

$$T_{22} = (K_{22} + i K_{23} \times (1 - i K_{33})^{-1} \times K_{32}) (2 + i g_2 T_{22}) \quad (2.17)$$

and get with the abbreviation

$$B_{22} = i K_{23} \times (1 - i K_{33})^{-1} \times K_{32} \quad (2.18)$$

the solution

$$T_{22} = \frac{2 (K_{22} + B_{22})}{1 - i g_2 (K_{22} + B_{22})} \quad (2.19)$$

Remember that B_{22} is, in the presently treated case (2.6), a complex number. It might be worth while to check the unitarity condition which states that

$$S_{22} = 1 + i g_2 T_{22} \quad (2.20)$$

has a modulus less or equal to 1. Inserting eq. (2.19) we find

$$S_{22} = \frac{1 + i \hat{\rho}_2 (K_{22} + B_{22})}{1 - i \hat{\rho}_2 (K_{22} + B_{22})}, \quad (2.21)$$

i.e., S_{22} is less or equal to 1 if and only if $\text{Im } B_{22} \geq 0$. Looking at (2.18) one sees that

$$\text{Im } B_{22} = K_{23} \times (1 + (\times K_{33})^2)^{-1} \times K_{32}. \quad (2.22)$$

Because of the symmetry conditions (2.12), $(\times K_{33})^2$ is a positive operator and therefore $\text{Im } B_{22}$ is positive.

If there are several two-particle channels present we have instead of (2.19)

$$T_{22} = 2 (1 - i (K_{22} + B_{22}) \hat{\rho}_2)^{-1} (K_{22} + B_{22}), \quad (2.23)$$

K_{22} and T_{22} are now the matrices (1.10); $\hat{\rho}_2$ has been given in (2.5). B_{22} is obviously a complex matrix of the same dimension as K_{22} and T_{22} .

With the state normalization (5.18) below, $\hat{\rho}_2$ is a unit matrix, and (2.23) simplifies to

$$T_{22} = 2 (1 - i (K_{22} + B_{22}))^{-1} (K_{22} + B_{22}). \quad (2.24)$$

3. DYNAMICAL ASSUMPTIONS

As already discussed in the introduction, we expect a focusing effect on angular distributions if we "unitarize" the Born terms. Usually this is done by N/D calculations. Instead we construct unitary amplitudes by the identification $K_{22} = \frac{1}{2}T^{\text{Born}}$. The first part of this chapter is devoted to the comparison of both methods. In the second part, we propose a simple model for the influence of uncorrelated three-particle states which is represented by the matrix B_{22} in the general formula.

a) To compare the N/D method with the K-matrix formalism, let us treat, as a particularly simple example, the one channel case. The left-hand singularities are supposed to be given by a one-particle exchange term $t_B(s, t)$, the partial waves $t_B^J(s)$ of which behave like

$$t_B^J(s) \sim s^\alpha (\ln s)^\beta \quad (3.1)$$

at infinity.

For most of the cases we are interested in, one finds $\alpha \geq 1$. This is due to the high spin of the particles like N^* , K^* , etc., occurring in peripheral interactions. The N/D solutions to that problem would, by construction, reproduce exactly the left-hand cuts of $t_B^J(s)$. It is, however, not clear whether solutions for our 'force functions' $t_B^J(s)$ do exist¹⁴⁾ and, if so, what their physical meaning would be. Therefore, our procedure is best compared to the determinantal approach usually used for unitarizing singular OPE graphs¹⁵⁾. The latter gives for the partial wave amplitude $t^J(s)$

$$\begin{aligned} t^J(s) &= t_B^J(s) \left[1 - \frac{s-s_0}{\pi} \int_{s_{th}}^{s_c} \frac{ds' \rho_2(s')}{s'-s_0} \frac{t_B^J(s')}{s'-s} \right]^{-1} \\ &= t_B^J(s) \left[1 - \frac{i}{2} \rho_2 t_B^J(s) - \frac{s-s_0}{\pi} P \int_{s_{th}}^{s_c} \frac{ds' \rho_2(s')}{s'-s_0} \frac{t_B^J(s')}{s'-s} \right]^{-1} \quad (3.2) \end{aligned}$$

An additional cut-off parameter s_c is necessary.
Unitarization by means of the K matrix gives the expression:

$$t^J(s) = \frac{t_B^J(s)}{1 - \frac{i}{2} \rho_2 t_B^J(s)} \quad (3.3)$$

Formally both expressions agree if we neglect the principal value integral in (3.2). But even more, for high partial waves ($J > J_0$, J_0 depending on s), both expressions (3.2) and (3.3) give for an arbitrary cut-off s_c

$$t^J(s) \approx t_B^J(s) \quad J > J_0, \quad (3.4)$$

s fixed.

This is because $t_B^J(s)$ is for fixed s a decreasing function of J , which finally becomes small compared to 1. Furthermore, for s_c chosen reasonably above s_{th} the principal value integral in (3.2) decreases more slowly below one, so that J_0 is in general higher for model (3.2). These considerations lead us to the following statement: both formulae (3.2) and (3.3) reproduce the feature of peripheral interactions that for sufficiently high partial waves the interaction is given by the long range, i.e., OPE forces. For low partial waves no definite predictions can be made. The question, which of the two expressions is nearer to reality, has to be settled by comparison with experiment.

If several two-particle channels are present the analogous remarks hold, only some care is necessary to assure the symmetry of the scattering matrix in the formulation of the determinantal method.

b) Now we come to our next problem, namely, to learn something about the contribution of uncorrelated three-particle states represented by B_{22} (2.18). From the field-theoretic side almost nothing is known about their structure. There is, however, one feature which may allow a treatment of these contributions. This is the notion that the matrices K_{23} , K_{32} and K_{33} have a very high dimension in actual cases. The angular momenta ℓ , ℓ' (recall the definitions (1.6)) estimated from the available energy $\sqrt{s}$ together with the number of possible final and initial states, e.g., $N\pi K$, $N\pi\pi K$, $NK\bar{K}K$, $\Lambda^0 K\bar{K}$, etc., for e.g., K^+p scattering at 3 GeV/c give a very high number (around 100 in this case). Furthermore, all matrix elements should be small and of roughly the same order of magnitude. These facts bring us to suggest¹⁶⁾ that a statistical treatment of these matrices might be possible. More precisely, we assume that the values of the matrix elements of K_{23} , K_{32} , K_{33} are distributed at random around the mean value zero. In the following chapter it is shown that in this case B is a multiple of the unit matrix with purely imaginary elements:

$$B_{22} = ib \cdot 1 \quad (3.5)$$

Moreover, from the reasoning following (2.22) it is evident that $b > 0$. Inserting (3.5) into (2.24) and making the denominator real, we obtain our final formula

$$T_{22} = 2 \frac{K_{22} + iK_{22}^2 + ib(1+b)}{(1+b)^2 + K_{22}^2} \quad (3.6)$$

The damping of the Born terms, as displayed by (3.6), has two origins: due to the term K_{22}^2 in the denominator the Born terms damp themselves, and due to the matrix $b \cdot 1$, they are damped by the three-particle channels.

Remember that all quantities in (3.6) depend on total angular momentum, isospin, etc. Specifically, the influence of all more-particle channels is given by one parameter b^J for each partial wave. Taken together, the parameters b^J result in one unknown function of the scattering angle (and the total energy, of course), which in our forthcoming numerical calculation will be determined from the elastic differential cross-section.

4. THE STATISTICAL ASSUMPTION

Our basic assumption for the derivation of eq. (3.5) is that all matrix elements of K , except those of K_{22} are statistically independent. For mathematical convenience we choose the Gaussian probability distribution

$$P(K_{ij}) = \frac{\tau}{\sqrt{2\pi}} \exp\left(-\frac{K_{ij}^2}{2\tau^2}\right). \quad (4.1)$$

The dependence on τ and τ' (see eq. (2.12)) is not spelled out until the end of this section. Statistical assumptions of the type (4.1) have been used in nuclear physics (see for example Dyson¹⁷), where reference to earlier work is found). One then obtains

$$\sum_{j=1}^N K_{ij}^2 = c_i \approx (N-1)\tau^2 = c, \quad (4.2)$$

$$\sum_{j=1}^N K_{ij} K_{jk} = c_{ik} \approx 0 \quad \text{for } i \neq k. \quad (4.3)$$

The summation goes over all three-particle states. The other indices i, k refer to both two- and three-particle states. If the widths for the elements of K_{23} and K_{33} are different, τ_2 and τ_3 say, one has to replace c in (4.2) by c_2 or c_3 , respectively. Observing (2.12), we have

$$K_{33} \times K_{33} = c_3 \cdot 1, \quad K_{23} \times K_{32} = c_2 \cdot 1, \quad (4.4)$$

and eq. (2.18) can be transformed into

$$\begin{aligned} B_{22} &= i K_{23} \times (1 + i K_{33}) \times (1 + K_{33} \times K_{33})^{-1} \times K_{32} \\ &= i K_{23} \times (1 + i K_{33}) \times K_{32} \cdot \frac{1}{1 + c_3} \\ &= i \frac{c_2}{1 + c_3} \cdot 1 = i b \cdot 1. \end{aligned} \quad (4.5)$$

In addition to (4.4), eq. (4.5) makes use of $K_{23} \times K_{33} = 0$, and assumes that the expectation value of $(1 + K_{33} \times K_{33})^{-1}$ is the same as $[1 + \text{Expectation}(K_{33} \times K_{33})]^{-1}$. This equality holds true if the power series expansion (4.8) below converges. On the other hand, (4.5) does certainly not depend on the specific distribution (4.1) chosen here.

Next, we investigate how accurate the approximate equalities in (4.2) and (4.3) may be. Let us introduce $\Delta \tau^2$, the mean error in τ^2 , and write

$$C_i = (N-1)(\tau^2 \pm \Delta \tau^2) \quad (4.6)$$

From (4.1), $\Delta \tau^2 = \tau^2 \sqrt{2/N}$, and the final results are

$$\begin{aligned} C_i &= C(1 \pm \sqrt{2/N}), \\ C_{ik} &= C(0 \pm \sqrt{1/N}). \end{aligned} \quad (4.7)$$

For $N = 200$, for example, the deviations of $K_{33} \times K_{33}$ from a multiple of the unit matrix will be of the order of 10% in the diagonal elements, and a factor $2^{-1/2}$ less in the other elements. Unfortunately, (4.7) is not sufficient to determine the accuracy of (4.5), since the error in the inverse matrix may be different from the error in the matrix itself. Let us expand $x(1 + K_{33} \times K_{33})^{-1}$ in a power series:

$$\begin{aligned} x(1 + K_{33} \times K_{33})_{ik}^{-1} &= \delta_{ik} d(\sigma - \sigma') - \sum_j \int d\sigma'' K_{ij}(\sigma, \sigma'') \cdot K_{jk}(\sigma'', \sigma') \\ &+ \sum_{jlm} \int d\sigma^4 K_{im}(\sigma, \sigma^4) \int d\sigma^3 K_{me}(\sigma^4, \sigma^3) \int d\sigma'' K_{ej}(\sigma^3, \sigma'') K_{jk}(\sigma'', \sigma') \\ &- + \dots \end{aligned} \quad (4.8)$$

Here we have displayed all indices and arguments in order to show the full complexity of the problem. In order to solve (4.8), we have to know how the elements of K behave as functions of the effective masses, σ and σ' . Here we can make two different assumptions:

i) they are independent of σ and σ' . This type of assumption is always made when experimentalists try to separate an effective mass distribution into a "resonance" and a "phase space background". Since we have included the resonances among the two-particle states, we are really dealing with the "background" only.

ii) there does exist an essential variation of K with σ , say. This is approximated by dividing the range of σ into a few intervals, such that in each interval K may be taken as a constant, whereas the values of K in different intervals may be taken to be statistically independent. In case (i) the solution of (4.8) is simple, and case (ii) is reduced to (i) by taking the number of the interval just as another index. Since this increases the dimension N of the statistical matrix, one might conclude that case (ii) gives smaller fluctuations of (4.7) than case (i). This conclusion need not hold true, however, since the above statistical step function may be a poor approximation to $K(\sigma)$.

Concluding then, it appears difficult to estimate the error of (4.5). It is possible that the method is practicable only when the series (4.8) converges; this means that all eigenvalues of K must be smaller than unity. In the statistical calculation this means that

$$c_3 < 1. \quad (4.9)$$

5. CONSTRUCTION OF K-MATRIX ELEMENTS BETWEEN EIGENSTATES OF TOTAL ANGULAR MOMENTUM AND PARITY

As mentioned in the introduction, we shall insert Born terms (one-meson-exchange graphs) as the elements of K_{22} . These are given by Feynman's rules in the form

$$T^{\text{Born}}(s, t, \lambda_c \lambda_d, \lambda_a \lambda_b). \quad (5.1)$$

Our notation follows closely that of Jacob and Wick (J.W.), Jackson and Pilkuhn²⁾, and Gottfried and Jackson⁵⁾. The incident particles are a and b, their helicities are λ_a , λ_b , and so on (see Fig. 1). The magnitudes of the three-momenta in the c.m.s. will be denoted by q and q' , and x will be the cosine of the c.m.s. scattering angle.

For the solution of our matrix equation (3.6), it is important to decouple it to as many independent matrix equations as possible, i.e., to employ a representation in which in addition to the total angular momentum J , also the parity P and the total isospin are diagonal. In this paper we shall omit isospin. Parity eigenstates are constructed below. Putting in (1.5) $\Theta = \Phi = \Phi' = 0$, we get for (5.1)

$$T^{\text{Born}}(s, t, \lambda_c \lambda_d, \lambda_a \lambda_b) = \sum_J \left(J + \frac{1}{2}\right) T_J^{\text{Born}}(s, \lambda_c \lambda_d, \lambda_a \lambda_b) \cdot d_{\lambda_\mu}^J(x). \quad (5.2)$$

The connection between t and x is

$$t = \mu^2 - 2qq'(z-x), \quad (5.3)$$

where μ is the mass of the exchanged meson, and

$$Z = \frac{1}{qq'} \left[E_b E_d - \frac{1}{2} (M_b^2 + M_d^2 + \mu^2) \right] = \frac{1}{qq'} \left[E_a E_c - \frac{1}{2} (M_a^2 + M_c^2 + \mu^2) \right] \quad (5.4)$$

is introduced for later convenience.

Instead of (5.2) we need the inverse relation,

$$T_J^{\text{Born}}(\lambda_c \lambda_d, \lambda_a \lambda_b) = 2\pi \int_{-1}^1 dx T^{\text{Born}}(t, \lambda_c \lambda_d, \lambda_a \lambda_b) \cdot d_{\lambda\mu}^J(x). \quad (5.5)$$

Here and in the following equations, we have dropped the argument s .

The integral (5.5) can be solved explicitly for any Born term and arbitrary J . The point is that $T^{\text{Born}}(t, \lambda_c \lambda_d, \lambda_a \lambda_b)$ can always be put into the form

$$T^{\text{Born}}(t, \lambda_c \lambda_d, \lambda_a \lambda_b) = \text{Pol}_n(x, \lambda_c \lambda_d, \lambda_a \lambda_b) \cdot d_{\lambda\mu}^S(x) \cdot [-2qq'(z-x)]^{-1} \quad (5.6)$$

where $\text{Pol}_n(x)$ is a polynomial in x of degree n . The expression in the square bracket is of course $t - \mu^2$, which comes from the propagator. The spin S of the d -function can always be taken as the smallest meaningful value, i.e.,

$$S = \max(|\lambda|, |\mu|). \quad (5.7)$$

For some pion-exchange terms contributing to meson-nucleon collisions, the decomposition (5.6) is done explicitly in the appendix.

To perform the x-integration, the product $d_{\lambda\mu}^S \cdot d_{\lambda\mu}^J$ is expanded in terms of Legendre polynomials (see J.W., eq. 32):

$$d_{\lambda\mu}^S \cdot d_{\lambda\mu}^J(x) = \sum_L C(JSL; \lambda, -\lambda) \cdot C(JSL; \mu, -\mu) \cdot P_L(x), \quad (5.8)$$

where the sum extends from $|J-S|$ to $J+S$.

Using expressions (5.6) and (5.8), the integral (5.5) becomes a sum of integrals of the type

$$\frac{1}{2} \int_{-1}^1 \frac{dx}{z-x} P_{0L}(x) P_L(x) = P_{0L}(z) \cdot Q_L(z) \quad (5.9)$$

where $Q_L(z)$ are the Legendre functions of the second kind,

$$\begin{aligned} Q_0 &= \frac{1}{2} \ln \frac{z+1}{z-1}, \quad Q_1 = z \cdot Q_0 - 1, \\ Q_L &= \frac{1}{L} [(2L-1)zQ_{L-1} - (L-1)Q_{L-2}] \quad \text{for } L > 1. \end{aligned} \quad (5.10)$$

Equation (5.9) is not valid for $L < n$. For $n = 1$ and 2, for example, one has

$$\begin{aligned} \frac{1}{2} \int \frac{dx}{z-x} \cdot x \cdot P_0(x) &= Q_1(z), \\ \frac{1}{2} \int \frac{dx}{z-x} x^2 P_0(x) &= zQ_1(z), \\ \frac{1}{2} \int \frac{dx}{z-x} x^2 P_1(x) &= z^2 Q_1(z) - \frac{1}{3} \end{aligned} \quad (5.11)$$

Thus, except for the lowest values of J , (5.5) gives

$$T_J^{\text{Born}}(\lambda_c \lambda_d, \lambda_a \lambda_b) = \frac{2\pi}{99} P_{0L}(z, \lambda_c \lambda_d, \lambda_a \lambda_b) \cdot P_{\lambda\mu}^J(z) \quad (5.12)$$

$$P_{\lambda\mu}^J(z) \equiv \sum_L C(JSL; \lambda, -\lambda) \cdot C(JSL; \mu, -\mu) \cdot Q_L(z).$$

Next, we construct matrix elements between states of definite parity. The parity operation on helicity states is (J.W. eq. 41):

$$P |j, \lambda_a, \lambda_b\rangle = \eta_a \eta_b (-1)^{j-S_a-S_b} |j, -\lambda_a, -\lambda_b\rangle \quad (5.13)$$

where S_a and S_b are the spins of particles a and b, and η_a and η_b are their intrinsic parities (for baryons and the spin-3/2 decuplet, $\eta = +1$, while for pseudoscalar and vector mesons, $\eta = -1$).

From here on we specialize in the case of meson-baryon collisions, where $S_a + S_b$ is always half-integer. For this case we define the following eigenstates of P:

$$P |j, \pm |\lambda_a, \lambda_b|\rangle = (-1)^{j \pm \frac{1}{2}} |j, \pm |\lambda_a, \lambda_b|\rangle. \quad (5.14)$$

From (5.13) we find

$$|j, \pm |\lambda_a, \lambda_b|\rangle = \frac{1}{\sqrt{2}} |j, \lambda_a, \lambda_b\rangle \pm \frac{1}{\sqrt{2}} \eta_a \eta_b (-1)^{S_a+S_b+\frac{1}{2}} |j, -\lambda_a, -\lambda_b\rangle \quad (5.15)$$

Thus, although the parity eigenstates are not helicity eigenstates, they are still characterized by $|\lambda_a|, |\lambda_b|$, and the sign of $\lambda_a \cdot \lambda_b$.

This is our reason for using the symbol $|\lambda_a, \lambda_b|$.

The matrix elements between parity eigenstates are

$$\begin{aligned} T_j^{\pm}(\lambda_c \lambda_d, \lambda_a \lambda_b) &= T_j(\lambda_c \lambda_d, \lambda_a \lambda_b) \\ &\pm \eta_a \eta_b (-1)^{S_a+S_b+\frac{1}{2}} \cdot T_j(\lambda_c \lambda_d, -\lambda_a -\lambda_b). \end{aligned} \quad (5.16)$$

Despite its asymmetric form, (5.16) is symmetric in the index pairs (c, d) and (a, b). Matrix elements between states of opposite parity are, of course, zero, therefore the matrix T_J has been reduced to two matrices T_J^+ and T_J^- , each having half the dimension of T_J .

Putting pieces together, we finally obtain as our K matrix elements,

$$K_J^{\pm}(\lambda_c \lambda_d, \lambda_a \lambda_b) = \frac{1}{2} (4 q q' s)^{-\frac{1}{2}} \times \quad (5.17)$$

$$\times \left[\text{Pol}_n(z, \lambda_c \lambda_d, \lambda_a \lambda_b) \cdot P_{\lambda, \mu}^{JS}(z) \pm \eta_a \eta_b (-1)^{S_a + S_b + \frac{1}{2}} \text{Pol}_n(z, \lambda_c \lambda_d, -\lambda_a - \lambda_b) \cdot P_{-\lambda, \mu}^{JS}(z) \right].$$

In this equation an additional factor

$$\sqrt{q q' / 4 s} \quad (5.18)$$

appears, which is the square root of the product of the initial and final momentum spaces. With this convention, the coupling constants in T^{Born} are rationalized ones, e.g. $G^2 = 14$ in eq. (A.3).

6. THE SICK GRAPH

As explained in the last chapter, the matrix equation (3.6) holds for each J and either parity separately. From now on, let us label by T_{if} the matrix elements of T_{22} , by K_{if} those of K_{22} , etc. (i stands for initial and f for final). Let $i = 1$ denote the experimental incident state (Kp or πp). In the notation of Jackson and Pilkuhn²⁾, this is a PB state (since $P \equiv$ pseudoscalar has zero spin and $B \equiv$ baryon has spin $\frac{1}{2}$, there is just one state for either parity). Experimentally, only $|T_{if}|^2$ can be measured, since T_{if} with $i \neq 1$ contains at least one unstable particle in the initial state (V or B^*). Still, in order to obtain T_{if} from (3.6), we have to know the whole matrix K_{if} . We thus have to consider, in addition to the reactions $PB \rightarrow PB$, $PB \rightarrow PB^*$, $PB \rightarrow VB$ and $PB \rightarrow VB^*$ which appear in the first row (and column) of table 1, also reactions like $PB^* \rightarrow PB^*$, which appear in the other rows. The Born terms for these reactions contain couplings which have not appeared in any previous model, for example the $PB^* B^*$ (read $\pi N N^*$) couplings. Some of these couplings are given in the appendix.

Among the reactions of table 1, reaction 6 ($PB^* \rightarrow VB$) needs a special discussion in case both B^* and V are unstable against decay into the

T A B L E 1

Matrix of quasi-two-particle
transitions occurring in PB collisions

$i \backslash f$	PB	PB^*	VB	VB^*
PB	1	2	3	4
PB^*	2	5	6	7
VB	3	6	8	9
VB^*	4	7	9	10

same particle, which in all practical cases will be pseudoscalar, either π or K. In this case namely, there will be a "sick" Born term^{*)}, containing the decay particle as the exchanged particle (Fig. 2).

The exchanged particle may then be real, which means that $z < 1$, so that the amplitude (5.6) becomes infinite in the physical region ($-1 < x < 1$), and the partial wave expansion (5.2) does not converge uniformly.

In fact, T_J^{Born} is not even real. According to Feynman's rule, z contains an infinitesimal $-i\epsilon$,

$$z = y - i\epsilon,$$

and (5.9) reads

$$\frac{1}{2} \int_{-1}^1 \frac{dx}{z-x} \text{Pol}_n(x) \cdot P_L(x) = \text{Pol}_n(z) \left[\bar{Q}_L(y) + i \frac{\pi}{2} P_L(y) \right], \quad (6.1)$$

where now

$$\bar{Q}_L(y) = \text{principal value} \int_{-1}^1 dx \frac{P_L(x)}{y-x}$$

Since T^{Born} is now complex, the identification $K = \frac{1}{2} T^{\text{Born}}$ is no longer possible. However, the integral equation defining the K matrix¹⁸⁾ tells us that its elements in lowest order perturbation theory are obtained by performing a principal value integration in (6.1), instead of using the $-i\epsilon$ device. The proper expression for K is therefore

$$K_J = \frac{1}{2} \text{Re } T_J^{\text{Born}}. \quad (6.2)$$

*) We got this name from Dr. E. Abers.

All other graphs are not affected by this change because they have $y > 1$, so they are real anyway. The non-uniform convergence of the partial wave expansion for $K(6)$ need not worry us, since the elements T_{1f} as calculated from (3.6) will still converge.

Of course, it is difficult to judge whether the exceptional role played by the sick graph is justified or not. What to do, for example, if $y = 1$? Alternatively, one could adopt the prescription of ref. 19: if one of the external particles is unstable, replace its mass by $M - i\Gamma/2$, where Γ is the width of the particle, and take for z the average of the two expressions of (5.4):

$$Z = y + i\gamma \quad (6.3)$$

$$y = \frac{1}{2qq'} \left[E_b E_d + E_a E_c - \frac{1}{2} (M_a^2 + M_b^2 + M_c^2 + M_d^2) + \mu^2 \right] \quad (6.4)$$

$$\gamma = \frac{1}{4qq'} (M_a \Gamma_a + M_b \Gamma_b + M_c \Gamma_c + M_d \Gamma_d). \quad (6.5)$$

In this case, all Born terms except that of the elastic channel would become complex. Since the recurrence relation (5.10) holds true even for complex z , (5.12) is easily evaluated also in this case. Again, in (5.17) only the real part should be taken.

A P P E N D I X

Helicity matrix elements for pseudoscalar exchange

With the helicity indices written out, the 4 x 4 matrix of table 1 becomes a 12 x 12 matrix. Remembering that the spins of P, B, V and B^{*} are 0, 1/2, 1 and 3/2, respectively, the PB^{*} state splits into two states, the VB state into three states, and the VB^{*} state into six states. The number of different helicity matrix elements is thus 1/2 * 12 * 13 for either parity, giving a total of 156. It would be inappropriate to write down all possible Born terms for these 156 transitions, say, for P and V exchange.

In the case of P exchange, Born terms between helicity (i.e., not parity) eigenstates decompose into the product of the propagator, one P-baryon coupling and one P-meson coupling. We now give these couplings explicitly, starting with the mesonic vertices (see in Fig. 1):

$$PP'V: \quad \langle \lambda_c | 0 \rangle = i g'' (a \cdot e) \cdot \epsilon^{\lambda_c}, \quad (A.1)$$

$$PV'V: \quad \langle \lambda_c | \lambda_a \rangle = \frac{f}{M_c} \epsilon^{\mu\nu\rho\sigma} a_\mu a_\nu \epsilon_\rho^{\lambda_a} \epsilon_\sigma^{\lambda_c}. \quad (A.2)$$

Except for a factor 2ig'', (A.1) gives

$$\langle 1 | 0 \rangle = - \langle -1 | 0 \rangle = -\sqrt{2} \, q \cdot \omega \cdot \sin, \quad (A.1a)$$

$$\langle 0 | 0 \rangle = \frac{1}{M_c} (q' E_a - q E_c X), \quad (A.1b)$$

and except for a factor if, (A.2) gives

$$\langle 1|1 \rangle = -\langle -1|-1 \rangle = -\frac{1}{M_c} (qE_c - q'E_a) \cos^2, \quad (\text{A.2a})$$

$$\langle -1|1 \rangle = -\langle 1|-1 \rangle = -\frac{1}{M_c} (qE_c - q'E_a) \sin^2, \quad (\text{A.2b})$$

$$\langle 1|0 \rangle = \langle -1|0 \rangle = -\sqrt{2} q' \cos \cdot \sin, \quad (\text{A.2c})$$

$$\langle 0|1 \rangle = \langle 0|-1 \rangle = -\sqrt{2} q \cdot \cos \sin, \quad (\text{A.2d})$$

$$\langle 0|0 \rangle = 0. \quad (\text{A.2e})$$

Here we have abbreviated

$$X = \cos \theta_s, \quad \cos = \cos \frac{\theta_s}{2}, \quad \sin = \sin \frac{\theta_s}{2}.$$

These abbreviations are useful for the construction of the $d_{\lambda\mu}^S$ to be discussed below. The signs of (A.1a, b) and (A.2a-e) include already the inversion of momentum (J.W. Eq. 13). We have chosen the incident baryon momentum along the +z axis, so the meson goes into the -z direction.

For the baryon couplings,

$$BPB': \quad G \bar{u}(\vec{d}, \lambda_d) \gamma_5 u(\vec{b}, \lambda_b), \quad (\text{A.3})$$

$$BPB^*: \quad i \frac{G^*}{\mu} \bar{u}_\mu(\vec{d}, \lambda_d) b_\mu u(\vec{b}, \lambda_b), \quad (\text{A.4})$$

$$B^*PB^*: \quad G^{**} \bar{u}_\mu(\vec{d}, \lambda_d) \gamma_5 u(\vec{b}, \lambda_b), \quad (\text{A.5})$$

we use the abbreviations^{5,7)}

$$\xi_{\pm} = \sqrt{(E_b + M_b)(E_d + M_d)} \left(\frac{q}{E_b + M_b} \pm \frac{q'}{E_d + M_d} \right),$$

$$\eta_{\pm} = \sqrt{(E_b + M_b)(E_d + M_d)} \left(1 \pm \frac{qq'}{(E_b + M_b)(E_d + M_d)} \right),$$
(A.6)

and obtain (neglecting the factors iG , iG_{μ}^* , and iG^{**}),

$$\langle \frac{1}{2} | \frac{1}{2} \rangle = -\langle -\frac{1}{2} | -\frac{1}{2} \rangle = \xi_- \omega, \quad (A.3a)$$

$$\langle -\frac{1}{2} | \frac{1}{2} \rangle = \langle \frac{1}{2} | -\frac{1}{2} \rangle = \xi_+ \sin, \quad (A.3b)$$

$$\langle \frac{3}{2} | \frac{1}{2} \rangle = -\langle -\frac{3}{2} | -\frac{1}{2} \rangle = -\sqrt{2} q \eta_- \omega^2 \sin, \quad (A.4a)$$

$$\langle \frac{3}{2} | -\frac{1}{2} \rangle = \langle -\frac{3}{2} | \frac{1}{2} \rangle = -\sqrt{2} q \eta_+ \omega \sin^2, \quad (A.4b)$$

$$\langle \frac{1}{2} | \frac{1}{2} \rangle = \langle -\frac{1}{2} | -\frac{1}{2} \rangle = \sqrt{\frac{2}{3}} \omega \left[\frac{1}{2}(1-x) q \eta_+ + q \eta_- \frac{E_d}{M_d} - x E_b \frac{q'}{M_d} \eta_- \right], \quad (A.4c)$$

$$\langle \frac{1}{2} | -\frac{1}{2} \rangle = -\langle -\frac{1}{2} | \frac{1}{2} \rangle = \sqrt{\frac{2}{3}} \sin \left[(1-x) \frac{E_d}{M_d} q \eta_+ - \frac{1}{2}(1+x) q \eta_- \right], \quad (A.4d)$$

$$\langle \frac{3}{2} | \frac{3}{2} \rangle = -\langle -\frac{3}{2} | -\frac{3}{2} \rangle = -\xi_- \omega^3, \quad (A.5a)$$

$$\langle -\frac{3}{2} | \frac{3}{2} \rangle = \langle \frac{3}{2} | -\frac{3}{2} \rangle = \xi_+ \sin^3, \quad (A.5b)$$

$$\langle \frac{3}{2} | \frac{1}{2} \rangle = \langle -\frac{3}{2} | -\frac{1}{2} \rangle = \frac{1}{\sqrt{3}} \left(\xi_+ - 2 \frac{E_b}{M_b} \xi_- \right) \omega^2 \sin, \quad (A.5c)$$

$$\langle \frac{3}{2} | -\frac{1}{2} \rangle = -\langle -\frac{3}{2} | \frac{1}{2} \rangle = -\frac{1}{\sqrt{3}} \left(\mathcal{S}_- - 2 \frac{E_b}{M_b} \mathcal{S}_+ \right) \cos \sin^2, \quad (\text{A.5d})$$

$$\langle \frac{1}{2} | \frac{3}{2} \rangle = \langle -\frac{1}{2} | -\frac{3}{2} \rangle = \frac{1}{\sqrt{3}} \left(\mathcal{S}_+ + 2 \frac{E_d}{M_d} \mathcal{S}_- \right) \cos^2 \sin, \quad (\text{A.5e})$$

$$\langle -\frac{1}{2} | \frac{3}{2} \rangle = -\langle \frac{1}{2} | -\frac{3}{2} \rangle = -\frac{1}{\sqrt{3}} \left(\mathcal{S}_- + 2 \frac{E_d}{M_d} \mathcal{S}_+ \right) \cos \sin^2, \quad (\text{A.5f})$$

$$\langle \frac{1}{2} | \frac{1}{2} \rangle = -\langle -\frac{1}{2} | -\frac{1}{2} \rangle \quad (\text{A.5g})$$

$$= -\frac{\omega}{3} \left[(1-x) \mathcal{S}_+ \left(\frac{E_d}{M_d} - \frac{E_b}{M_b} \right) - \mathcal{S}_- \left(\frac{2q q'}{M_b M_d} + \frac{1}{2} \right) + x \mathcal{S}_- \left(2 \frac{E_b E_d}{M_b M_d} - \frac{1}{2} \right) \right],$$

$$\langle -\frac{1}{2} | \frac{1}{2} \rangle = \langle \frac{1}{2} | -\frac{1}{2} \rangle \quad (\text{A.5h})$$

$$= -\frac{\sin}{3} \left[(1-x) \mathcal{S}_- \left(\frac{E_d}{M_d} - \frac{E_b}{M_b} \right) + \mathcal{S}_+ \left(\frac{2q q'}{M_b M_d} - \frac{1}{2} \right) - x \mathcal{S}_+ \left(2 \frac{E_b E_d}{M_b M_d} - \frac{1}{2} \right) \right].$$

In addition to the coupling (A.5), there also exists a derivative coupling for $B^* PB^*$, which we do not write down.

As an example of how to divide the Born terms into polynomials and d-functions eq. (5.6), we discuss for reaction $4(PB \rightarrow VB^*)$ the amplitudes $T^{\text{Born}}(t, \lambda_c 3/2, 0 \frac{1}{2})$, which contain, for $\mu = \lambda_c - \lambda_d = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}$, the products

$$\langle -1|0 \rangle \langle \frac{3}{2} | \frac{1}{2} \rangle = -\sqrt{\frac{2}{5}} q^2 \eta_- d_{-\frac{1}{2}-\frac{5}{2}}^{\frac{5}{2}}, \quad (\text{A.7})$$

$$\langle 0|0 \rangle \langle \frac{3}{2} | \frac{1}{2} \rangle = \sqrt{\frac{2}{3}} q \eta_- \frac{1}{M_c} (q' E_a - q E_c x) \cdot d_{-\frac{1}{2}-\frac{3}{2}}^{\frac{3}{2}}, \quad (\text{A.8})$$

$$\langle 1|0 \rangle \langle \frac{3}{2} | \frac{1}{2} \rangle = \frac{1}{2} q^2 \eta_- (1-x^2) d_{-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}}, \quad (\text{A.9})$$

respectively. Expressions (A.7) and (A.9) are identical, the formal

difference being due to the different splitting into polynomial and d-function:

$$\cos^3 \sin^2 = \frac{1}{\sqrt{10}} d_{-\frac{1}{2}-\frac{5}{2}}^{\frac{5}{2}} = \frac{1}{4} (1-x^2) d_{-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}} \quad (\text{A.10})$$

We finish by discussing the degree n of the polynomial in (5.6). With pseudoscalar exchange and the couplings (A.1)-(A.5),

$$n \leq \frac{5}{2} - S$$

where S is given by (5.7). Since S has to be at least $\frac{1}{2}$ we get $n \leq 2$. Similarly, for vector exchange with the simplest coupling, $n \leq \frac{7}{2} - S$.

R E F E R E N C E S

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F I G U R E C A P T I O N S

Figure 1 Feynman graph for the Born terms.

Figure 2 Quasi-two-particle graph, in which the exchanged
pseudoscalar may be real.

FIG. 1

FIG. 2