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MUON CAPTURE IN NUCLEI

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A B S T R A C T

Theoretical calculations of the total capture rate for muons in certain light nuclei (^{40}Ca , ^{16}O , ^{12}C , ^4He) have been carried out with cognizance taken of the important role played by giant dipole resonances in the capture process. The calculation involves the following steps: (1) Relating the dipole contribution of the vector interaction to its unretarded value by the use of the ground state elastic form factor. (2) Relating the unretarded dipole part of the vector interaction to an integral over the photo disintegration cross-section (by the use of isotopic spin invariance) and using empirical photo-disintegration data to evaluate this contribution. (3) Using the Wigner supermultiplet theory to relate the matrix elements of the axial vector and induced pseudoscalar interaction to those of the vector interaction. Using the shell model (with and without the closure approximation) to evaluate other multipolar contributions and recoil correction terms. The assumptions implicit in the above steps have been examined in as much detail as present knowledge permits. In particular, arguments are given on the basis of several models to justify step (1). The basic assumption involved in the supermultiplet theory used in step (3) is that of weak spin dependence of the forces. This theory predicts that the giant electric dipole resonance is one of a family (vector supermultiplet) of giant resonances involving spins and isospins. Evidence for the existence of at least one other such resonance is shown to exist in inelastic electron scattering data on ^{16}O . The effect of spin dependence of the forces in splitting this giant resonance supermultiplet is investigated by examining the 0^- , 1^- , and 2^- particle-hole states calculated by Lewis for ^{16}O ; its effect in our calculations is relatively small. Agreement between theory and experiment is within the uncertainties of both (but with the possibility of a real discrepancy in ^4He) thus exhibiting general consistency of muon capture with the universality of weak interactions. Inelastic electron scattering data can be a rich source of information to serve as the basis for better calculations and for exploring the giant resonance supermultiplet.

1. INTRODUCTION

The problem of fitting the phenomenon of muon capture into the general framework of a universal theory of weak interactions is an important and challenging one and has therefore been the subject of considerable theoretical and experimental work. The primary aim of such investigations is the determination of the nature of the fundamental muon capture interaction and its consistency with some appropriately formulated universal Fermi interaction theory (U.F.I.), although one can also hope eventually to turn the picture around and use muon capture as a tool for studying nuclear structure. To achieve these aims, one must determine from experimentally observed capture rates the basic parameters entering into the fundamental interaction. Here one is faced with two problems. First there are questions of an atomic or molecular nature since the rate is determined in part by the atomic or molecular wave function of the muon before capture. This part of the problem presents no real difficulties except in the case of capture in hydrogen where the formation of μ -molecules plays an important role. Secondly, however, there is the necessity of evaluating nuclear matrix elements of appropriate operators. Except in the case of hydrogen, this is a complicated problem since one must generally depend on the use of some kind of nuclear model and the question of its suitability for the calculations required becomes a critical one. As always in such cases, it is helpful to be able to correlate the required nuclear structure information with information which can be obtained from other experiments.

Leaving aside the case of capture in hydrogen with its own special problems, there are, generally speaking, three experimental approaches to the muon capture problem and for each an associated theoretical problem concerning nuclear matrix elements : one may observe muon capture to a definite final state of the daughter nucleus in which case one requires specific information on the initial and final state nuclear wave functions. Or one may study the systematics

of total capture rates in heavy nuclei with the aim of determining the "capture properties" of extended "nuclear matter" in the hope of thus freeing oneself from the details of individual wave functions for nuclear states. Such an approach was initiated by Primakoff ¹⁾ and considered further by Tolhoek and his collaborators ²⁾, Klein and Wolfenstein ³⁾, and Bell and Løvseth ⁴⁾. Finally, one can consider total capture rates of selected nuclei, usually the lighter nuclei, where a particular nuclear model may be employed. Here it is hoped that by suitable selection the deficiencies of any such model may be minimized when transitions to many final states are summed even if the transition rate to any one particular final state is not quantitatively reliable. A study of this last kind is the recent work of Luyten, Rood, and Tolhoek ⁵⁾ (hereafter referred to as LRT) on the nuclei ^{16}O and ^{40}Ca to which we shall refer in more detail below.

In the present paper our attention will be principally directed to the last of these approaches though our conclusions have an important bearing on the second approach as well ^{*)}. We shall pay particular attention to the doubly magic nuclei ^4He , ^{16}O , and ^{40}Ca but shall also give results for ^{12}C and indicate how our approach can be extended to other nuclei where the mass number is a multiple of 4. The study of ^{16}O and ^{40}Ca by LRT on the basis of the single particle model has yielded results which appear to be at variance with expectations based on the universal Fermi interaction. While it has not been seriously suggested that this is evidence against UFI, nevertheless it is clearly important to establish whether there is reason for concern, particularly since Klein and Wolfenstein ³⁾ as well as Bell and Løvseth ⁴⁾ find discrepancies in the method employing systematics of heavy nuclei as well.

Our principal thesis here is that in real nuclei, the interactions between nucleons are known to give rise to collective excited states, and these

^{*)} This will be discussed in a forthcoming publication.

collective states can play an important, and sometimes, at least, a dominant role in the capture process. For example, one knows ^{1),5)} that on the average roughly 20 MeV is left in the nucleus in these light nuclei so that the excitation is in the region in which the giant dipole resonance is the predominant feature of nuclear structure ^{*)}. We shall see, moreover, that since the allowed transitions are greatly suppressed by the Pauli principle, most of the capture takes place through the first forbidden dipole matrix elements (this was already pointed out in the original fundamental paper on muon capture in nuclei by Tiomno and Wheeler ⁷⁾). In fact, in the shell model everything can be reduced to matrix elements of the vector interaction involving nucleon operators $\tau^{(-)}_e - i\vec{p}_\nu \cdot \vec{x}$ ($\vec{p}_\nu$ is the neutrino momentum) and LRT find that the first forbidden contributions, those leading to final 1^- states of the nuclei, make up 90% of the capture rate in ^{16}O and 75% in ^{40}Ca . Now it is known that isotopic spin is a good quantum number for these light nuclei so that one can very simply relate matrix elements of $\tau^{(-)}$ to those of $\tau^{(3)}$. In this manner one can reduce the operator describing the first forbidden part of the vector interaction $\tau^{(-)}_x$ to $\tau^{(3)}_x$ which is exactly the same operator as describes photo-absorption and it is known that the dominant feature of photo-absorption is the giant dipole resonance. Here, however, the effects of the internucleon forces become crucial for while such a giant dipole resonance appears in a simple harmonic oscillator shell model, it appears at much too low an energy and it is now known through the work of

*) The significance of giant resonances in the μ capture process has been independently noted by Balashov, Belyaev, Kabachnik, and Eramzhian ⁶⁾ who use the resonance picture as a basis for estimating the neutron spectrum from μ capture in ^{16}O and ^{40}Ca but make no attempt to calculate total capture rates. We thank Professor D. Wilkinson for bringing this work to our attention.

Elliott and Flowers ⁸⁾ and Brown and his collaborators ⁹⁾ that it is the interactions between the particles which lead to the raising of this state to its correct energy. The energies of the participating states in muon capture play an important quantitative role because of the strong dependence of the capture rate on the neutrino energy ^{*)}. We shall use the above observations to relate the muon dipole capture rate to an integral over the experimental photo-absorption cross-section. It is also true that the effect of retardation of the dipole operators is summarized completely in the harmonic oscillator shell model calculation in terms of the ground state form factor of the nucleus and we shall give some arguments to show this is very likely also the case in the presence of internucleon interactions. This then leaves only the higher multipole contributions to be evaluated. Since nothing is known concerning collective states which would be important for these and since their contribution is relatively small, we have simply employed a closure approximation evaluated in the shell model. This gives, we feel, a very reliable procedure for computing the vector contribution to total muon capture rates ^{**)}.

^{*)} Using our ideas discussed above, Sens et al. ¹⁰⁾ have recently shown that by simply shifting the energies of the shell model dipole states of LRT to their empirically observed values, approximate consistency with UFI is established for ^{16}O and ^{40}Ca . The extent to which this is a valid procedure is discussed in the conclusion.

^{**)} We note that the conserved vector current theory ¹¹⁾ allows one to exactly relate the vector part of muon capture to electron scattering from the $T=0$ ground state to the $T=1$ excited states with the same energy loss and momentum transfer to the nucleus but that the existing electron scattering data is not much help in evaluating the total capture rates.

The main remaining problem is to discuss the relation between the matrix elements of the axial vector operators $\vec{\sigma} \tau^{(-)} e^{-i\vec{p}_V \cdot \vec{x}}$ and those of the vector operators $\tau^{(-)} e^{-i\vec{p}_V \cdot \vec{x}}$ in the presence of inter-nucleon forces. In the basic single particle model the contributions of the squares of these matrix elements stand in the simple ratio of 3:1. Here we can get some insight by assuming that the inter-nucleon potential is primarily of a Wigner and Majorana type with only a weak spin dependence. Such a potential is known to give the main features of low energy nucleon-nucleon scattering. In this case one can employ Wigner's supermultiplet approximation ¹²⁾ to show that the axial vector matrix elements stand in the same ratio to the vector matrix elements as above, giving us some confidence in the calculation of total muon capture rates. The states to which transitions are possible via the axial vector interaction are members of the same supermultiplet in each case as the states to which transitions via the vector interaction are possible. This has the important implication that there exist other giant resonances in the form of collective modes of nuclei which together with the giant dipole electromagnetic resonance comprise a supermultiplet in Wigner's sense. The existence of such collective states of nuclei has been speculated upon by Glassgold, Heckrotte, and Watson ¹³⁾ and by Fallieros, Ferrell, and Pal ¹⁴⁾ and others and indeed may be thought of naively as just the generalizations to spin and isotopic spin of the Goldhaber and Teller ¹⁵⁾ model of the giant dipole electromagnetic resonance where the protons oscillate against the neutrons. We shall discuss these states in more detail in the text and shall see that there is already some evidence from electron scattering for their existence.

It still remains, however, to ascertain the role of spin dependent forces, including the spin-orbit force in nuclei, in possibly upsetting our conclusions. For information on this point we rely on wave functions computed by Lewis ¹⁶⁾ *) for the states arising from the $p^{-1}d$ and $p^{-1}s$ configurations of $^{16}_0\text{O}$ and use these to study the relation between the distribution of vector and axial vector strength.

*) These calculations are similar in spirit to those carried out by Elliott and Flowers ⁸⁾ and Brown ⁹⁾.

2. PRELIMINARY REMARKS

If we make the following assumptions about the nature of the weak interactions :

- (1) the conserved vector current theory ¹¹⁾ (CVC) is valid;
- (2) the strong interaction vector current is even under $G = Ce^{-i\pi T_2}$ while the axial vector current is odd (i.e., only "first class currents" exist ¹⁷⁾);

then the matrix element of the interaction for the basic μ capture process

$$\mu^- + p \rightarrow \nu_\mu + n \quad (2.1)$$

illustrated in Fig. 1 is given by ^{*)}

$$q = p_\nu - p_\mu = p_1 - p_2$$

Fig. 1

Muon capture on a nucleon

^{*)} We use a metric such that $a_\mu = (\vec{a}, ia_0)$ and the γ matrices are Hermitian $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$.

$$H_{fi}^W = \frac{1}{\Omega} \frac{G}{\sqrt{2}} j_{\lambda}^{(l)} J_{\lambda}^{(s)} \quad (2.2)$$

Ω is our normalization volume, $G = (1.01 \times 10^{-5})/(M^2) (\hbar^3)/c$ is the usual weak interaction coupling constant (M = mass of proton) and

$$j_{\lambda}^{(l)} = \bar{u}(\vec{p}_\nu) \gamma_{\lambda} (1 + \gamma_5) u(\vec{p}_\mu) \quad (2.3)$$

$$J_{\lambda}^{(s)} = \bar{u}(\vec{p}_2) \left[F_1(q^2) \gamma_{\lambda} + F_2(q^2) \sigma_{\lambda\rho} q_{\rho} + F_A(q^2) \gamma_5 \gamma_{\lambda} - i g_{\lambda} \gamma_5 F_P(q^2) \right] \tau^{(-)} u(\vec{p}_1) \quad (2.4)$$

where

$$\tau^{(-)} \equiv \frac{1}{2} (\tau^{(1)} - i \tau^{(2)}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$q = p_\nu - p_\mu = p_1 - p_2$$

In μ -capture on a proton at rest the four momentum transfer squared is given by

$$q^2 = \frac{(m_{\mu}c)^2}{1 + \frac{m_{\mu}}{M}} = .90 (m_{\mu}c)^2 \quad (2.5)$$

In general, neglecting the momentum of the initial muon which is of the order of $|\vec{p}_{\mu}| \sim (\alpha Z) m_{\mu}$, one has

$$q^2 = 2(m_{\mu}c) p_z - (m_{\mu}c)^2 \quad (2.6)$$

Using the fact that on the average in nuclei $\langle p \rangle \approx 85 \text{ MeV}^{1),5)}$, one has

$$q^2 \approx .61 (m_\mu c)^2 \quad (2.7)$$

We therefore make the following evaluations of the form factors in (2.4)

$$i) \quad F_1[.61(m_\mu c)^2] \approx 1 - \frac{1}{6} [.61 \left(\frac{m_\mu c}{\hbar} \right)^2] \langle r_p^2 \rangle = .983 \quad (2.8)$$

This follows from the CVC of Ref. ¹¹⁾ and a proton radius of $\langle r_p^2 \rangle = 0.60 \text{ f}^2$ ¹⁸⁾

$$ii) \quad F_2[.61(m_\mu c)^2] = \left(\frac{\lambda_p - \lambda_n}{2Mc} \right) F_1[.61(m_\mu c)^2] = \frac{3.69}{2Mc} \quad (2.9)$$

This also follows from CVC and the fact that the momentum transfer dependence of F_2 is experimentally that of F_1 for low q^2 ¹⁸⁾.

$$iii) \quad F_A[.61(m_\mu c)^2] \approx F_A(0) = -1.15 \pm 0.02 \quad (2.10)$$

This follows from the fact that F_A can be expected to have only a weak q^2 dependence for small q^2 ¹⁹⁾ and from an identification of $F_A(0)$ with the value obtained from neutron decay ^{*)}. The value (2.10) is a recent evaluation of the data on the $f\tilde{\tau}$ value of the neutron ²⁰⁾. Finally, we make the further assumption that the induced pseudoscalar coupling constant is given by the relation ^{21),22),23)}.

*) Even if one takes $F_A(q^2) = -1.15 F_1(q^2)$, the momentum transfer dependence of $F_1(q^2)$ introduces only a 2% correction to F_A , as seen in (2.8).

$$\text{iv) } F_p[.61(m_\mu c)^2] = \frac{2Mc}{(m_\pi c)^2 + .61(m_\mu c)^2} F_A(0) = \frac{7.5}{m_\mu c} F_A(0) \quad (2.11)$$

This is the same set of assumptions as used in LRT⁵⁾ except for the use of the value of q^2 given by (2.7) instead of (2.5) and the new value of F_A in (2.10). We shall be primarily concerned with seeing whether the total capture rate in nuclei is consistent with this set of assumptions on the coupling constants.

We can now make the reduction of (2.4) to two-component spinors [we assume non-relativistic muons in (2.3) in making the reduction of (2.2)] and then to a sum over all nucleons treated as free nucleons to obtain the nuclear operator in the usual fashion^{19),5)} and find

$$\begin{aligned} G_{op}^{\vec{J}(s)} &= -i \sum_{j=1}^A e^{-\frac{i}{\hbar} \vec{p}_\mu \cdot \vec{x}(j)} \phi_\mu(\vec{x}(j)) \tau^{\pm}(j) \left\{ G_A \vec{\sigma}(j) - G_P \hat{p}_\nu (\vec{\sigma}(j) \cdot \hat{p}_\nu) + g_V \frac{\vec{p}(j)}{Mc} - g_A \hat{p}_\nu \frac{\vec{\sigma}(j) \cdot \vec{p}(j)}{Mc} \right\} \\ G_{4op}^{\vec{J}(s)} &= \sum_{j=1}^A e^{-\frac{i}{\hbar} \vec{p}_\mu \cdot \vec{x}(j)} \phi_\mu(\vec{x}(j)) \tau^{\pm}(j) G_V \end{aligned} \quad (2.12)$$

In this expression $\phi_\mu(\vec{x}(j))$ is the muon wave function. Some of the $1/M$ corrections are written explicitly as $\vec{p}(j)/M$ [they give contributions $\sim (v_N)/c$] and the remaining terms which have the form $(p_N)/M$ have been lumped into the "effective coupling constants"

$$G_V = G F_1[.61(m_\mu c)^2] \left(1 + \frac{p_\nu}{2Mc}\right) = 1.03 G$$

$$G_A = G \left[F_A(0) - \frac{p_\nu}{2Mc} \left\{ F_1[.61(m_\mu c)^2] + 2M F_2[.61(m_\mu c)^2] \right\} \right] = -1.36 G$$

$$G_P = G \frac{p_\nu}{2Mc} \left\{ 7.5 F_A(0) - F_A(0) - (F_1 [.61(m_\mu c)^2] + 2M F_2 [.61(m_\mu c)^2]) \right\} = -.54G$$

$$g_V = G F_1 [.61(m_\mu c)^2] = .98G$$

$$g_A = G F_A(0) = -1.15G$$

$$G = \frac{1.01 \times 10^{-5}}{M^2} \frac{\hbar^3}{c} \quad (2.13)$$

Following Refs. 18) and 5) we have used an average value of $p_\nu = 85 \text{ MeV}$
 $\overline{L(p_\nu)}/(2Mc) = 0.045$ to evaluate the correction terms.

We can now proceed to the total capture rate in the standard fashion.

We make one more assumption along the way :

We take $|\phi_\mu|_{av}^2$ out of the nuclear matrix element. Since the muon wave function varies only slowly over the nucleus, this is a very good assumption as has been shown by Sens 24).

Summing over final nuclear spin states and averaging over initial states one arrives at the total capture rate 5), 19)

$$\Lambda_{\mu c} = \frac{p_\mu^2 |\phi_\mu|_{av}^2}{2\pi \hbar^2 c} \left[G_V^2 M_V^2 + 3G_A^2 M_A^2 + (G_P^2 - 2G_P G_A) M_P^2 \right] + \Lambda'_{\mu c} \quad (2.14)$$

Here

$$M_V^2 = \overline{\sum_a \sum_b \left(\frac{v_{ab}}{v_\mu} \right)^2} \int \frac{d\hat{v}}{4\pi} \left| \langle b | \sum_{i=1}^A \tau^{(-)}(i) e^{-i \vec{v}_{ab} \cdot \vec{x}(i)} | a \rangle \right|^2 \quad (2.15)$$

$$M_A^2 = \frac{1}{3} \sum_a \sum_b \left(\frac{\nu_{ab}}{\nu_\mu} \right)^2 \int \frac{d\hat{\nu}}{4\pi} \left| \langle b | \sum_{i=1}^A \tau^{(-)}(i) \vec{\sigma}(i) e^{-i \vec{\nu}_{ab} \cdot \vec{x}(i)} | a \rangle \right|^2 \quad (2.16)$$

$$M_P^2 = \sum_a \sum_b \left(\frac{\nu_{ab}}{\nu_\mu} \right)^2 \int \frac{d\hat{\nu}}{4\pi} \left| \langle b | \sum_{i=1}^A \tau^{(-)}(i) \hat{\nu} \cdot \vec{\sigma}(i) e^{-i \vec{\nu}_{ab} \cdot \vec{x}(i)} | a \rangle \right|^2 \quad (2.17)$$

The notation is as follows :

$$\begin{aligned} |a\rangle &= \text{initial state of the nuclear system,} \\ |b\rangle &= \text{final state of the nuclear system,} \\ \vec{\nu}_{ab} &= \nu_{ab} \hat{\nu} = \vec{p}_\nu / \hbar \text{ for the nuclear transition } a \rightarrow b, \\ \hbar \nu_\mu c &\equiv m_\mu c^2. \end{aligned}$$

The remainder of the nucleon recoil corrections, those terms contained in $\Lambda'_{\mu c}$ (and not in the effective coupling constants), are discussed in section 6. They represent corrections of the order of 10% of the terms we have written. Our immediate concern will be the evaluation of the sums over nuclear matrix elements which comprise M_V^2 , M_A^2 , and M_P^2 .

3. ANALYSIS OF THE SHELL MODEL CALCULATIONS

In this section we analyze the shell model calculations of LRT in order to extract certain features which will serve as the basis for our further considerations. LRT show that for $^{16}_0, ^{40}_{\text{Ca}}$ as well as in certain other cases the following relations hold :

$$M_V^2 = M_A^2 = M_P^2. \quad (3.1)$$

In fact these relations have been assumed in almost all calculations of total capture rates. We shall not reiterate the arguments for their validity in the shell model since we shall discuss them later under less restrictive assumptions. Accepting them, our problem is thus reduced to evaluating M_V^2 .

With isotopic spin a good quantum number and the ground state having $T = 0$, the sum over $|b\rangle$ in Eq. (2.15) for M_V^2 runs effectively only over states for which $T = 1, T_3 = -1$. If we introduce

$$T_{\pm} \equiv T_1 \pm iT_2 = \frac{1}{2} \sum_{j=1}^A [\tau^{(1)}(j) \pm i\tau^{(2)}(j)] \quad (3.2)$$

and note that

$$2\tau^{(\pm)}(j) = \mp [T_{\pm}, \tau^{(3)}(j)] \quad (3.3)$$

then we may write

$$M_V^2 = \frac{1}{4} \sum_a \sum_{b, b'} \left(\frac{v_{ab}}{v_{\mu}} \right)^2 \int \frac{d\vec{v}}{4\pi} \langle a | \sum_{i=1}^A \tau^{(3)}(i) e^{i\vec{v}_{ab} \cdot \vec{\chi}(i)} | b' \rangle \langle b' | T_+ | b \rangle \langle b | T_- | b \rangle \langle b | \sum_{i=1}^A \tau^{(3)}(i) e^{-i\vec{v}_{ab} \cdot \vec{\chi}(i)} | a \rangle$$

where we have used the fact that $T_{\pm} |a\rangle = 0$. Noting the fact that the states

$|b'\rangle$ and $|b''\rangle$ must be $T = 1, T_3 = 0$ states and that $|b\rangle, |b'\rangle, |b''\rangle$ must belong to the same isospin multiplet to get a non-vanishing contribution to the sums, we find that

$$M_V^2 = \frac{1}{2} \sum_a \sum_{b'} \left(\frac{\nu_{ab}}{\nu_\mu} \right)^2 \int \frac{d^3x}{4\pi} \left| \langle b' | \sum_{i=1}^A \tau^{(3)}(i) e^{i \vec{\nu}_{ab} \cdot \vec{x}(i)} | a \rangle \right|^2. \quad (3.5)$$

The sum on $|b'\rangle$ now effectively runs over states with $T = 1, T_3 = 0$, i.e., excited $T = 1$ states of the parent nucleus. The neutrino momentum in this expression is still $\vec{\nu}_{ab}$, corresponding to the correct transition energy. We may thus write

$$\hbar \nu_{ab} c = E_m - E_{b'a} \quad (3.6)$$

where the energy E_m is defined to be the energy which the neutrino would have if in the capture process the nucleus were to make a transition between two states belonging to the same isobaric multiplet. It differs from the muon rest energy $m_\mu c^2$ by the binding energy of the muon in the atom E_B , the neutron-proton rest energy difference $(M_n - M_p)c^2$, and by the Coulomb energy difference of corresponding isobaric states in the initial and final nucleus. To the accuracy we require it can be represented by the expression :

$$E_m = \hbar \nu_{ab} c = m_\mu c^2 - E_B - (M_n - M_p)c^2 + \frac{0.0115(Z-1)}{A^{1/3}} m_\mu c^2 \quad (3.7)$$

where Z and A are the atomic and mass numbers of the parent nucleus.

It is now convenient to divide the sum in Eq. (3.5) into the contributions of the various multipoles. If we define

$$M_{JM}(i) \equiv \int \nu_{ab} x_i Y_{JM}(\Omega_{x_i}) \quad (3.8)$$

and assume a $J = 0^+$ ground state, then

$$M_V^2 = 2\pi \sum_{b'} \left(\frac{\nu_{ab}}{\nu_\mu} \right)^2 \left| \langle b'_{J_{b'}} \| \sum_{i=1}^A \tau^{(3)}(i) m_{\frac{J_{b'}}{2}}(i) \| 0^+, T=0 \rangle \right|^2 \quad (3.9)$$

and we can classify the transitions as monopole ("allowed") if $J_{b'} = 0^+$, dipole ("first forbidden") if $J_{b'} = 1^-$, quadrupole ("second forbidden") if $J_{b'} = 2^+$, etc. LRT have noticed from their shell model calculations the following regularities in ^{16}O and ^{40}Ca :

- (a) Monopole transitions make a relatively small contribution to the sum (less than 4% in ^{16}O and less than 8% in ^{40}Ca). This can be understood in terms of the exclusion principle for if we look at

$$\sum_{i=1}^A \tau^{(3)}(i) m_{00}(i) \xrightarrow{\nu_{ab} \rightarrow 0} \frac{1}{\sqrt{4\pi}} \sum_{i=1}^A \tau^{(3)}(i) = \frac{2}{\sqrt{4\pi}} T_3 \quad (3.10)$$

which vanishes on the ground state. Thus the entire contribution to the monopole transitions comes from the retardation effects, and the calculations show these are small *).

*) For the nuclei in question, using $\langle \nu_{ab} \rangle \approx 85 \text{ MeV}$ one has

$$\begin{array}{ll} \langle \nu_{ab} \rangle R \cong 1.3 & \text{for } ^{16}\text{O} \\ \cong 1.8 & \text{for } ^{40}\text{Ca} \end{array}$$

as estimates of the maximum contributing angular momentum. Thus there is an eventual reduction of the higher multipoles and retardation terms, the reduction being greater in the lighter nuclei.

- (b) The principal contribution to the transition rate arises from dipole terms which account for 90% of the total in ^{16}O and 75% of the total in ^{40}Ca .
- (c) The quadrupole terms are also relatively small representing about 7% of the total in ^{16}O and about 15% in ^{40}Ca . Higher multipoles account for less than 3% in both cases.

One infers from the above that a reasonable evaluation of the capture rate requires primarily an accurate evaluation of the dipole contribution. One can also see, although not explicitly noted by these authors, that the entire dipole contribution to M_V^2 in the LRT calculation (single particle oscillator model) can be written

$$(M_V^2)_D = (M_V^2)_{\text{U.D.}} \left| F_{\text{el}}(\nu_{ab} = \nu_m - \nu_{\text{osc}}) \right|^2 \quad (3.11)$$

to within 0.3% in ^{16}O and 2% in ^{40}Ca . $(M_V^2)_{\text{U.D.}}$ is the unretarded dipole contribution obtained by letting $\nu_{ab} \rightarrow 0$:

$$(M_V^2)_{\text{U.D.}} = \frac{\nu_\mu^2}{6} \sum_a \sum_{b'} \left(\frac{\nu_{ab}}{\nu_\mu} \right)^4 \left| \langle b' | \sum_{i=1}^A \tau^{(3)}(i) \vec{x}(i) | a \rangle \right|^2 \quad (3.12)$$

The retardation effects, i.e., the effect of keeping the full $j_1(\nu_{ab}x)$ rather than $(\nu_{ab}x)/3$ in (3.8), are summarized in the ground state elastic form factor computed in the same model and evaluated for a neutrino momentum corresponding to a nuclear excitation of $\hbar\omega_{\text{osc}}$. This is just the position of the giant dipole resonance in the basic shell model. We will return to a discussion of the generality of this result later; we merely note it here as a feature of the LRT single particle model calculation.

4. EVALUATION OF $(M_V^2)_{\text{U.D.}}$

The matrix elements occurring in Eq. (3.12) will be recognized as just those which enter into the cross-section for photo-absorption by a nucleus in the unretarded dipole approximation. The unretarded dipole approximation is in fact a very good one for the photo-effect since the energies at which the major part of the photo-absorption cross-section occurs are sufficiently low that the photon wave-length is indeed long compared to nuclear dimensions. The total cross-section for absorption of a photon of energy E by a nucleus (in this approximation) can be written

$$\sigma_{\gamma}(E) = \sum_a \sum_{b'} \sigma_{a \rightarrow b'}(E) \quad E = \text{photon energy} \quad (4.1)$$

with

$$\sigma_{a \rightarrow b'}(E) = \frac{\pi^2 \alpha}{3} (E_{b'} - E_a) \left| \langle b' | \sum_{i=1}^A \tau^{(3)}(i) \vec{x}(i) | a \rangle \right|^2 \delta(E - E_{b'} + E_a), \quad (4.2)$$

the cross-section associated with the nuclear transition $a \rightarrow b'$. Here $\alpha = 1/137$ is the fine structure constant. Thus one can easily write $(M_V^2)_{\text{U.D.}}$ as an integral over $\sigma_{\gamma}(E)$

$$(M_V^2)_{\text{U.D.}} = \frac{Z_{\mu}^2}{2\pi^2 \alpha} \left(\frac{Z_m}{Z_{\mu}} \right)^4 \int_0^{E_m} \frac{(E_m - E)^4}{E_m^4} \frac{\sigma_{\gamma}(E)}{E} dE, \quad (4.3)$$

Experimental information on the photo-absorption cross-section would allow us to calculate directly the important quantity $(M_V^2)_{\text{U.D.}}$. In fact

the occurrence of the "guillotine factor" $\sqrt{(E_m - E)/(E_m)}^4 \frac{1}{E}$ in the integrand is very fortuitous : it makes the result less sensitive to the high energy behaviour of the photo-absorption cross-section which is experimentally less well known than its low energy behaviour. Thus for example, (4.3) is much better known than just $\int \sigma_f(E) dE$, the integrated photo-absorption cross-section. Furthermore, the important contribution to the integrand comes from sufficiently low energies that one does not have to worry about representing the total photo-absorption cross-section by its unretarded dipole approximation. Finally, the effect of including $|F_{el}(E_m - E)|^2$ under the integrand in (4.3) amounts to only a few % change in (3.11) where one uses $|F_{el}(E_m - E_{res})|^2$ for both ^{16}O and ^{40}Ca (note for large E , $E_m - E \rightarrow 0$ so that $F_{el} \rightarrow 1$).

A comparison of empirical information on nuclear photo-absorption with calculations based on the oscillator shell model immediately reveals an important inadequacy of the latter. According to the oscillator shell model of LRT the electric dipole photo-absorption cross-section would be concentrated in a single peak at an energy corresponding to the oscillator spacing which is 12.7 MeV in ^{16}O and 10.7 MeV in ^{40}Ca , for example. In fact, it is observed that very little photo-absorption occurs at these energies and that the major part of the cross-section is concentrated in a "giant dipole resonance" which lies at about 23 MeV in ^{16}O and 20 MeV in ^{40}Ca . Thus the single particle shell model severely underestimates the energies at which dipole transitions occur. This has been recognized for a long time in the theory of nuclear photo-disintegration and the reason for it has been clarified by the work of Elliott and Flowers⁸⁾, Brown, and others⁹⁾. The large energy shift is a result of dynamical correlations arising from residual two-particle interactions (or what is the same thing, particle-hole interactions) which act to concentrate the dipole oscillator strength in the highest energy states arising from the pertinent shell model configuration (there is of course also some shift in the configuration energies due to the spin-orbit force). The same phenomenon should then occur in muon capture, the important final energy states of the nucleus should lie considerably higher in energy than predicted by the shell model. This will have an important effect on the capture

rate as a result of the strong (approximately fourth power) dependence of the capture rate on the neutrino energy.

The dynamical correlations arising from residual forces could, and probably do, have an effect not only on the energies of the states to which transitions occur, but also on the matrix elements for the transitions, particularly when the two-particle forces are of an exchange character. It is well known that the effect of exchange forces in photo-disintegration is to increase the integrated total photo-absorption cross-section $\int_0^\infty \sigma_T(E) dE$ from its classical Thomas-Reiche-Kuhn sum rule value (which is the value obtained in the harmonic oscillator shell model) by perhaps as much as 100%. Some of this arises from the raising of the energy differences $(E_b, -E_a)$ in Eq. (4.2), but it is not clear that the entire change in the sum rule arises from this cause.

In any case, the use of Eq. (4.3) with an empirical photo-absorption cross-section will automatically take all dynamical correlation effects into account, and suggests that one uses a formula of the type

$$M_V^2 = |F_{el}(E_m - E_{res})|^2 \frac{\nu_\mu^2}{2\pi^2 \alpha} \left(\frac{\nu_m}{\nu_\mu} \right)^4 \int_0^{E_m} \left(\frac{E_m - E}{E_m} \right)^4 \frac{\sigma_T(E)}{E} dE + (M_V^2)_{O.M.} \quad (4.4)$$

In this expression $(M_V^2)_{O.M.}$ represents capture through the other multipoles $T = 1, J = 0^+, 2^+, 3^-, 4^+, 5^-$, etc. Since nothing is known considering the collective properties of these states, and since this is now a correction to the leading term, one can evaluate it in either a closure approximation or use the results of LRT for an explicit sum over partial transitions. We shall discuss the evaluation of (4.4) in more detail in section 6 but we feel there is little doubt concerning the superiority of such a procedure over blind dependence on the shell model calculations for the dipole transitions.

Granting this we may then claim that using (4.4) with $(M_V^2)_{O.M.}$ evaluated as indicated above that we should obtain a superior calculation of the

capture rate since we have made no other assumptions than those usually made but only have used a more reliable estimate for the term of major importance in the result.

This is not enough, however, to assure us that the values which one thus calculates are indeed accurate. The reason for this lies in the following considerations :

- (a) The relation $M_V^2 = M_A^2 = M_P^2$ as well as the relation between $(M_V^2)_{U.D.}$ and $(M_V^2)_D$ involving the elastic form factor were originally derived on the basis of the harmonic oscillator shell model. Now that one has cast doubts on the reliability of the shell model for evaluating $(M_V^2)_{U.D.}$ it is clear that these relations should be re-examined to determine whether dynamical correlations may appreciably modify them.
- (b) The effect of dynamical correlations is such as to decrease the dipole contributions to the capture rate from the values computed on the basis of the shell model. This tends to make the contributions from other multipoles (in particular the $T = 1, J = 2^+$) relatively more important. To what extent will dynamical correlations modify the contributions of other multipoles in the same ways as they modify dipole contributions ?

We now examine these questions.

5. RE-EXAMINATION OF ASSUMPTIONS IN THE PRESENCE OF INTERACTIONS

A. Elastic form factor.

The relation (3.11)

$$(M_V^2)_D = (M_V^2)_{u.D.} |F_{el}(E_m - E_{res})|^2 \quad (3.11)$$

where $E_m - E_{res}$ is the momentum transferred in exciting the nucleus to the giant resonance $T = 1$ states has been suggested by many people. It is a simple consequence of the basic idea of Goldhaber and Teller¹⁵⁾ that the neutrons make small oscillations as a unit against the protons, with the two groups each maintaining their ground state spatial distributions. This is shown in Ref.²⁵⁾. The same basic idea was formulated earlier by Fallieros, Ferrell, and Pal¹⁴⁾ who discussed collective states, in particular the giant resonance, in terms of the generator co-ordinate technique. Brink²⁶⁾ has also shown that for particles in an oscillator potential with harmonic oscillator forces between them, the giant dipole resonance corresponds simply to an excitation in the relative motion of the neutrons and protons, with each group maintaining its ground state charge excitation. The form factor for this transition (provided that the change in the relative co-ordinate is small compared to the size of the nucleus) again leads to the relation (3.11) for this is just what was shown in Ref.²⁵⁾. Thus, theoretically, Eq. (3.11) has some basis even in the presence of interactions. There is also experimental evidence that (3.11) is valid from the electron scattering experiments of Bishop and Isabelle²⁷⁾ who find that the form factor for electron excitation of the giant resonance in $^{16}_0$ is consistent with relation (3.11).

B. Supermultiplets and the relation $M_V^2 = M_A^2 = M_P^2$.

In this section we shall explore the question as to how well the relation $M_V^2 = M_A^2 = M_P^2$ may be expected to hold when the effects of residual interparticle forces on shell model states are taken into account. As the basis of a first essay into this question we consider the situation when the nuclear forces are almost spin-independent, that is where the basic interparticle interactions are essentially of the Wigner and Majorana type only. In this case the Wigner theory of supermultiplets is applicable. The basis of this theory is the observation that under the quoted conditions, the following fifteen Hermitian operators will commute with the Hamiltonian :

$$\begin{aligned} T^\alpha &= \frac{1}{2} \sum_{i=1}^A \tau^\alpha(i) & (\alpha=1,2,3) \\ S_\lambda &= \frac{1}{2} \sum_{i=1}^A \sigma_\lambda(i) & (\lambda=x,y,z) \\ Y_\lambda^\alpha &= \frac{1}{2} \sum_{i=1}^A \tau^\alpha(i) \sigma_\lambda(i) \end{aligned} \quad (5.1)$$

The first three of these are simply the components of the total isotopic spin of the nucleus, the second three the components of the total spin angular momentum of the nucleus, while the last nine are less familiar quantities. One can show that these operators are the infinitesimal generators of a 15 parameter Lie group, namely the special unitary group in four dimensions, SU_4 . Thus an eigenstate of the Hamiltonian will in general belong to some irreducible representation of the group SU_4 . A set of states

It will be noted that the operators entering the matrix elements involved in M_V^2 , M_A^2 , and M_P^2 are such operators with

$$\omega(i) = e^{-i \vec{v}_{ab} \cdot \vec{X}(i)}$$

and that the operators entering into $(M_V^2)_{U.D.}$ are also of this character but with

$$\omega(i) = \vec{X}(i)$$

Now we turn to the question of representations. The simplest irreducible representation of the group SU_4 is the one-dimensional identity representation. In this representation we have only one linearly independent state which is left invariant by all the transformations of the group, i.e., is a scalar under the group. In this case all the fifteen operators (5.1) when applied to the state annihilate it. Thus if $|0\rangle$ is such a state, we shall say that it constitutes a scalar supermultiplet, and we will have :

$$T^\alpha |0\rangle = S_\lambda |0\rangle = Y_\lambda^\alpha |0\rangle = 0. \quad (5.5)$$

Wigner¹²⁾ has shown that if the nuclear forces are short range and attractive in relative two-particle S-states, then the ground states of nuclei for which the mass number is a multiple of four and for which $T^{(3)} = 0$ (such as ${}^4\text{He}$, ${}^{12}\text{C}$, ${}^{16}\text{O}$, ${}^{28}\text{Si}$, ${}^{40}\text{Ca}$, for example) will be scalar supermultiplets. This implies from (5.5) that they are also isospin and spin singlets, of course.

Suppose that we have a state $|0\rangle$ which is a scalar supermultiplet and that we apply to it the fifteen operators which are the components of a vector operator. If the state is not annihilated by them, then one can show that one generates fifteen

linearly independent states. The application of any of the fifteen generators of the group to each of these states can then easily be shown to generate a linear combination of them simply by using the relations (5.3) defining a vector operator. Therefore these states constitute a basis for a representation of the group SU_4 . In fact one can show that the resultant 15-dimensional representation is irreducible. We shall call a set of 15 states which form a basis for this irreducible representation simply a vector supermultiplet. It will be noted from (5.3) that three of the states constitute an isospin triplet and spin singlet, three constitute a spin triplet and isospin singlet, while the remaining nine constitute a spin and isospin triplet. So long as the nuclear forces are spin-independent, the existence of any one of these states as eigenstates of the Hamiltonian implies the existence of the remainder as eigenstates degenerate with the original. The introduction of spin-dependent forces will first split the degeneracy of the states belonging to a supermultiplet into non-degenerate isobaric multiplets, and further, it will mix states belonging to different supermultiplets.

If we accept for the present the assumption of weak spin dependence of the nuclear forces, we can readily see the significance of supermultiplet structure in the theory of muon capture. Let us assume for the moment that there truly exists a sharp giant electric dipole resonance state of a nucleus which is a true eigenstate of the Hamiltonian. This state will then contain all the dipole oscillator strength, which means that it is generated by applying the operator

$$\sum_{i=1}^A \tau^{(3)}(i) \vec{X}(i) \quad (5.6)$$

to the ground state of the nucleus. Suppose that this ground state is a scalar supermultiplet as it would be for the nuclei in which we are interested. Then since (5.6) is a component of a vector operator, such a state must be one belonging to a vector supermultiplet and hence there exist 14 other states degenerate with it that are obtained from the ground state by applying the other 14 components of the vector operator to the ground state. In particular 3 of the other components of the vector operator are just the operators whose matrix elements enter into the expressions for M_A^2 and M_P^2 . In fact the only transitions from the ground state induced by each of these operators will be into the single state of the supermultiplet which it generates and with a transition matrix element which is simply related to the matrix element of (5.6). Indeed we shall show shortly that these relations are just such that the relation $M_V^2 = M_A^2 = M_P^2$ holds.

To see the physical basis of these collective states, we need only remember that the giant electric dipole resonance corresponds to a state in which protons oscillate against neutrons. The other states belonging to the "giant dipole" vector supermultiplet may then be identified as states in which protons with spin "up" and neutrons with spin "down" oscillate against protons with spin "down" and neutrons with spin "up", and states in which nucleons with spin "up" oscillate against nucleons with spin "down" (spin waves). Since the nuclear forces are spin and charge independent, the "force constants" for all these oscillations will be the same and they will all have the same frequencies and energies. The existence of such collective modes of motion of a nucleus have been considered in the past by Glassgold, Heckrotte, and Watson ¹³⁾, and by Fallieros, Ferrell, and Pal ¹⁴⁾, although we are not aware that the supermultiplet theory of Wigner has been brought to bear on the question before.

Even if a well-defined giant resonance does not exist, the general conclusions we draw above continue to hold when the spin-dependence of the forces is weak. The nuclear states $|b\rangle$ which give non-vanishing contributions to M_V^2 in the sum (2.15) will each still belong to a vector supermultiplet. For each such state there will be certain other states belonging to the same supermultiplet which will give contributions to the sums occurring in M_A^2 and M_P^2 . The associated matrix elements will all be related by a generalization of the Wigner-Eckart theorem. Thus the transition energies and matrix elements for the vector part of the μ capture interaction can be directly related to corresponding transition energies and matrix elements for the axial vector and induced pseudoscalar interactions in the multiplet theory. It remains only to show that these relations are such as to yield the desired equality between M_V^2 , M_A^2 , and M_P^2 ; this is accomplished by the use of a simple trick.

We remark first that the matrix elements entering into M_A^2 and M_P^2 are all of the form

$$\langle b | \sum_{i=1}^A \tau^{(-)}(i) \sigma_{\lambda}(i) e^{-i \vec{r}_{ab} \cdot \vec{x}(i)} | a \rangle$$

where $\sigma_{\lambda}(i)$ is some component of the spin of the i th nucleon. Let us define the three operators

$$U^1 \equiv Y_{\lambda}^1 \quad U^2 \equiv Y_{\lambda}^2 \quad U^3 \equiv T^3 \quad (5.7)$$

One can readily verify from (5.2) that they satisfy the commutation relations for the generators of the rotation group :

$$[U^i, U^j] = i \epsilon_{ijk} U^k \quad (5.8)$$

and if the state $|a\rangle$ is a scalar supermultiplet, then the application of any of them to this state annihilates it. Now with

$$U^- \equiv U^1 - i U^2 \quad (5.9)$$

we note that

$$\sum_{i=1}^A \tau^{(-)}(i) \sigma_{\lambda}(i) e^{-i \vec{\nu}_{ab} \cdot \vec{\chi}(i)} = \frac{1}{2} \left[U^-, \sum_{i=1}^A \tau^{(3)}(i) e^{-i \vec{\nu}_{ab} \cdot \vec{\chi}(i)} \right] \quad (5.10)$$

Thus an exact repetition of the arguments leading from (3.2) to (3.5) (the states $|b'\rangle$ and $|b''\rangle$ must now have $U = 1$, $U^3 = 0$ and $|b\rangle$, $|b'\rangle$, $|b''\rangle$ must now belong to the same "U spin" multiplet) leads to

$$\begin{aligned} \sum_b \left(\frac{\nu_{ab}}{\nu_{\mu}} \right)^2 \left| \langle b | \sum_{i=1}^A \tau^{(-)}(i) \sigma_{\lambda}(i) e^{-i \vec{\nu}_{ab} \cdot \vec{\chi}(i)} | a \rangle \right|^2 = \\ = \frac{1}{2} \sum_{b'} \left(\frac{\nu_{ab}}{\nu_{\mu}} \right)^2 \left| \langle b' | \sum_{i=1}^A \tau^{(3)}(i) e^{-i \vec{\nu}_{ab} \cdot \vec{\chi}(i)} | a \rangle \right|^2 \end{aligned} \quad (5.11)$$

or (since $\int \frac{d^3 \nu}{4\pi}$ and $\sum_{b'}$ can obviously be interchanged)

$$M_A^2 = M_P^2 = M_V^2. \quad (5.12)$$

Unfortunately, these results are not quite enough for our needs since the nuclear forces are spin-dependent and, in particular, the spin-orbit force in nuclei is not particularly weak. The effect of spin-dependence, as we noted earlier is to split the degeneracy of

states belonging to a supermultiplet so that the transitions induced by the axial vector and induced pseudoscalar interactions will be to states differing in energy from those to which the vector interaction induces transitions. Furthermore it causes mixing of the supermultiplets, i.e., the actual energy eigenstates will be linear combinations of states belonging to different supermultiplets. The resultant matrix element changes and energy shifts may then invalidate the equality of M_V^2 , M_A^2 , and M_P^2 which is so important for our argument. In order to obtain some information on this point, we have investigated what we obtain on the basis of more realistic wave functions for the case of $^{16}_0$.

C. Particle-hole theory of the giant resonance in $^{16}_0$.

Lewis ¹⁶⁾ has calculated the states arising from the $(2s)(1p)^{-1}$ and the $(1d)(1p)^{-1}$ configurations in $^{16}_0$ using the particle-hole theory ^{8),9)} of collective states. We are particularly interested in the 0^- , 1^- , 2^- states since these are the states which are connected by a first-forbidden orbital matrix element in M_V , M_A , and M_P ^{*)}. The allowed matrix elements in M_A are again expected to be small because of the results of section B or simply because the operator $\sum_{i=1}^A \sigma_{\lambda}(i) \tau^{(-)}(i)$ applied to the ground state vanishes because of the Pauli principle in a doubly closed shell nucleus. Lewis has calculated these states by taking configuration energies from neighbouring nuclei, thus taking into account the spin-orbit force, and by using a two-nucleon potential

$$V(1,2) = \left[{}^1V(r_{12}) {}^1P + {}^3V(r_{12}) {}^3P \right] \left[\frac{1 + P_M(1,2)}{2} \right] \quad (5.13)$$

*) We note that $M_A = M_P$ as long as $\vec{S} |a\rangle = 0$ where $\vec{S}$ is the total spin operator. We shall assume this to be true for $^{16}_0$ which is doubly magic.

fit to low energy nucleon-nucleon scattering. In this expression :

$$\begin{aligned} {}^1P &= \frac{1}{4} (1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2) \quad \text{singlet projection operator} \\ {}^3P &= \frac{1}{4} (3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2) \quad \text{triplet projection operator} \\ P_{ij} &= \text{Majorana space exchange operator} \end{aligned} \quad (5.14)$$

and ${}^1V(r_{12})$, ${}^3V(r_{12})$ were taken as Yukawa potentials with parameters chosen to give the correct singlet and triplet effective ranges and scattering lengths ¹⁶⁾.

$$V(r_{12}) = V_0 \frac{e^{-\mu r_{12}}}{\mu r_{12}} \quad (5.15)$$

$$\begin{aligned} {}^1V_0 &= -46.87 \text{ MeV} & {}^1\mu &= 0.8547 \text{ f}^{-1} \\ {}^3V_0 &= -52.13 \text{ MeV} & {}^3\mu &= 0.7261 \text{ f}^{-1} . \end{aligned}$$

The resulting energies and wave functions are given in Table 1 (together with the unperturbed configuration energies as obtained from neighbouring nuclei). In Table 2 we have listed the values of the matrix elements $\frac{1}{\eta^2} D$ and $\frac{1}{\eta^2} S_J$ associated with each transition. Here D and S_J are unretarded or long wave-length limit parts of

$$D = 4\pi \left| \left(T=1, J=1^- \parallel \sum_{i=1}^A \tau^{(-)}(i) M_1(\Omega_i) \parallel T=0, J=0^+ \right) \right|^2 \quad (5.16)$$

$\nu_{ba} \rightarrow 0$

$$S_J = 4\pi \left| \left(T=1, J^- \parallel \sum_{i=1}^A \tau^{(-)}(i) X_J''(\Omega_i) \parallel T=0, J=0^+ \right) \right|^2$$

$\chi_{ba} \rightarrow 0$

(5.17)

J^π	ENERGY (MeV)	$1d_{3/2} (1p_{3/2})^{-1}$ (22.76 MeV)	$1d_{5/2} (1p_{3/2})^{-1}$ (17.68 MeV)	$1d_{3/2} (1p_{1/2})^{-1}$ (16.60 MeV)	$1d_{5/2} (1p_{1/2})^{-1}$ (11.52 MeV)	$2s_{1/2} (1p_{3/2})^{-1}$ (18.55 MeV)	$2s_{1/2} (1p_{1/2})^{-1}$ (12.39 MeV)
0^-	27.28	0.998					-0.070
0^-	14.41	0.070					0.998
1^-	26.63	0.936	-0.072	-0.301		-0.165	0.012
1^-	23.89	0.214	0.881	0.322		0.245	0.119
1^-	21.01	0.143	-0.302	0.002		0.942	0.030
1^-	18.65	0.238	-0.339	0.897		-0.145	-0.051
1^-	14.63	-0.029	-0.113	-0.011		-0.063	0.991
2^-	24.52	0.979	0.135	0.042	-0.110	0.100	
2^-	21.34	-0.173	0.673	-0.185	-0.178	0.672	
2^-	20.01	0.017	-0.654	0.177	0.086	0.730	
2^-	18.69	-0.077	0.248	0.965	-0.020	-0.008	
2^-	13.85	0.076	0.201	-0.025	0.974	0.069	

T A B L E 1

Energies and wave functions for the $T=1$, $J^\pi=0^-, 1^-$, and 2^- states in ^{16}O calculated by Lewis ¹⁶⁾. We have indicated the unperturbed configuration energies under the basis states.

J^π	ENERGY	$\frac{1}{\eta^2} D$	$\frac{1}{\eta^2} S_0$	$\frac{1}{\eta^2} S_1$	$\frac{1}{\eta^2} S_2$
0^-	27.28		1.18		
0^-	14.41		0.14		
1^-	26.63	0.93		2.21	
1^-	23.89	2.91		0.79	
1^-	21.01	0.02		0.29	
1^-	18.65	0.06		0.43	
1^-	14.63	0.07		0.29	
2^-	24.52				1.27
2^-	21.34				3.88
2^-	20.01				0.17
2^-	18.69				0.00
2^-	13.85				1.33

T A B L E 2

The strengths $\frac{1}{\eta^2} D$ and $\frac{1}{\eta^2} S_J$, defined in Eqs. (5.10) and (5.11), for the states of Table 1. η is defined to be $\eta = \frac{1}{2} \nu_b a b_{osc}$ where b_{osc} is given by $\frac{\hbar^2}{Mb_{osc}^2} = \hbar \omega_{osc}$.

We have used the notation $\eta = b_{\text{osc}} \mathcal{V}_{ba}$ with $\frac{\hbar^2}{M \omega_{\text{osc}}^2} = \hbar \omega_{\text{osc}}$. S_J just represents the multipole analysis of the axial matrix elements and

$$X_{JM}''(\Omega_i) = \sum_{M'q} (1M'1q | 11JM) M_{1M'}(\Omega_i) \sigma_{1q}(i) \quad (5.18)$$

is the tensor product of η_1 and $\vec{\sigma}$ *). Thus we have

$$(M_V^2)_{\text{u.d.}} = \sum_{1^- \text{ states}} \left(\frac{\mathcal{V}_{ab}}{\mathcal{V}_\mu} \right)^2 D \quad (5.19)$$

and

$$(M_A^2)_{\text{u.d.}} = \frac{1}{3} \left\{ \sum_{0^- \text{ states}} \left(\frac{\mathcal{V}_{ab}}{\mathcal{V}_\mu} \right)^2 S_0 + \sum_{1^- \text{ states}} \left(\frac{\mathcal{V}_{ab}}{\mathcal{V}_\mu} \right)^2 S_1 + \sum_{2^- \text{ states}} \left(\frac{\mathcal{V}_{ab}}{\mathcal{V}_\mu} \right)^2 S_2 \right\} \quad (5.20)$$

for the contribution of each state to M_V^2 and M_A^2 . We note from Table 2 that

$$\sum_{0^-} \frac{1}{\eta^2} S_0 : \sum_{1^-} \frac{1}{\eta^2} S_1 : \sum_{2^-} \frac{1}{\eta^2} S_2 = 1 : 3 : 5 \quad (5.21)$$

*) The angular momentum notation follows Edmonds²⁸⁾.

$$\sum_{1^-} \frac{1}{\eta^2} D : \sum_{1^-} \frac{1}{\eta^2} S_1 = 1:1 \quad (5.22)$$

$$\sum_J \frac{1}{\eta^2} S_J : \sum_{1^-} \frac{1}{\eta^2} D = 3:1 \quad (5.23)$$

exactly as in the single particle harmonic oscillator case. This must be true since all we have done is to make a unitary transformation among these simple basis states. What is different, however, is now the distribution in energy of these various states. It will be seen that the 1^- states have almost all their vector strength in the upper two states (a general feature of this type of calculation according to Brown ⁹⁾). The interesting thing is that almost all the axial vector strength lies in these two states also. One might think of the upper 1^- state as the axial vector collective mode while the second 1^- state is the vector collective mode in our supermultiplet terms. There is experimental evidence that this is actually the case in $^{16}_0$. Lewis ¹⁶⁾ has analyzed the electron scattering data of Barber and Goldemberg ²⁹⁾ in terms of these wave functions. By looking in the backward direction, these experiments pick out the transverse electric dipole matrix elements. The transverse electric dipole operator has two parts, one is from the convection current and reduces to the familiar $\tau^{(3)}_{\vec{x}}$ in the long wave-length limit while the other comes from the nucleon magnetic moments and is of the form $\frac{|\vec{q}| \lambda}{2Mc} \tau^{(3)} \left[\vec{\sigma}_\lambda \cdot \frac{\vec{q}}{q} \right]$ where $|\vec{q}|$ is the momentum transfer and λ represents the difference of magnetic moments of the proton and neutron. This term is negligible for photons, but for electrons can be quite clearly seen. The fact that the upper two 1^- states have such large $\tau^{(3)} \left[\vec{\sigma}_\lambda \cdot \frac{\vec{q}}{q} \right]$ strengths results in two predictions :

- i) There is a pronounced minimum in the sum of the transverse electric dipole form factors for these two states as a function of momentum transfer.
- ii) There is an upward shift in the peak of the giant resonance of about 3 MeV as the momentum transfer is increased because of the shift in the dipole strength from the second to the first 1^- state. This is a result of the interference of the two terms contributing to the transverse electric dipole operator.

Both of these effects are clearly seen experimentally²⁹⁾. The situation is very similar in ^{12}C where both of these effects are also seen³⁰⁾. Unfortunately, the experiments are not yet accurate enough to get good values of S_1 , but the particle-hole picture is completely consistent with the experiments. We also see from Table 2, that the 0^- states have almost all their strength in exactly the same region that the 1^- do. For the 2^- states we see that the strength is predominantly in the "collective" state of 21.34 MeV. There is, however, a good deal of strength in the state at 13.85 MeV, which corresponds to the ground state of ^{16}N . The present calculation over-estimates the strength of this state since it would give a branching ratio of about 15.7% to this state instead of $\sim 7\%$ as is observed³¹⁾. After weighting the figures in Table 2 by $(\hbar c \nu_{ab})^4 = (108 \text{ MeV} - E)^4$ to get the contributions to (5.19) and (5.20) one finds the following results :

$$M_A^2(0^-)/M_A^2(1^-)/M_A^2(2^-) = 0.92/3.00/5.85$$

$$3M_A^2(1^-)/M_V^2 = 1.03 \quad (5.24)$$

$$M_A^2/M_V^2 = 1.12$$

in an obvious notation. Thus we find $(M_A^2)_{U.D.} = (M_V^2)_{U.D.}$ to within 12% using the Lewis wave functions. One may be doing better than this by assuming $M_A^2 = M_V^2$ and evaluating M_V^2 from the observed photo-absorption cross-section since experimentally there is somewhat more 1^- vector strength at low energies than indicated in Table 2, while there is also somewhat less 2^- strength there as discussed above. It is difficult to say conclusively, however.

We should also emphasize that the Lewis calculations include the effect of the spin orbit force and a spin dependent central force. If the spin dependence of the nuclear forces comes from a very strong tensor force as in the more realistic nucleon-nucleon potentials, the assumption $M_V^2 = M_A^2 = M_P^2$ can be further affected. So far, we have no further information on this point.

The above calculations take into account only the effect of the effect of the spin dependent forces on those states arising from the $(2s)(1p)^{-1}$ and $(1d)(1p)^{-1}$ configurations, but not their effect on the ground state wave function. The latter effect means that the ground state is also not a pure supermultiplet scalar state but is admixed with states which the operators U^i do not annihilate. For ^{16}O and ^{40}Ca this effect will be much smaller than the first effect since the admixed configurations lie much higher in energy, and hence are not expected to be important. In ^{12}C the admixtures may be more significant and could lead to a non-negligible contribution from the allowed monopole part of the axial vector interaction. The special case of ^4He will be discussed in the conclusion.

D. Contribution of other multipoles.

The last point which requires re-examination is the possible effect of collective states on multipole contributions other than the dipole, in particular, the quadrupole contribution which is quite

sizeable in ^{40}Ca . It is important to note that the familiar collective 2^+ associated with an oscillation of the whole nucleus between prolate and oblate spheroidal shapes is not significant since it is a $T = 0$ state. A collective $T = 1, 2^+$ state would be one in which neutrons and protons separately undergo such oscillations between prolate and oblate shapes, but in which the protons "prolate" when the neutrons "oblate" and vice versa. To our knowledge, nothing is known concerning the existence of such collective states in nuclei. If they exist they are probably at higher energies than the giant dipole states and could result in quadrupole contributions different from what we have calculated from the shell model. In the present state of ignorance with regard to such collective motions, we have no resource but to rely on the shell model results.

6. NUMERICAL CALCULATIONS

On the basis of the preceding discussions, we shall try to calculate the total muon capture rates in ^{40}Ca , ^{16}O , ^{12}C , and ^4He on the basis of Eqs. (2.14), (3.1) and (4.4). (^{12}C is not doubly magic but we think we can get a fairly good estimate of the capture rate as discussed below). We introduce the following definitions ^{*)}:

$$G_{\mu c}^2 \equiv G_V^2 + 3G_A^2 + (G_P^2 - 2G_P G_A) = \frac{5.54}{M^4} \times 10^{-10} \frac{\hbar^6}{c^2}$$

$$|\phi_\mu|_{av}^2 \equiv (\phi_\mu)_{point}^2 R = \frac{1}{\pi} (Z\alpha v_\mu)^3 R \quad (R \text{ is given by Sens }^{24}) \quad (6.1)$$

$$\Lambda'_{\mu c} \equiv \text{contribution of nucleon recoil terms}$$

$$(M_V^2)_{O.M.} \equiv \text{contribution to } M_V^2 \text{ of all other multipoles beside } J = 1^-.$$

With these we can write

$$\Lambda_{\mu c} = \Lambda_{\mu c}^0 + \Lambda'_{\mu c} \quad (6.2)$$

$$\begin{aligned} \Lambda_{\mu c}^0 &= \frac{v_\mu^2 G_{\mu c}^2}{2\pi \hbar^2 c} \left[\frac{1}{\pi} (Z\alpha v_\mu)^3 R \right] [M_V^2] \\ &= 281 \text{ sec}^{-1} [R Z^3] [M_V^2] \end{aligned} \quad (6.3)$$

^{*)} LRT use $G_{\mu c}^2 = \frac{5.60 \times 10^{-10}}{M^4} \frac{\hbar^6}{c^2}$.

$$M_V^2 = |F_{el}(E_m - E_{res})|^2 \frac{V_\mu^2}{2\pi^2\alpha} \left(\frac{V_m}{V_\mu}\right)^4 \int_0^{E_m} \left(\frac{E_m - E}{E_m}\right)^4 \sigma_\gamma(E) \frac{dE}{E} + (M_V^2)_{o.m.} \quad (6.4)$$

The problem is then to evaluate the two terms in M_V^2 and $\Lambda'_{\mu c}$. We shall discuss these one at a time

(a) Evaluation of $|F_{el}(E_m - E_{res})|^2 \frac{V_\mu^2}{2\pi^2\alpha} \left(\frac{V_m}{V_\mu}\right)^4 \int_0^{E_m} \left(\frac{E_m - E}{E_m}\right)^4 \sigma_\gamma(E) \frac{dE}{E}$.

There are some troublesome aspects to the evaluation of the integral given above in terms of experimentally observed cross-sections. In most cases, only the energy variation of the cross-section for (γ, p) and (γ, n) processes are known with any accuracy. There is evidence in the case of $^{16}_0$ for a substantial high energy contribution to the cross-section from processes like (γ, pn) and a host of more complicated disintegrations. In high energy region, it is not clear how much of the observed cross-section is of electric dipole character. Fortunately, the emission of more than one nucleon generally requires sufficient energy that the contribution to the integral from these is relatively small because of the presence of the guillotine factor. There exists the possibility, however, that our results for $^{16}_0$, $^{40}_{Ca}$, and $^{12}_C$ may be somewhat low owing to our ignorance in this matter. We summarize below the manner in which we have obtained our values :

$^4_{He}$

For $^4_{He}$ we have used the photo-disintegration data of Gorbunov and Spiridonov³²⁾ which extend to 170 MeV. A check on the absolute normalization of the experimental cross-section is provided by using the relation between the bremsstrahlung integrated

cross-section and the mean square radius of ${}^4\text{He}$ derived by Foldy³³⁾. This relation is satisfied to within 7% suggesting that the absolute normalization of the cross-section has this accuracy. The straightforward evaluation of M_V^2 from the data yields a value of 0.094 with the form factor evaluated at $E_m - E_{\text{res}} = 79$ MeV. If the form factor were included under the integral and evaluated therefore at the appropriate neutrino momentum for each transition, the result differs from that above by less than $\frac{1}{2}\%$. On the basis of the completeness of the data, the check on its normalization, and the small effect of the form factor (which ranges only from 0.85 to 1.0 over the energies of interest) we believe the result we obtain should have an accuracy of the order of 10%.

${}^{12}\text{C}$

The photo-absorption cross-section of ${}^{12}\text{C}$ shows a single narrow peak below 25 MeV situated at 22.5 MeV with an integral under the peak of 95 MeV-mb^{*)}. Above 25 MeV we have assumed that the (γ, n) cross-section is equal to the observed (γ, p) cross-section, since they agree in the neighbourhood of 25 MeV. On this basis we obtain a value of 0.324 for M_V^2 by evaluating the form factor at $E_m - E_{\text{res}} = 83$ MeV. Since 75% of the contribution comes from the peak, there will be only a very small difference on including the form factor under the integral. We estimate that the uncertainty in the result is not likely to be larger than about 15%.

${}^{16}\text{O}$

There exists so many different (and in some cases, conflicting) measurements on the photo disintegration cross-section for ${}^{16}\text{O}$, below 30 MeV that we have chosen not to rely on the direct

*) See Refs. 34), 35) for an analysis of the extensive experimental data.

experimental results but on an analysis of them (for the energy range up to 26 MeV) made by Fuller and Hayward ³⁴⁾. There is now further evidence from recent work of Gorbunov and Osipova ³⁶⁾ for a substantial cross-section above 26 MeV. The energy variation of this high energy cross-section is known only for the (γ, p) and (γ, n) reactions. There is a considerable cross-section for (γ, pn) and other more complex modes of disintegration at higher energies. It is difficult to assess how much of this high energy cross-section is dipole in character. We have proceeded on the assumption that the disintegrations in which more than one nucleon is emitted lie at sufficiently high energies as to make a relatively minor contribution to the integral we require and thus have used the sum of the (γ, p) and (γ, n) cross-sections as representing the important contribution for our purposes. On this basis we calculate a value of 0.45 for M_V^2 . Including the form factor under the integral leads to a value about 3% lower. About 12% of the result comes from energies above 26 MeV. It is difficult to estimate the absolute accuracy of the result from the data but we believe it is not likely to be more than 10% too high, but because of the uncertainties in the high energy cross-section, it may be as much as 20% too low.

⁴⁰Ca

The available data show the photo-absorption cross-section as a single peak situated at 20 MeV with an integrated cross-section of 500 MeV-mb ³⁴⁾. In the absence of any high energy data, we have used this peak alone to calculate M_V^2 obtaining a result of 1.09 with the form factor evaluated at $E_m - E_{res} = 91.5$ MeV. It is difficult to assess reliability of this value since not only are we again ignorant of higher energy contributions, but the square of the form factor is here 0.40 which is sufficiently small that we can envisage a substantial inaccuracy arising from it as well.

(b) Evaluation of $(M_V^2)_{OM}$

We discuss here the evaluation of the contribution to M_V^2 from all other multipoles besides the dipole. This is now a correction to our principal result. We have therefore evaluated it in the following two ways :

- i) We can explicitly take the sum over these partial transitions from the work of LRT. The results for $^{16}_0$ and $^{40}_{Ca}$ are given in Table 4.
- ii) We can try to use a closure approximation like that of Primakoff ¹⁾ and LRT ⁵⁾ to evaluate this correction term. To do this we first define a quantity

$$\mathcal{M}^2 \equiv \frac{1}{2} \int \frac{d\vec{v}}{4\pi} \langle 0 | \sum_{i=1}^A \sum_{j=1}^A \tau^{(3)}(i) \tau^{(3)}(j) e^{i\vec{v} \cdot (\vec{x}(i) - \vec{x}(j))} | 0 \rangle \quad (6.5)$$

where $|0\rangle$ is the ground state and $\vec{v}$ is an average neutrino momentum to be evaluated later. This is just the entire contribution of all the matrix elements of M_V^2 in the closure approximation. The unretarded dipole contribution to M_V^2 goes as

$$\mathcal{D} \equiv \frac{1}{2} \int \frac{d\vec{v}}{4\pi} \langle 0 | \sum_{i=1}^A \sum_{j=1}^A \tau^{(3)}(i) \tau^{(3)}(j) [\vec{v} \cdot \vec{x}(i)] [\vec{v} \cdot \vec{x}(j)] | 0 \rangle \quad (6.6)$$

$$= \frac{A}{4} \eta^2 \quad (\eta \equiv v b_{osc}) \quad (6.7)$$

where we have used the single particle oscillator model for

	^{40}Ca Ref. 5)	$^{16}_0$ Ref. 5)	^{12}C Ref. 30)	^4He Ref. 37) *)
b_{osc}	2.03 f	1.80 f	1.60 f	1.38 f
$\sqrt{\langle r^2 \rangle}$	3.52 f	2.70 f	2.36 f	1.46 f
$r_0 \equiv \sqrt[5]{\frac{3}{A}} \frac{\sqrt{\langle r^2 \rangle}}{A^{1/3}}$	1.33 f	1.38 f	1.33 f	1.19 f
$\hbar\omega_{\text{osc}} = \frac{\hbar^2}{M b_{\text{osc}}^2}$	10.1 MeV	12.8 MeV	16.2 MeV	21.8 MeV
E_m	110 MeV	108 MeV	107 MeV	107 MeV
$\hbar\omega = E_m - 2\hbar\omega$	89.8 MeV	82.4 MeV	74.6 MeV	63.4 MeV

TABLE 3

Oscillator parameters used for ^{40}Ca , $^{16}_0$, ^{12}C , and ^4He . The corresponding root-mean-square radius $\sqrt{\langle r^2 \rangle}$ and radius parameter for an equivalent uniform density sphere, r_0 , are also given. E_m is computed from Eq. (3.7).

*) The value $\sqrt{\langle r^2 \rangle} = 1.46$ for ^4He is obtained after correcting the observed mean square radius for that of the individual protons. The result $e^{-\eta^2/4}$ for (6.8) is incorrect because of the effect of centre-of-mass motion (which is negligible for the heavy nuclei). Thus in comparing with experiment one should really take ³⁸⁾

$$F_{\text{el}}^{(\text{exp})} = e^{\frac{\eta^2}{4A}} e^{-\frac{\eta^2}{4}} = e^{-\frac{3}{16}\eta^2}$$

We have just used $e^{-\eta^2/4}$ in Eqs. (6.4) and (6.9) [in order to reproduce (6.13)]. The difference in $|F_{\text{el}}|^2$ is very small.

ELEMENT	R	$(M_V^2)_{U.D.}$ [Eq. (4.3)]	$E_m - E_{res}$ (MeV)	$ F_{el}(E_m - E_{res}) ^2$	$ F_{el} ^2 (M_V^2)_{U.D.}$	$(M_V^2)_{O.M.}$	M_V^2	$\Lambda_{\mu c}^0$ [in sec^{-1}]	$\Lambda'_{\mu c}$ [in sec^{-1}]
^{40}Ca	0.44	2.73	91.5	0.40	1.09	$\begin{cases} 1.67 & \text{closure} \\ 1.16 & \text{partial trans.} \\ 1.41 & \text{average} \end{cases}$ ⁵⁾	2.50	24.7×10^5	3.6×10^5 Ref. ⁵⁾
^{16}O	0.79	0.77	85	0.63	0.485	$\begin{cases} 0.231 & \text{closure} \\ 0.178 & \text{partial trans.} \\ 0.204 & \text{average} \end{cases}$ ⁵⁾	0.69	0.79×10^5	0.16×10^5 Ref. ⁵⁾
^{12}C	0.83	0.45	83	0.72	0.324	0.073	0.40	0.20×10^5	$\sim 0.06 \times 10^5$
^4He	0.98	0.108	79	0.87	0.094	0.004	0.098	195	54 Ref. ³⁸⁾

T A B L E 4

Contributions to the theoretical values for the total capture rates in ^{40}Ca , ^{16}O , ^{12}C , and ^4He . R is defined in Eq. (6.1) and taken from Ref. 24). The quantities appearing in the second column are defined in Eqs. (4.4), (3.7), and (2.14). The experimental data for $\sigma_\gamma(E)$ are discussed in the text. E_{res} in the third column is the peak energy in $\sigma_\gamma(E)$ and F_{el} is taken from Eqs. (6.10) to (6.12). The contributions from other multipoles have been evaluated by the closure approximation discussed in the text and by taking the sums over partial transitions given in LRT. $\Lambda'_{\mu c}$ is taken from Ref. 5) and Ref. 38).

$|0\rangle$ in the evaluation of D . Finally, defining

$$F_{el}(\nu) \equiv \frac{1}{A} \langle 0 | \sum_{i=1}^A e^{i\vec{\nu} \cdot \vec{x}(i)} | 0 \rangle \quad (6.8)$$

we can write for $(M_V^2)_{OM}$, following our discussion in section 3,

$$(M_V^2)_{OM} = \left(\frac{\nu}{\nu_\mu} \right)^2 [M^2 - D |F_{el}(\nu)|^2] \quad (6.9)$$

This yields the following expressions

^{40}Ca :

$$\begin{aligned} M^2 &= 20 - 20 \left[1 + \frac{1}{8}\eta^4 - \frac{1}{80}\eta^6 + \frac{1}{640}\eta^8 \right] e^{-\frac{1}{2}\eta^2} \\ D|F_{el}|^2 &= 10\eta^2 \left[1 - \frac{1}{4}\eta^2 + \frac{1}{80}\eta^4 \right]^2 e^{-\frac{1}{2}\eta^2} \end{aligned} \quad (6.10)$$

^{16}O :

$$\begin{aligned} M^2 &= 8 - 8 \left[1 + \frac{1}{16}\eta^4 \right] e^{-\frac{1}{2}\eta^2} \\ D|F_{el}|^2 &= 4\eta^2 \left[1 - \frac{1}{8}\eta^2 \right]^2 e^{-\frac{1}{2}\eta^2} \end{aligned} \quad (6.11)$$

^4He :

$$\begin{aligned} M^2 &= 2 - 2e^{-\frac{1}{2}\eta^2} \\ D|F_{el}|^2 &= \eta^2 e^{-\frac{1}{2}\eta^2} \end{aligned} \quad (6.12)$$

We note that in each of the above cases, if one takes the limit of $M^2 - D|F_{el}|^2$ for small η

$$M^2 - D|F_{el}|^2 \underset{\eta \rightarrow 0}{=} Q = \frac{1}{6} \eta^2 D \left[\frac{\langle r^2 \rangle}{b^2} \right] \quad (6.13)$$

where we have defined

$$Q \equiv \frac{1}{8} \int \frac{d\nu}{4\pi} \left(0 \left| \sum_{i=1}^A \sum_{j=1}^A \tau^{(3)}(i) \tau^{(3)}(j) |\vec{\nu} \cdot \vec{x}(i)|^2 |\vec{\nu} \cdot \vec{x}(j)|^2 \right| 0 \right) \quad (6.14)$$

and $\langle r^2 \rangle$ is the mean square radius of the nucleus calculated in the oscillator model (it can be obtained from Eq. (6.8) - $F_{el} = 1 - \frac{1}{6} \nu^2 \langle r^2 \rangle + \dots$). Since the correction Q goes as η^4 , it gets very small for light nuclei. (For this reason we have also extended our results to ^{12}C by simply using the empirical relation (6.13) to evaluate Q^* .)

The remaining question is what value of ν to use in evaluating (6.9). Since the predominant contributions from other multipoles come from the 0^+ and 2^+ transitions, and since these are at least at an energy $2\hbar\omega_{osc}$ in the oscillator model, we have used this energy to evaluate the closure sum. The parameters used in this evaluation are given in Table 3 and the results in Table 4. The partial transition sum is very likely an underestimate of $(M_V^2)_{OM}$ while, since the difference between the closure and partial sum evaluation all

*) For ^{12}C , $|F_{el}|^2 = (1 - \frac{1}{9} \eta^2)^2 e^{-\frac{1}{2} \eta^2}$.

comes from transitions with $E \geq 2\hbar\omega$. The closure approximation is probably an overestimate of the correction. We have therefore used an average of the two. This leads to a 10% uncertainty in the final result for ^{40}Ca and a 5% uncertainty for ^{16}O . Since we know nothing about the collective states of the other multipoles, it is difficult to assess the absolute accuracy of our calculation of $(M_V^2)_{\text{OM}}$. For ^{12}C and ^4He this term plays a very small role.

- (c) The term $\Lambda'_{\mu c}$ in Eq. (6.1) represents the contribution of the nucleon recoil corrections. Since it is a small correction to our final result we will use the results of LRT for ^{40}Ca and ^{16}O (with an appropriate extrapolation to ^{12}C) and the results of Primakoff et al. for ^4He ³⁷⁾). The term $\Lambda'_{\mu c}$ begins to play a larger role for the lighter elements contributing 25% to the final answer in ^4He while only contributing 14% to ^{40}Ca . This is not because the recoil corrections are getting larger for light nuclei, but rather that the dominant capture term, which is "first-forbidden" or higher order, goes to zero more quickly as the size of the system is decreased.

$\Lambda_{\mu c}^0$ and $\Lambda'_{\mu c}$ contain two other corrections for ^4He due to its light mass :

- i) A factor $(1 + \frac{m_\mu}{M_{\text{He}}})^{-3}$ coming from the reduced mass in $|\phi_\mu|^2_{\text{point}}$,
- ii) A factor $(1 + \frac{p_\nu}{M_{\text{He}}})^{-1}$ with $p_\nu \approx 85 \text{ MeV}$ coming from a correct treatment of the neutrino phase space.

The final theoretical productions for $\Lambda_{\mu c}$ are given in Table 5 and compared with the experimental results.

*) Equations (8.21) to (8.24) of LRT give the same result as Primakoff et al. ³⁷⁾ for ^4He .

ELEMENT	Λ THEORETICAL μ^c (in sec^{-1})	Λ EXPERIMENTAL μ^c (in sec^{-1})	REFERENCES
^{40}Ca	28.3×10^5	$25.5 \pm 0.5 \times 10^5$	24)
^{16}O	0.95×10^5	$0.98 \pm 0.05 \times 10^5$ $0.97 \pm 0.03 \times 10^5$	10) 39)
^{12}C	0.26×10^5	$0.39 \pm 0.01 \times 10^5$ $0.37 \pm 0.01 \times 10^5$ $0.36 \pm 0.01 \times 10^5$ $0.30 \pm 0.08 \times 10^5$	39) 40) 41) 10)
^4He	249	375 ± 46 336 ± 75	42) 43)

T A B L E 5

Comparison of theoretical and experimental total capture rates in ^{40}Ca , ^{16}O , ^{12}C , and He^4 . The experimental data are taken from Refs. 10), 39), 40), 41), 42).

7. CONCLUSIONS

Before stating our conclusions it would be well to reiterate the sources of uncertainty in our calculated results :

- (1) Uncertainties associated with the photo-disintegration data itself, particularly the absolute normalization of the cross-section and the absence of detailed information on the high energy dipole cross-section.
- (2) Uncertainties associated with the effect of spin dependence of the forces (particularly, tensor forces), in disturbing the relation $M_V^2 = M_A^2 = M_P^2$. Here there are two effects, one arising from the fact that the ground state is no longer a pure scalar supermultiplet, and the second arising from the fact that final states forming a vector supermultiplet are split and admixed with other supermultiplets even within a given set of configurations.
- (3) Uncertainties with respect to the accuracy with which the ground state form factor relates the retarded and unretarded dipole matrix elements.
- (4) Uncertainties associated with the possible influence of collective states on multipole transitions other than dipole.

An examination of Table 5 shows immediately that the agreement between theory and experiment achieved in the cases ^{16}O and ^{40}Ca is much better than should be expected in the light of the uncertainties mentioned above. In the case of ^{12}C the discrepancy which remains is quite tolerable. In this connection it should be remembered that ^{12}C is not doubly magic and hence its ground state may deviate somewhat from a pure scalar supermultiplet state. To the extent that it does, there will be an additional allowed monopole contribution from the axial vector interaction which has not been taken into account. It is not unlikely that the

discrepancy present may result from this effect ^{*)}.

The discrepancy in the case of ${}^4\text{He}$ is more disturbing. Here retardation effects and higher multipole contributions are relatively so small that they cannot be blamed for the discrepancy. The available photo-disintegration data is as complete as one could desire and its absolute normalization is subject to a check which suggests that it is not likely to be more than 10% too low. The spin orbit force is weaker in ${}^4\text{He}$ than in the other cases so that it is difficult

^{*)} After completing this manuscript, the authors realized that an estimate of the allowed contributions is possible. The allowed transitions can be written as a sum over transitions to $T = 1, 1^+$ states. Such a level with a strong M1 transition to the ground state in ${}^{12}\text{C}$ is known to exist at 15.1 MeV (it is the isobaric partner of the ground state of ${}^{12}\text{B}$). The transition is sufficiently strong that it is unlikely to arise purely from the orbital magnetic moment. Thus presumably there is a substantial spin-flip contribution which can only arise if the ground state is not a pure scalar supermultiplet. The relatively rapid β decay of ${}^{12}\text{B}$ is further evidence for this. Inelastic electron scattering data ²⁹⁾ allow us to estimate the transition moment at the momentum transfer appropriate to muon capture and hence to estimate the allowed axial vector contribution to muon capture for the transition from the ground state of ${}^{12}\text{C}$ to the ground state of ${}^{12}\text{B}$. This contribution amounts to $(0.07 \pm 0.02) \times 10^5 \text{ sec}^{-1}$, thus representing about 20% of the total capture rate. If this is added to our previous estimate for forbidden transitions, we obtain $0.33 \times 10^5 \text{ sec}^{-1}$ for the capture rate in ${}^{12}\text{C}$ in satisfactory agreement with the experimental value. However, our previous estimate for forbidden transitions is now less reliable since we have evidence that the supermultiplet approximation is breaking down more seriously than was anticipated for ${}^{12}\text{C}$.

to imagine that the relation $M_V^2 = M_A^2 = M_P^2$ is breaking down seriously from the effect of spin dependent forces on the final states. In fact, even if the photo-disintegration cross-section were raised by 10% we would still require $M_A^2 > 1.5 M_V^2$ to achieve agreement between theory and experiment. Finally the recoil corrections can be evaluated quite accurately by a closure approximation and are not likely to be appreciably in error.

However, it should be remarked that in spite of the fact that ${}^4\text{He}$ is doubly magic, tensor forces will admix some 5D_0 state to the dominantly 1S_0 ground state. Estimates of the admixture of D state are quite uncertain ⁴⁴⁾ but are in the neighbourhood of a few percent. While normally this would not be of much concern, ${}^4\text{He}$ has such a small radius that the "forbiddenness" factor relating successive multipoles is quite large. Furthermore the raising in energy of the dipole states reduces the first forbidden contribution. Thus even a small allowed contribution from the axial vector interaction arising from the fact that the ground state is not pure scalar supermultiplet can be significant. Caine and Jones ⁴⁵⁾ have made an admittedly crude estimate of the effect of D-state admixture and conclude that its contribution to the capture rate is $38 \pm 20 \text{ sec}^{-1}$. If we take this contribution to be near its upper limit, say 60 sec^{-1} , and increase the photo-disintegration cross-section by 10%, we can bring our results in line both with the experiments and with the calculated capture rates of Goulard et al. ³⁷⁾ and Caine and Jones ⁴⁵⁾. The possibility of a real discrepancy or that the experimental capture rates are too high cannot be excluded, however. If the vector supermultiplet admixed to the ground state consists primarily of two particle excitations, then we can show that the percentage contribution of the allowed monopole in ${}^{16}\text{O}$ and ${}^{40}\text{Ca}$ is much smaller than in ${}^4\text{He}$ (where it amounts to approximately 15% of the total). Unfortunately any more quantitative statement depends on a detailed model of the ground state of these nuclei.

All in all the clearest conclusion that we can draw from our results is that there is no reason at present to question the consistency of muon capture in light nuclei with the Universal Fermi Interaction assumptions. A more

quantitative check is still certainly desirable. The brightest path for achieving greater quantitative accuracy would seem to be through the agency of extensive and detailed inelastic electron scattering data. While this data cannot be directly used to provide the matrix elements required for calculating muon capture rates (since both $T = 0$ and $T = 1$ transitions are mixed in electron inelastic scattering and there is no unambiguous way of separating the two) still the required matrix elements are so similar that one could be confident that a nuclear model which can explain the details of scattering data quantitatively would also serve as a reliable basis on which to calculate the matrix elements required for muon capture.

It is perhaps worth commenting on the fact that Sens et al.¹⁰⁾ were able to obtain good agreement between predicted and observed muon capture rates in ^{16}O and ^{40}Ca by simply modifying the transition energies in the shell model calculations of LRT. In the shell model, the integrated photo-absorption cross-section will have the classical (Thomas-Reiche-Kuhn) value. The effect of particle interactions will not only raise the energy of the giant resonance to its observed position but will substantially increase the integrated photo-absorption cross-section. In the case of ^{16}O (and probably in the case of ^{40}Ca as well, though there is no evidence on this point) the increase takes place principally at energies considerably above the giant resonance region. In both ^{16}O and ^{40}Ca the observed integrated cross-section up to 30 MeV is about 80% of the classical sum rule value. Thus a modification of the transition energies alone in the shell model accounts for the principal effects of particle interaction, but it is not clear that this is a dependable rule in other nuclei.

The fact that photo-disintegration data give one information on the strengths of transitions to various final states in energy means that one is also in a position to make some predictions of the neutrino energy spectrum in muon capture and hence of the distribution of nuclear excitation energies after capture. In fact, Balashov et al.⁶⁾ attempt a prediction of the neutron spectrum following muon capture in ^{16}O and ^{40}Ca on the basis of the resonance model for capture^{*)}. It should be clear that the general techniques that we have used can be employed as well in the case of radiative muon capture.

*) A recently reported measurement by Kaplan et al.⁴⁶⁾ of the average neutrino momentum for capture in ^{40}Ca yields 90 ± 2 MeV/c in precise agreement with what we would expect on the basis of our method of calculation since both the dipole and quadrupole transitions will give a relatively narrow peak at this value.

Perhaps the most interesting conclusion that can be drawn from our results is the suggestion that there exists a giant resonance supermultiplet in nuclei of which the well-known giant electric dipole resonance is a member. A detailed search for and investigation of the other members of the supermultiplet by the techniques of inelastic electron scattering, and perhaps inelastic nucleon scattering as well, is clearly indicated.

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REFERENCES

- 1) H. Primakoff, Rev. Mod. Phys. 31, 802 (1959).
- 2) H.A. Tolhoek and J.R. Luyten, Nuclear Physics 3, 679 (1957);
H.A. Tolhoek, Nuclear Physics 10, 327 (1959).
- 3) R. Klein and L. Wolfenstein, Phys. Rev. Letters 9, 408 (1962).
- 4) J.S. Bell and J. Løvseth, Nuovo Cimento 17, 408 (1964).
- 5) J.R. Luyten, H.P.C. Rood and H.A. Tolhoek, Nuclear Physics 41, 236 (1963).
- 6) V.V. Balashov, V.B. Belyaev, N.M. Kabachnik and R.A. Eramzhian, Dubna preprint P 1530 (1964).
- 7) J. Tiomno and J.A. Wheeler, Rev. Mod. Phys. 21, 153 (1949).
- 8) J.P. Elliott and B.H. Flowers, Proc. Roy. Soc. 242A, 57 (1957).
- 9) G.E. Brown and M. Bolsterli, Phys. Rev. Letters 3, 472 (1959);
G.E. Brown, L. Castillejo and J.A. Evans, Nuclear Physics 22, 1 (1961);
N. Vinh Mau and G.E. Brown, Nuclear Physics 29, 89 (1962).
- 10) J. Barlow, J.C. Sens, P.J. Duke and M.A.R. Kemp, Physics Letters 9, 84 (1964).
- 11) R.P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).
- 12) E. Wigner, Phys. Rev. 51, 106 (1937).
- 13) A.E. Glassgold, W. Heckrotte and K.M. Watson, Annals of Physics 6, 1 (1959).
- 14) S. Fallieros, R. Ferrell and M.K. Pal, Nuclear Physics 15, 363 (1960).
- 15) M. Goldhaber and E. Teller, Phys. Rev. 74, 1046 (1948).

56.

- 16) F.H. Lewis, Jr., Phys. Rev. (to be published) and private communication.
- 17) S. Weinberg, Phys. Rev. 112, 1375 (1958).
- 18) C. de Vries, R. Hofstadter, and A. Johansson, Proceedings of Stanford University Conference on Nucleon Structure, 1963 (to be published).
- 19) A. Fujii and H. Primakoff, Nuovo Cimento 12, 327 (1959).
- 20) Communication from C.S. Wu to F. Gürsey, T. D. Lee and M. Nauenberg, quoted in Columbia University preprint.
- 21) M.L. Goldberger and S.B. Treiman, Phys. Rev. 111, 354 (1958).
- 22) Y. Nambu, Phys. Rev. Letters 4, 380 (1960).
- 23) L. Wolfenstein, Nuovo Cimento 8, 882 (1958).
- 24) J.C. Sens, Phys. Rev. 113, 679 (1958).
- 25) J. Goldemberg, Y. Torizuka, W.C. Barber and J.D. Walecka, Nuclear Physics 43, 242 (1963).
- 26) D.M. Brink, Nuclear Physics 4, 215 (1960).
- 27) G.R. Bishop and D.B. Isabelle, Nuclear Physics 45, 209 (1963).
- 28) A.R. Edmonds, "Angular Momentum in Quantum Mechanics", Princeton University Press, Princeton, New Jersey (1957).
- 29) W.C. Barber and J. Goldemberg (to be published).
- 30) F.H. Lewis, Jr. and J.D. Walecka, Phys. Rev. 133, B849 (1964).
- 31) R. Cohen, S. Devons and A.D. Kanaris, Phys. Rev. Letters 11, 134 (1963).
- 32) A.N. Gorbunov and V.M. Spiridonov, Zh. Eksp. Theor. Fiz. 34, 866 (1958); JETP 7, 600 (1958).

- 33) L.L. Foldy, Phys. Rev. 107, 1303 (1957); see also M.L. Rustgi and J.S. Levinger, Phys. Rev. 106, 530 (1957).
- 34) E.G. Fuller and E. Hayward, "Nuclear Reactions" edited by P.M. Endt and P.B. Smith (North-Holland Publishing Co., Amsterdam 1962), Vol. II, Chapter III.
- 35) E. Hayward, Rev. Mod. Phys. 35, 324 (1963).
- 36) A.N. Gorbunov and V.A. Osipova, Zh. Eksp. Theor. Fiz. 34, 866 (1958); JETP 16, 27 (1963).
- 37) B. Goulard, G. Goulard and H. Primakoff, Phys. Rev. 133, B186 (1964).
- 38) L.J. Tassie and F.C. Barker, Phys. Rev. 111, 940 (1958).
- 39) M. Eckhouse, Carn. Inst. of Techn., Dept. of Physics, Pittsburgh, Thesis (NYO-9286, February 1962).
- 40) R.A. Reiter, T.A. Romanowski, R.B. Sutton and B.G. Chidley, Phys. Rev. Letters 5, 22 (1960).
- 41) J.L. Lathrop, R.A. Lundy, V.L. Telegdi, R. Winston and D.D. Yovanovitch, Phys. Rev. Letters 7, 107 (1961).
- 42) M. Bloch (private communication).
- 43) R. Bizzarri, E. Di Capua, U. Dore, G. Gialanelli, P. Guidoni and I. Laakso (to be published in Nuovo Cimento).
- 44) G. Abraham, L. Cohen and A.S. Roberts, Proc. Phys. Soc. A68, 265 (1955).
- 45) C.A. Caine and P.S.H. Jones, Nucl. Phys. 44, 177 (1963).
- 46) S.N. Kaplan, B. Macdonald and R.V. Pyle, Bull. Am. Soc. 9, 393 (1964).