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ON LOCAL QUANTUM FIELD THEORY WITHOUT ULTRA-VIOLET DIVERGENCES
- THIRD ORDER -

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ABSTRACT

It was shown ^{1),2)} in the non-linear local quantum theory of the scalar one-component field that the difficulties of the local theory associated with the presence of the ultra-violet divergences can be removed in second order perturbation theory by the introduction of an interaction Lagrangian satisfying certain requirements. In the present paper the third order of perturbation theory is investigated and it is shown that no new restrictions appear on the interaction Lagrangian. Unitarity is fulfilled for the Green's function in this order.

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I. INTRODUCTION

In the preceding paper ^{1),2)} the class of non-linear Lagrangians in quantum field theory of one component scalar field was considered. The Lagrangian density is written down as

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_I(x) \quad (\text{I.1})$$

$$\mathcal{L}_0(x) = -\frac{1}{2} \left(m^2 \varphi^2(x) - \frac{\partial \varphi(x)}{\partial x_\mu} \frac{\partial \varphi(x)}{\partial x^\mu} \right) \quad (\text{I.2})$$

$$\mathcal{L}_I(x) = -g U(\varphi(x)) \quad (\text{I.3})$$

It was shown that ultra-violet divergences are absent in the second order of perturbation theory if the interaction functions $U(\alpha)$, treated as functions of the complex variable α , possess the following properties :

- i) $U(\alpha)$ are continuous and have no singularities along the real axis, and, near the point $\alpha = 0$, can be expanded in a Taylor series with a certain convergence radius ρ :

$$U(\alpha) = \sum_{m=0}^{\infty} \frac{u_m}{m!} \alpha^m \quad (\text{I.4})$$

- ii) in the complex α plane the functions $U(\alpha)$ may only have cuts with degree less than one, i.e., the integral of $|U(\alpha)|^2$ must exist over any bounded region in the complex α plane.
- iii) at infinity the functions $U(\alpha)$ satisfy the condition

$$\lim_{|\alpha| \rightarrow \infty} \frac{U(\alpha)}{\alpha^2} = 0 \quad (\text{I.5})$$

For the interactions possessing these properties, it was possible to construct the amplitudes for physical processes in the second order of perturbation theory satisfying the unitarity condition and having the correct singularities in the complex momentum plane.

But, for a final answer to the question as to the possibility of constructing a local field theory without ultra-violet divergences, it is necessary to investigate the higher approximations of perturbation theory. Here there are two essential questions : firstly, will new restrictions on the class of permissible non-linear interaction Lagrangians appear (i.e., restrictions on the behaviour of functions $U(\alpha)$ in the complex α plane) ? Secondly, will it be possible to determine the amplitudes for physical processes as analytical functions of external momenta in accordance with the demands of unitarity and analyticity ?

In this article the third order of perturbation theory will be considered.

II. FORMULATION OF THE PROBLEM

The formulation of the problem in third order perturbation theory coincides completely with that of the second order.

The S matrix in the third order in the interaction representation can be written as

$$\begin{aligned}
 S_3 &= \frac{(-ig)^3}{3!} \iiint d^4x_1 d^4x_2 d^4x_3 T \{ U(\varphi(x_1)) U(\varphi(x_2)) U(\varphi(x_3)) \} = \\
 &= \frac{(-ig)^3}{3!} \iiint d^4x_1 d^4x_2 d^4x_3 \exp \left\{ \frac{1}{2} \iint d^4y_1 d^4y_2 \Delta(y_1 - y_2) \frac{\delta^2}{\delta \varphi(y_1) \delta \varphi(y_2)} \right\} \times \\
 &\quad \times U(\varphi(x_1)) U(\varphi(x_2)) U(\varphi(x_3)) = \\
 &= \frac{(-ig)^3}{3!} \iiint d^4x_1 d^4x_2 d^4x_3 \exp \left\{ \Delta(x_1 - x_2) \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} + \Delta(x_2 - x_3) \frac{\partial^2}{\partial \alpha_2 \partial \alpha_3} + \Delta(x_3 - x_1) \frac{\partial^2}{\partial \alpha_3 \partial \alpha_1} \right\} \times \\
 &\quad \times U(\alpha_1) U(\alpha_2) U(\alpha_3) \Big|_{\alpha_j = \varphi(x_j)}
 \end{aligned} \tag{II.1}$$

where

$$\Delta(x_i - x_j) = \langle 0 | T(\varphi(x_i) \varphi(x_j)) | 0 \rangle$$

In order to go over to the normal product in (II.1) it is necessary to find the function

$$\begin{aligned}
 Y_3(\Delta_1, \Delta_2, \Delta_3; \alpha_1, \alpha_2, \alpha_3) &= \exp \left\{ \Delta_3 \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} + \Delta_1 \frac{\partial^2}{\partial \alpha_2 \partial \alpha_3} + \Delta_2 \frac{\partial^2}{\partial \alpha_3 \partial \alpha_1} \right\} U(\alpha_1) U(\alpha_2) U(\alpha_3) \\
 &= \sum_{m_1, m_2, m_3=0}^{\infty} Y_{m_1, m_2, m_3}(\Delta_1, \Delta_2, \Delta_3) \frac{\alpha_1^{m_1}}{m_1!} \frac{\alpha_2^{m_2}}{m_2!} \frac{\alpha_3^{m_3}}{m_3!}
 \end{aligned} \tag{II.2}$$

where

$$\Delta_3 = \Delta(x_1 - x_2), \quad \Delta_1 = \Delta(x_2 - x_3), \quad \Delta_2 = \Delta(x_3 - x_1)$$

The functions $Y_{m_1 m_2 m_3}(\Delta)$ in the expansion in powers of $\alpha_1, \alpha_2, \alpha_3$ are related to the radiation operators and are the expansion coefficients of the S matrix in a series of normal products of the operator $\varphi(x)$.

The amplitudes of physical processes are the Fourier transforms of the radiation operators $Y_{m_1 m_2 m_3}(\Delta)$.

The first problem is that of defining the following formal integral

$$\mathcal{K}_{m_1 m_2 m_3}(p_j^2) = \iiint d^4x_1 d^4x_2 d^4x_3 e^{ip_j x_j} Y_{m_1 m_2 m_3}(\Delta(x_1 - x_2), \Delta(x_2 - x_3), \Delta(x_3 - x_1)) \quad (\text{II.3})$$

where Y is a function of $\Delta(x)$.

The functions $\mathcal{K}(p_j^2)$ must have the correct analytical properties ³⁾ in the complex planes of $z_1 = p_1^2$; $z_2 = p_2^2$; $z_3 = p_3^2$. In addition we have to find the conditions on the functions $Y_{m_1 m_2 m_3}$ such that the ultra-violet divergences are absent.

The second problem is that of finding the functions $Y_{m_1 m_2 m_3}$ (II.2) if the Lagrangian interaction $U(\alpha)$ is known.

The result of the action of the operator in (II.2) is equivalent to the solution of the following system of partial differential equations :

$$\frac{\partial}{\partial \Delta_3} y_3 = \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} y_3 ; \frac{\partial}{\partial \Delta_2} y_3 = \frac{\partial^2}{\partial \alpha_1 \partial \alpha_3} y_3 ; \frac{\partial}{\partial \Delta_1} y_3 = \frac{\partial^2}{\partial \alpha_2 \partial \alpha_3} y_3 \quad (\text{II.4})$$

with the initial condition

$$y_3(0, 0, 0; \alpha_1, \alpha_2, \alpha_3) = U(\alpha_1) U(\alpha_2) U(\alpha_3) \quad (\text{II.5})$$

As in the second order it is sufficient, firstly, to have a solution to this equation which is only a formal series (II.2) (convergence of the series in the neighbourhood of the point $(\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0)$ is not necessary) and, secondly, to fulfil an initial condition weaker than (II.5) :

$$y_{m_1 m_2 m_3}(0, 0, 0) = u_{m_1} u_{m_2} u_{m_3} \quad (\text{II.6})$$

where u_m is determined by (I.4).

In addition, it is very important to determine the ranges of the variables $\Delta_1, \Delta_2, \Delta_3$ since the solutions of equations such as (II.4) have singularities at the points $\Delta_j = 0$.

Thus the following problems are posed :

- A) we have to define the Fourier integral (II.3) and consequently find the range of variation of the complex parameters Δ_j ;
- B) we have to solve the equations (II.4) with initial condition (II.6). Furthermore it is necessary to check that the properties of the functions $U(\alpha)$ enumerated in the introduction are such that the radiation operators $Y_{m_1 m_2 m_3}(\Delta)$ possess the necessary properties.

III. DEFINITION OF THE FOURIER INTEGRAL

Let us write the integral (II.3) in a simpler form :

$$\mathcal{K}(p_j^2) = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3) T(p_j^2) \quad (\text{III.1})$$

where

$$T(p_j^2) = \iiint d^4x_1 d^4x_2 d^4x_3 \delta^{(4)}(x_1 + x_2 + x_3) e^{i(p_1 x_2 - p_2 x_1)} y_{m_1 m_2 m_3}(\Delta(x_1), \Delta(x_2), \Delta(x_3)) \quad (\text{III.2})$$

Note that the functions $y_{m_1 m_2 m_3}$ can be obtained easily from (II.2) in the form of a formal series in $\Delta(x_j)$ by substituting in (II.2) the function $U(\alpha)$ in the form (I.4) and expanding in $\Delta(x_j)$:

$$y_{m_1 m_2 m_3}(\Delta(x_j)) = \sum_{n_1, n_2, n_3=0}^{\infty} u_{m_1+n_2+n_3} u_{m_2+n_3+n_1} u_{m_3+n_1+n_2} \frac{\Delta(x_1)^{n_1}}{n_1!} \frac{\Delta(x_2)^{n_2}}{n_2!} \frac{\Delta(x_3)^{n_3}}{n_3!} \quad (\text{III.3})$$

First, let us define the integral

$$T_{n_1 n_2 n_3}(p_j^2) = \iiint d^4x_1 d^4x_2 d^4x_3 \delta^{(4)}(x_1 + x_2 + x_3) e^{i(p_1 x_2 - p_2 x_1)} \Delta(x_1)^{n_1} \Delta(x_2)^{n_2} \Delta(x_3)^{n_3} \quad (\text{III.4})$$

As usual by the function $\Delta(x)$ we mean the causal function regularized by the Pauli-Villars method, so that the integral (III.4) is convergent. We take the limit which eliminates the regularization only in the final equations.

This integral is known to possess certain analytical properties ³⁾ in the complex plane of the variables

$$Z_1 = p_1^2, \quad Z_2 = p_2^2, \quad Z_3 = p_3^2 \quad (\text{III.5})$$

Because of the complicated nature of the analytical properties of the integral (III.4) in the variables z_1, z_2, z_3 we will proceed as follows. At first, we will obtain this integral in the so-called Euclidean region in the space of z_j , i.e., the range that consists of real z_j satisfying the inequalities

$$Z_1 < 0, \quad Z_2 < 0, \quad Z_3 < 0$$

$$2Z_1Z_2 + 2Z_2Z_3 + 2Z_3Z_1 - Z_1^2 - Z_2^2 - Z_3^2 > 0 \quad (\text{III.6})$$

In this range the amplitudes with three vertices are real.

The same result can be obtained from (III.4) if we change over to a Euclidean metric in the co-ordinate representation. By displacing the contour of integration in the complex t plane from $(-\infty, +\infty)$ to $(i\infty, -i\infty)$ and integrating over the angles in the four dimensional spaces x_j in (III.4) it is possible to obtain

$$T_{n_1 n_2 n_3}(Z_j) = \int_0^\infty \int_0^\infty \int_0^\infty d\zeta_1 d\zeta_2 d\zeta_3 \Delta_2^{n_1}(\zeta_2 + \zeta_3) \Delta_2^{n_2}(\zeta_3 + \zeta_1) \Delta_2^{n_3}(\zeta_1 + \zeta_2) V(\zeta_1, \zeta_2, \zeta_3; Z_1, Z_2, Z_3)$$

(III.7)

where

$$V(\sigma_j; z_i) = (\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \times \\ \times \frac{32\pi^3}{\sqrt{R}} \left\{ J_0\left(\sqrt{\frac{P - \sqrt{QR}}{2}}\right) - J_0\left(\sqrt{\frac{P + \sqrt{QR}}{2}}\right) \right\} \quad (\text{III.8})$$

$$P = P(\sigma_j, z_i) = (\sigma_2 + \sigma_3)^2 (z_1 - z_2 - z_3) + (\sigma_3 + \sigma_1)^2 (-z_1 + z_2 - z_3) + (\sigma_1 + \sigma_2)^2 (-z_1 - z_2 + z_3)$$

$$Q = Q(\sigma_j) = 16 \sigma_1 \sigma_2 \sigma_3 (\sigma_1 + \sigma_2 + \sigma_3)$$

$$R = R(z_j) = 2z_1 z_2 + 2z_2 z_3 + 2z_3 z_1 - z_1^2 - z_2^2 - z_3^2$$

J_0 is the Bessel function and $\Delta_2(\sigma)$ is real and is the causal function in the space-like region ^{4), 1)}

$$\Delta_2(\sigma) = \frac{1}{4\pi^2 \sigma} \left[\mu K_1(\mu \sigma) - M K_1(M \sigma) \right] \quad (\text{III.9})$$

The integral (III.7) is convergent in the region (III.6) because there the arguments of the Bessel functions in (III.8) are always real.

The function (III.2) in the Euclidean region can be determined as

$$T(z_j) = \int_0^\infty \int_0^\infty \int_0^\infty d\sigma_1 d\sigma_2 d\sigma_3 V(\sigma_1, \sigma_2, \sigma_3; z_1, z_2, z_3) \times \\ \times \mathcal{Y}(\Delta_2(\sigma_2 + \sigma_3), \Delta_2(\sigma_3 + \sigma_1), \Delta_2(\sigma_1 + \sigma_2)) \quad (\text{III.10})$$

Thus we must consider the function Y as a function of real positive variables. This function must be expandable in an asymptotic series (III.3).

The conditions for the absence of the ultra-violet divergences, i.e., the conditions for the convergence of the integral (III.10) near point $\sigma_j = 0$ are

$$\lim_{\Delta_j \rightarrow +\infty} \frac{Y(\Delta_1, \Delta_2, \Delta_3)}{\Delta_j^2} = 0 \quad (j=1, 2, 3) \quad (\text{III.11})$$

$$\lim_{\Delta \rightarrow +\infty} \frac{Y(\Delta, \Delta, \Delta)}{\Delta^4} = 0$$

where Δ_j is real and positive. These relations can be obtained by investigating the convergence of the integral (III.10) near the point $\sigma_1 = \sigma_2 = \sigma_3 = 0$.

Note that the integral (III.10) decreases when the z_j increase (z_j in the Euclidean region).

Then the problem consists in the analytical continuation of the function $T(z_j)$ into the physical region of the variables z_j . It is necessary to show that unitarity and analyticity will be fulfilled.

In this article we shall show that this is so in the case of the Green's function.

IV. T PRODUCT

In order to find the functions $Y_{m_1 m_2 m_3}(\Delta_1, \Delta_2, \Delta_3)$ let us use the same method as was used for the investigation of the second order ^{1), 2)}. We have

$$Y_3 = \exp \left\{ \Delta_3 \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} + \Delta_1 \frac{\partial^2}{\partial \alpha_2 \partial \alpha_3} + \Delta_2 \frac{\partial^2}{\partial \alpha_3 \partial \alpha_1} \right\} U(\alpha_1) U(\alpha_2) U(\alpha_3) \quad (\text{IV.1})$$

Every operator $\exp \left\{ \Delta \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \right\}$ can be represented as

$$\exp \left\{ \Delta \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \right\} = \frac{1}{\pi} \iint_{-\infty}^{\infty} d^2 t e^{-|t|^2 + \sqrt{\Delta} \left(t \frac{\partial}{\partial \alpha_i} + t^* \frac{\partial}{\partial \alpha_j} \right)} \quad (\text{IV.2})$$

where $t = x + iy$; $t^* = x - iy$; $d^2 t = dx dy$. By substituting (IV.2) in (IV.1) we obtain

$$Y_3 = \frac{1}{\pi^3} \iiint d^2 t_1 d^2 t_2 d^2 t_3 e^{-|t_1|^2 - |t_2|^2 - |t_3|^2} \times \quad (\text{IV.3})$$

$$\times U(\alpha_1 + t_3 \sqrt{\Delta_3} + t_2^* \sqrt{\Delta_2}) U(\alpha_2 + t_1 \sqrt{\Delta_1} + t_3^* \sqrt{\Delta_3}) U(\alpha_3 + t_2 \sqrt{\Delta_2} + t_1^* \sqrt{\Delta_1})$$

This integral is finite for the class of functions $U(\alpha)$ under investigation.

Let us show that it satisfies the Eqs. (II.4). We will use the following results. One can show that if in (IV.3) there are three different functions U_1, U_2 and U_3 instead of only the same U and if these functions have the following kind of singularity in the same point α_0

$$U_j(\alpha) \sim \frac{1}{(\alpha - \alpha_0)^{\gamma_j}} \quad (\gamma_j > 0)$$

then the integral (IV.3) is convergent if

$$\gamma_j < 2 \quad ; \quad \gamma_1 + \gamma_2 + \gamma_3 < 5$$

Now consider, for instance, the first equation in (II.4)

$$\begin{aligned} \frac{\partial}{\partial \Delta_3} y_3 &= \frac{1}{\pi^3} \iiint d^2 t_1 d^2 t_2 d^2 t_3 e^{-|t_1|^2 - |t_2|^2 - |t_3|^2} \times \\ &\times \frac{1}{2\sqrt{\Delta_3}} \left\{ t_3 U'(\beta_1) U(\beta_2) + t_3^* U(\beta_1) U'(\beta_2) \right\} U(\beta_3) \end{aligned} \quad (\text{IV.4})$$

where $\beta_1 = \alpha_1 + \sqrt{\Delta_3} t_3 + t_2^* \sqrt{\Delta_2}$ etc. The integral obtained is convergent for the class of functions under consideration because of the remark made above. By integrating in (IV.4) by parts we obtain

$$\begin{aligned} \frac{\partial}{\partial \Delta_3} y_3 &= \frac{1}{\pi^3} \iiint d^2 t_1 d^2 t_2 d^2 t_3 e^{-|t_1|^2 - |t_2|^2 - |t_3|^2} U'(\beta_1) U'(\beta_2) U(\beta_3) \\ &= \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} y_3 \end{aligned} \quad (\text{IV.5})$$

This integral is also finite. Thus, the first equation of (II.4) is satisfied. The other equations are analogously satisfied.

One can show from (IV.3) that the initial condition (II.5) is fulfilled.

To obtain the radiation operators $Y(3)_{m_1 m_2 m_3}(\Delta)$ we substitute

$$\begin{aligned} X_1 &= X'_1 - \frac{(-\alpha'_1 + \alpha'_2 + \alpha'_3)}{2\sqrt{\Delta_1}} \\ X_2 &= X'_2 - \frac{(\alpha'_1 - \alpha'_2 + \alpha'_3)}{2\sqrt{\Delta_2}} \\ X_3 &= X'_3 - \frac{(\alpha'_1 + \alpha'_2 - \alpha'_3)}{2\sqrt{\Delta_3}} \end{aligned} \quad (\text{IV.6})$$

and obtain

$$\begin{aligned} y_3 &= \frac{1}{\pi^3} \iiint d^2 t_1 d^2 t_2 d^2 t_3 \exp \left\{ - \left[\left(X_1 - \frac{-\alpha'_1 + \alpha'_2 + \alpha'_3}{2\sqrt{\Delta_1}} \right)^2 + \left(X_2 - \frac{\alpha'_1 - \alpha'_2 + \alpha'_3}{2\sqrt{\Delta_2}} \right)^2 + \right. \right. \\ &\quad \left. \left. + \left(X_3 - \frac{\alpha'_1 + \alpha'_2 - \alpha'_3}{2\sqrt{\Delta_3}} \right)^2 + y_1^2 + y_2^2 + y_3^2 \right] \right\} U(t_3 \sqrt{\Delta_3} + t_2^* \sqrt{\Delta_2}) \times \\ &\quad \times U(t_2 \sqrt{\Delta_2} + t_1^* \sqrt{\Delta_1}) U(t_1 \sqrt{\Delta_1} + t_3^* \sqrt{\Delta_3}) \end{aligned} \quad (\text{IV.7})$$

The functions $Y(3)_{m_1 m_2 m_3}$ can be easily obtained by expanding the exponent in a power series in $\alpha'_1, \alpha'_2, \alpha'_3$. The expression when m_1, m_2, m_3 are arbitrary is very lengthy and will not be given here.

It is also easy to check that the conditions (III.11) are fulfilled if the function $U(\alpha)$ satisfies (I.5).

We will give the expression for the S matrix in the third order of perturbation theory in the case of the interaction

$$U(\alpha) = \frac{\alpha^3}{(1+f\alpha^2)^{3/4}} \quad (\text{IV.8})$$

Let us represent Y_3 in the form

$$y_3 = e^{\Delta_j \frac{\partial^2}{\partial \alpha_i \partial \alpha_k}} (\alpha_1 \alpha_2 \alpha_3)^3 e^{-\Delta_j \frac{\partial^2}{\partial \alpha_i \partial \alpha_k}} \bar{y}_3 \quad (\text{IV.9})$$

$$\bar{y}_3 = e^{\Delta_j \frac{\partial^2}{\partial \alpha_i \partial \alpha_k}} \frac{1}{(1+f\alpha_1^2)^{3/4} (1+f\alpha_2^2)^{3/4} (1+f\alpha_3^2)^{3/4}} \quad (\text{IV.10})$$

After some calculations it is possible to obtain

$$\begin{aligned} \bar{y}_3 &= \frac{1}{\pi^3 2^{3/2}} \iiint_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(\alpha_1 \alpha_2 \alpha_3)^{3/4}} \int_0^1 d\sigma_1 \dots d\sigma_6 \delta\left(1 - \sum_{j=1}^6 \sigma_j\right) \times \\ &\times \int_0^\infty d\xi \xi^5 \exp\left\{-\frac{A^2 \xi}{1+a^2 \xi} - \frac{B^2 \xi}{1+b^2 \xi}\right\} \times \\ &\times \sum_{K_1+K_2+K_3=3} \frac{6! (-)^{K_2+K_3} B^{2K_1} (a^2-b^2)^{K_2} A^{2K_3}}{(2K_1)! K_2! (2K_3)! (1+b^2 \xi)^{2K_1+K_2+\frac{1}{2}} (1+a^2 \xi)^{2K_3+K_2+\frac{1}{2}}} \end{aligned} \quad (\text{IV.11})$$

where

$$B = (\sigma_1 + \sigma_2)\sqrt{x_1+1} + (\sigma_3 + \sigma_4)\sqrt{x_2+1} + (\sigma_5 + \sigma_6)\sqrt{x_3+1}$$

$$A = \sqrt{f} \left\{ (\sigma_1 - \sigma_2)\alpha_1 + (\sigma_3 - \sigma_4)\alpha_2 + (\sigma_5 - \sigma_6)\alpha_3 \right\}$$

$$a^2 = f \left\{ \Delta_1 (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6)^2 + \Delta_2 (\sigma_1 - \sigma_2 + \sigma_5 - \sigma_6)^2 + \Delta_3 (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4)^2 \right\}$$

$$b^2 = f \left\{ \Delta_1 (\sigma_3 - \sigma_4 - \sigma_5 + \sigma_6)^2 + \Delta_2 (\sigma_1 - \sigma_2 - \sigma_5 + \sigma_6)^2 + \Delta_3 (\sigma_1 - \sigma_2 - \sigma_3 + \sigma_4)^2 \right\}$$

This result is complicated but it is possible to show that this integral is convergent for any real and positive Δ_j and satisfies the initial condition (II.5) when $\Delta_j \rightarrow +0$.

Thus no new restrictions appear in the third order.

V. GREEN'S FUNCTION

It will be shown that the integral (III.7) in the case of the Green's function can be continued into the physical region in accordance with the conditions of unitarity and analyticity.

In the region $p^2 < 0$ the Green's function according to (III.10) (when $z_1 = z_2 = p^2$, $z_3 = 0$) equals

$$G(p^2) = \iiint_0^\infty d\sigma_1 d\sigma_2 d\sigma_3 a(\sigma_1, \sigma_2, \sigma_3) \frac{J_1((\sigma_1 + \sigma_2)\sqrt{-p^2})}{(\sigma_1 + \sigma_2)\sqrt{-p^2}} \times \\ \times Y_{110}(\Delta_2(\sigma_2 + \sigma_3), \Delta_2(\sigma_3 + \sigma_1), \Delta_2(\sigma_1 + \sigma_2)) \quad (V.1)$$

where

$$a(\sigma_1, \sigma_2, \sigma_3) = 64\pi^3 (\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \sqrt{\sigma_1 \sigma_2 \sigma_3 (\sigma_1 + \sigma_2 + \sigma_3)}$$

Here the function Y_{110} satisfies the conditions of (III.11) and can be expanded in an asymptotic series similar to (III.3)

$$Y_{110}(\Delta_2(\sigma_2 + \sigma_3), \Delta_2(\sigma_3 + \sigma_1), \Delta_2(\sigma_1 + \sigma_2)) = \\ = \sum_{n_1, n_2 = n_0}^\infty \sum_{n_3 = N_0}^\infty r_{n_1 n_2 n_3} \frac{\Delta_2^{n_1}(\sigma_2 + \sigma_3)}{n_1!} \frac{\Delta_2^{n_2}(\sigma_1 + \sigma_3)}{n_2!} \frac{\Delta_2^{n_3}(\sigma_1 + \sigma_2)}{n_3!} \quad (V.2)$$

where n_0 and N_0 are positive numbers, the powers with which the asymptotic series for the Green's function begins, and $r_{n_1 n_2 n_3} = r_{n_2 n_1 n_3}$ are the coefficients of the asymptotic series connected with u_m and introduced for the sake of convenience.

Y_{110} can be expanded in an asymptotic series in any two of the three variables $\Delta_2(\sigma_2 + \sigma_3)$, $\Delta_2(\sigma_1 + \sigma_3)$, $\Delta_2(\sigma_1 + \sigma_2)$ as can be seen from Chapter IV. For instance

$$Y_{110}(\Delta_2(\sigma_2 + \sigma_3), \Delta_2(\sigma_1 + \sigma_3), \Delta_2(\sigma_1 + \sigma_2)) = \sum_{n_1=n_0}^{\infty} \sum_{n_3=N_0}^{\infty} \frac{\Delta_2^{n_1}(\sigma_2 + \sigma_3)}{n_1!} \frac{\Delta_2^{n_3}(\sigma_1 + \sigma_2)}{n_3!} \mathcal{U}_{n_1+n_3} Y_{n_1+n_3, n_1}(\Delta_2(\sigma_1 + \sigma_3)) \quad (V.3)$$

where the function $Y_{k_1, k_2}(\Delta_2)$ is the radiation operator of the second order $(1, 2)$.

Since the asymptotic series begins with $n_1 = n_2 = n_0$, $n_3 = N_0$, $G(p^2)$ in (V.1) can easily be continued into the region $0 < p^2 < (n_0 + N_0)^2 \mu^2$ and in this region $G(p^2)$ will be real.

$$G(p^2) = \iiint_0 d\sigma_1 d\sigma_2 d\sigma_3 a(\sigma_1, \sigma_2, \sigma_3) \frac{I_1((\sigma_1 + \sigma_2)\sqrt{p^2})}{(\sigma_1 + \sigma_2)\sqrt{p^2}} \times Y_{110}(\Delta_2(\sigma_2 + \sigma_3), \Delta_2(\sigma_3 + \sigma_1), \Delta_2(\sigma_1 + \sigma_2)) \quad (V.4)$$

This integral is convergent. When $p^2 \geq (n_0 + N_0)^2 \mu^2$, it is divergent in the region of large σ . This means that $G(p^2)$ has a cut in the complex p^2 plane beginning at the point $p^2 = (n_0 + N_0)^2 \mu^2$.

To continue into the region

$$(n_0 + N_0)^2 \mu^2 < N^2 \mu^2 < p^2 < (N+1)^2 \mu^2 \quad (V.5)$$

let us proceed as follows. Let us represent the function Y_{110} in (V.4) in the

form

$$y = \bar{y} + A_1 + A_2 + B \quad (V.6)$$

where

$$\bar{y} = y - A_1 - A_2 - B \quad (V.7)$$

$$\begin{aligned} A_1 &= A_1(\Delta_2(\sigma_2 + \sigma_3), \Delta_2(\sigma_1 + \sigma_3), \Delta_2(\sigma_1 + \sigma_2)) = \\ &= \sum_{n_2 + n_3 \leq N} \frac{\Delta_2^{n_2}(\sigma_1 + \sigma_3)}{n_2!} \frac{\Delta_2^{n_3}(\sigma_1 + \sigma_2)}{n_3!} Q_{n_2 n_3}(\Delta_2(\sigma_2 + \sigma_3)) \\ A_2 &= A_2(\Delta_2(\sigma_2 + \sigma_3), \Delta_2(\sigma_1 + \sigma_3), \Delta_2(\sigma_1 + \sigma_2)) = \\ &= \sum_{n_1 + n_3 \leq N} \frac{\Delta_2^{n_1}(\sigma_2 + \sigma_3)}{n_1!} \frac{\Delta_2^{n_3}(\sigma_1 + \sigma_2)}{n_3!} Q_{n_1 n_3}(\Delta_2(\sigma_1 + \sigma_3)) \end{aligned} \quad (V.8)$$

$$Q_{n_2 n_3}(\Delta) = \mathcal{U}_{1+n_2+n_3} \left\{ \mathcal{Y}_{1+n_3, n_2}(\Delta) - \sum_{n_1=n_0}^{N-n_3} \frac{\Delta^{n_1}}{n_1!} \mathcal{U}_{n_1+1+n_3} \mathcal{U}_{n_1+n_2} \right\}$$

$$\begin{aligned} B &= B(\Delta_2(\sigma_2 + \sigma_3), \Delta_2(\sigma_1 + \sigma_3), \Delta_2(\sigma_1 + \sigma_2)) = \\ &= \sum_{\substack{n_1 + n_3 \leq N \\ n_2 + n_3 \leq N}} \sum_{n_1 n_2 n_3} \frac{\Delta_2^{n_1}(\sigma_2 + \sigma_3)}{n_1!} \frac{\Delta_2^{n_2}(\sigma_1 + \sigma_3)}{n_2!} \frac{\Delta_2^{n_3}(\sigma_1 + \sigma_2)}{n_3!} \end{aligned}$$

Let us substitute (V.6) in (V.4). The integral for $\bar{y}$ will be convergent in the region (V.5) and will be real. For continuation of the integrals containing A_1 , A_2 and B it is necessary to change the path of integration from the ray

in $\mathcal{Q}_{inj}$ then it will be valid in the case of an infinite sum (function of Δ_2).

Thus the imaginary part of the Green's function will be determined by the usual method : use

$$D^{(+)}(x) = \frac{1}{(2\pi)^3 i} \int d^4k \theta(k_0) \delta(k^2 - \mu^2) e^{ikx}$$

instead of the causal function $\Delta(x)$. Also, unitarity will hold because $\mathcal{Q}_{inj}(\Delta_2)$ is the radiation operator of the second order.

CONCLUSION

The investigation of the third order of the perturbation theory showed that no new restrictions appeared on the class of the interaction function $U(\alpha)$. It can be expected to be the same in the higher approximations, because the number of integrations (when we go from T product to N product in the S matrix) will grow as n^2 (n the order of perturbation theory), but at the same time the number of functions will grow as n .

The questions of unitarity and analyticity are more complicated. They have been demonstrated only for the Green's function in the third order, but the author hopes that they will be proved for all cases.

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R E F E R E N C E S

- 1) G.V. Efimov, preprint JINR (Dubna), E-1204, 1963, JETP 44, 2107 (1963).
- 2) G.V. Efimov, Phys. Letters 4, 314 (1963).
- 3) For example : R.E. Cutkosky, Lectures on Analytic Properties of the
S matrix in Perturbation Theory, NORDITA, 1962.
- 4) N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized
Fields, New York, 1959.
- 5) L.D. Landau, Nucl. Phys. 13, 181 (1959).
- 6) R.E. Cutkosky, Journ. Math. Phys. 1, 429 (1960).