CM-P00057183

ARCHIVES

A NEW APPROACH TO SCATTERING BY SINGULAR POTENTIALS *)**)

S. Fubini

CERN - Geneva

and

Istituto di Fisica dell'Università, Torino

R. Stroppolini

Istituto di Fisica Teorica dell'Università, Napoli

Sezione di Napoli dell'I.N.F.N.

*) The research reported in this document has been sponsored by the Air Force Office of Scientific Research OAR through the European Office Aerospace Research, United States Air Force.

***) Submitted to Nuovo Cimento as "Lettera all'Editore".

The problem of giving an unambiguous mathematical meaning to the solutions of non-renormalizable field theories has been the object of several investigations. One possible line of attack is to consider simple models given by singular non-relativistic or relativistic potential scattering equations as a guide to obtain new ideas and techniques to be generalized to the field theoretical case.

The new feature of singular potentials is that they lead to a differential equation with non-Fuchsian singularities both at $r \rightarrow 0$ and $r \rightarrow \infty$. This last ultra-violet singularity causes the break-down of the conventional perturbation expansion.

One way of treating the singular case is to consider the solution ψ^+ , ψ^- , ψ_R of the Schroedinger equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\lambda^2}{r^2} + k^2 \right) \psi(r) = V(r) \psi(r) \quad (1)$$

having the asymptotic behaviour

$$\psi^\pm(r) \rightarrow r^{-\frac{1}{2}} e^{\pm i k r} \quad r \rightarrow \infty \quad (2)$$

and *)

$$\psi_R(r) \rightarrow r^{-\frac{1}{2}} [V(r)]^{-\frac{1}{4}} e^{\int^r [V(r')]^{\frac{1}{2}} dr'} \quad r \rightarrow 0 \quad (3)$$

*) Such an asymptotic behaviour can be obtained by the well-known W.K.B. method.

2.

The ψ^+ , ψ^- , ψ_R solutions can be obtained by iterating Volterra equations starting from their asymptotic behaviour respectively at infinity and at the origin. These solutions can be joined by means of the Wronskian and in this manner an explicit expression for the S matrix can be obtained in several cases ¹⁾.

In this letter we wish to discuss a new point of view in the study of singular potentials which allows a systematic treatment of a large class of potentials. The approach we are proposing is essentially based on the study of the polydromy of the wave function $\psi(r)$ in the r variable. The important point is that both singular and regular potentials exhibit very simple and general features in this framework.

It is a well-known mathematical result ²⁾ that, if the potential $V(r)$ is a single valued function of r , it is possible to find two independent solutions of Eq. (1)

$$\begin{aligned}\psi_I(r) &= r^\gamma \chi_I(r) \\ \psi_{II}(r) &= r^{-\gamma} \chi_{II}(r)\end{aligned}\tag{4}$$

where χ_I and χ_{II} are single valued functions of r and γ is a parameter whose determination, in terms of an eigenvalue problem, will be discussed later. This result can be easily understood if one performs the transformation

$$r = e^{iZ}\tag{5}$$

Equation (1) becomes

$$\left(\frac{d^2}{dz^2} - \lambda^2 \right) \varphi(z) = \bar{U}(z) \varphi(z) \quad (6)$$

$$\bar{U}(z) = U(r) = r^2 [-k^2 + V(r)] \quad (7)$$

which coincides with the Schroedinger equation of a particle of energy λ^2 in a periodic potential $\bar{U}(z)$. Expression (4) represents the well-known Bloch solution and the determination of the exponent λ leads to the problem of the Brillouin zones.

The previous results can be understood from a more general standpoint by considering the operator

$$M = e^{-2\pi i \vec{x} \cdot \vec{p}} = e^{2\pi i r \frac{\partial}{\partial r}} = e^{2\pi i \frac{\partial}{\partial z}} \quad (8)$$

The operator performs a rotation of $2\pi i$ in the r space (or equivalently a translation of 2π in the z space) and therefore its eigenfunctions of the form (4) have well defined multivaluedness properties.

For a one-valued $V(r)$, i.e., for a periodic $\bar{U}(z)$ this operator commutes with the Hamiltonian, thus (4) simply states that the solution of (1) can be classified by means of the operator M . A general solution of Eq. (1) is therefore of the form

4.

$$\psi(r) = \alpha_I \psi_I(r) + \alpha_{II} \psi_{II}(r) \quad (9)$$

The decomposition (9) can be performed by means of the operators

$$P_I = \frac{M - e^{-2\pi i \delta}}{2i \sin 2\pi \delta} \quad (10)$$

$$P_{II} = \frac{M - e^{2\pi i \delta}}{-2i \sin 2\pi \delta}$$

which project on the I and II components respectively.

A possible use of the polydromy properties is to analyze the solutions ψ^+ , ψ^- , ψ_R by means of Eqs. (9) and (10). One obtains in this way a method of "joining" the two asymptotic behaviours which is explicitly (order by order) independent on the co-ordinate. This method is under investigation and leads to very promising results.

An alternative procedure, which will be discussed in more detail in this letter is to compute directly $\chi_I(r)$ and $\chi_{II}(r)$ and to study their behaviour at $r \rightarrow 0$ and $r \rightarrow \infty$. This can be achieved by expanding

$$U(r) = \sum_{-\infty}^{+\infty} u_n r^n \quad (11)$$

and

$$\chi(r) = \sum_{n=-\infty}^{+\infty} c_n r^n \quad (12)$$

Substituting Eqs. (11) and (12) into Eq. (1), one obtains an infinite system of coupled equations

$$\left[(\gamma + n)^2 - \lambda^2 \right] c_n = \sum_m u_{n-m} c_m \quad (13)$$

The possible values of the parameter γ are obtained by equating to zero the determinant of Eq. (13) ^{*)} (Hill's determinant)

$$\left| H_{nm} \right| = 0$$

where

$$H_{nm} = \delta_{nm} - \frac{u_{n-m}}{(\gamma + n)^2 - \lambda^2} \quad (13')$$

We can now realize the deep difference between regular and singular potential. In the first case U_n is non-vanishing only for positive n and is equal to the centrifugal term for $n = 0$; therefore, the determinant reduces

^{*)} Notice that Hill's determinant is a periodic function of γ with period 1; the γ which differ by an integer can be made to correspond to the same solution of Eq. (1) by a simple relabelling of the c_n . The two independent solutions of Eq. (1) correspond to the values γ and $-\gamma$, as Hill's determinant is an even function of γ .

simply to the product of the diagonal terms leading to $\gamma = \pm \lambda + n$, which coincides with the well-known Fuchs result. In the singular case, on the other hand, the presence of negative powers in Eq. (11) gives γ as the solution a transcendental equation which can be cast in the form ²⁾

$$\cos 2\pi \lambda - \cos 2\pi \gamma = \left[\Delta(0) - 1 \right] \left[1 - \cos 2\pi \lambda \right] \quad (14)$$

where $\Delta(0)$ is the Hill determinant for $\gamma = 0$.

On the other hand, we see here the reason for the great simplicity of problems with a singular potential in the limit $k^2 = 0$. In this case only negative powers appear and again we have $\gamma = \pm \lambda + n$, it is therefore not surprising that those problems can be reduced, by appropriate change of variables, to equivalent regular problems.

In order to illustrate the procedure for obtaining the scattering matrix we shall concentrate on the potential

$$V(r) = \frac{g^2}{r^4} \quad (15)$$

In this case the wave function $\psi_I(r)$ will behave for $r \rightarrow \infty$ as

$$\psi_I(r) \rightarrow A_I^+ r^{-\frac{1}{2}} e^{i\left[kr - \left(\lambda + \frac{1}{2}\right)\frac{\pi}{2}\right]} + A_I^- r^{-\frac{1}{2}} e^{-i\left[kr - \left(\lambda + \frac{1}{2}\right)\frac{\pi}{2}\right]} \quad (16)$$

and for $r \rightarrow 0$ as

$$\psi_I(r) \rightarrow B_I r^{\frac{1}{2}} e^{\frac{g}{r}} \quad (17)$$

analogous formulae for ψ_{II} do exist.

The S matrix is given by

$$S = \frac{\begin{vmatrix} A_I^+ & B_I \\ A_{II}^+ & B_{II} \end{vmatrix}}{\begin{vmatrix} A_I^- & B_I \\ A_{II}^- & B_{II} \end{vmatrix}} \quad (18)$$

The Hill equation becomes simple in this case giving for C_n *)

$$\begin{cases} C_n = \left(\frac{ik}{2}\right)^{2n} \frac{\Gamma\left(1 + \frac{\gamma+\lambda}{2}\right) \Gamma\left(1 + \frac{\gamma-\lambda}{2}\right)}{\Gamma\left(1 + \frac{\gamma+\lambda}{2} + n\right) \Gamma\left(1 + \frac{\gamma-\lambda}{2} + n\right)} \Delta_\gamma(-\infty, -1) \Delta_\gamma(n+1, \infty) \\ C_{-n} = \left(\frac{q}{2}\right)^{2n} \frac{\Gamma\left(1 + \frac{\gamma+\lambda}{2}\right) \Gamma\left(1 - \frac{\gamma-\lambda}{2}\right)}{\Gamma\left(1 - \frac{\gamma+\lambda}{2} + n\right) \Gamma\left(1 - \frac{\gamma-\lambda}{2} + n\right)} \Delta_\gamma(-\infty, -n-1) \Delta_\gamma(1, \infty) \end{cases} \quad (19)$$

$n \geq 0$

*) The choice of the normalization of C_n is completely arbitrary and does not affect the form of Eq. (18). For the potential g^2/r^4 only even powers of r will appear in $\chi(r)$, and we shall write

$$\chi(r) = \sum_{-\infty}^{\infty} C_n r^{2n}$$

where $\Delta_{\gamma}(a, b)$ are the determinants including the rows and columns with indexes $\geq a, \leq b$ (of course $\Delta_{\gamma}(-\infty, \infty)$ is the Hill determinant).

The quantities A and B appearing in the asymptotic Eqs. (16) and (17) can be determined by means of the saddle-point method leading

$$A^{\pm}(\gamma) = (2\pi)^{-\frac{1}{2}} e^{\pm i \frac{\pi}{2} (\lambda - \gamma)} \Gamma\left(1 + \frac{\gamma + \lambda}{2}\right) \Gamma\left(1 + \frac{\gamma - \lambda}{2}\right) \times \\ \times \left(\frac{k}{2}\right)^{-\gamma - \frac{1}{2}} \Delta_{\gamma}(-\infty, -1) \quad (20)$$

$$B(\gamma) = (2\pi)^{-\frac{1}{2}} \Gamma\left(1 - \frac{\gamma + \lambda}{2}\right) \Gamma\left(1 - \frac{\gamma - \lambda}{2}\right) \left(\frac{q}{2}\right)^{\gamma - \frac{1}{2}} \Delta_{\gamma}(1, \infty) \quad (21)$$

The final form of the S matrix is

$$S = \frac{J^{+}}{J^{-}}$$

$$J^{\pm} = e^{\pm i \frac{\pi}{2} (\gamma - \lambda)} - e^{\pm i \frac{\pi}{2} (\gamma + \lambda)} \left(\frac{q}{2}\right)^{4\gamma} \times \\ \times \left[\frac{\Gamma\left(1 - \frac{\gamma + \lambda}{2}\right) \Gamma\left(1 - \frac{\gamma - \lambda}{2}\right)}{\Gamma\left(1 + \frac{\gamma + \lambda}{2}\right) \Gamma\left(1 + \frac{\gamma - \lambda}{2}\right)} \right]^2 \cdot \left[\frac{\Delta_{\gamma}(1, \infty)}{\Delta_{\gamma}(-\infty, -1)} \right]^2 \quad (22)$$

where

$$f^2 = k g \quad (23)$$

The S matrix depends on the three parameters f , λ and γ which are interconnected by the implicit relation

$$\Delta_\gamma(-\infty, +\infty) = 0 \quad (24)$$

For γ and λ fixed $\Delta(-\infty, \infty)$, $\Delta(-\infty, -1)$, $\Delta(1, \infty)$ are power series of f^4 whose lowest terms can be easily computed.

One might wish to obtain from Eq. (22) an explicit expression of the S matrix in terms of f and λ only, by using Eq. (24) in order to eliminate γ . This can be done in a consistent manner for small values of (weak coupling limit). We want to remark, however, that this expansion (analogous to what one would obtain by direct limiting processes like peratization) is rather complicated and cumbersome and the number and kind of terms one has to keep in order to get a decent approximation depend in an essential manner on the value of λ since (apart from logarithmic terms) one has a series both in $f^4 \lambda$ and $f^4 *$.

*) This complicated dependence on λ could already be guessed by remembering that for each value of λ there is a finite number (increasing with λ) of convergent terms in the Born expansion.

In addition to this, one gets apparent poles for entire values of λ (i.e., when the terms of the double series coincide). These poles are completely eliminated in the final expression of the S matrix in a way which has some analogy with what happens in the theory of hypergeometric functions with coincident roots.

The conclusion of this letter is that the study of the polydromy of the wave function is of fundamental importance in the theory of singular potentials. The previous example shows explicitly that the introduction of the polydromy parameter allows us to obtain the S matrix in a neat and simple form, whereas conventional treatments would have led to tedious and not very illuminating calculations.

REFERENCES

- 1) L. Bertocchi, S. Fubini and G. Furlan, *Nuovo Cimento* **35**, 632 (1965);
E. Del Giudice and E. Galzenati, *Nuovo Cimento* (in press).
- 2) E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*,
412, Cambridge (1962).