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SOME REMARKS ON THE RENORMALIZATION CONSTANTS AND
THE BOUND STATE CONDITION

G. Furlan ^{*)} and G. Mahoux ^{**)}
CERN-Geneva

A B S T R A C T

We consider different examples of field theories, super-renormalizable, renormalizable and non-renormalizable ones, in order to evaluate the renormalization constant Z of the vertex between a bound state and the component particles. In ladder approximation, some properties of the Z 's are examined, with particular regard to the possibility of the cut-off factorization.

^{*)} On leave from the Istituto di Fisica dell'Università, Trieste, Italy.

^{**)} On leave from Centre d'Etudes Nucléaires de Saclay, Gif-sur-Yvette (S&O), France.

1. INTRODUCTION

Recent speculations have enlarged our confidence in considering the renormalization constants as quantities of physical value. A large amount of interest has been in fact devoted to the possibility that the vanishing of all the renormalization constants of a field theory could be a means to get the masses and coupling constants of all the interacting particles. Moreover it seems that also in badly divergent problems, like vector meson electrodynamics, the requirement of vanishing renormalization constants could make it possible to overcome the usual difficulties of the non-renormalizable theories ¹⁾. However, there is also a less ambitious and, maybe, more conservative approach where the first purpose is to get the bound states of two interacting particles, which are still considered as elementary (with known parameters) and are represented by quantized fields. As discussed in a previous paper ²⁾ a way to do this is to introduce the bound state as a fictitious particle and to represent it with a classical, unquantized field: the compositeness condition would be equivalent to the bound state "field" not entering into the interaction Lagrangian. This amounts to imposing the vanishing of the constant Z which renormalizes the vertex between the elementary and composite particles. This relation would enable us to determine the mass of the bound state as a function of the other parameters of the theory. Really, this statement lies on a somewhat heuristic basis and on the analogy with potential scattering (we remember that in the non-relativistic case the equivalent of Z is the Jost function ³⁾). In addition, the knowledge of Z , thought as a function of the external field mass, allows the determination of the S matrix. Thus it can be of help for a better understanding of the problem, to calculate explicitly, in some cases, this quantity Z . This will enable us to recognize some of its properties, beside verifying that the bound state spectrum (when present) is given by its vanishing. The aim of this note is to present the results of some calculations in the case of super-renormalizable, renormalizable and non-renormalizable theories.

2. SUPER-RENORMALIZABLE THEORIES

a.) We start from an interaction $L_I = g \psi^2 \phi$ of two identical (but distinguishable) scalar fields ψ , of mass m , and a scalar field ϕ of mass μ , that in the following we will put zero for simplicity. In ladder approximation the integral equation for the vertex part is

$$\Gamma(p, q) = Z + \frac{g^2}{(2\pi)^4} \int \frac{d^4 p' \Gamma(p', q)}{[(p' + q/2)^2 + \mu^2][p'^2 + m^2](p - p')^2} \quad (2.1)$$

$$p = \frac{p_1 - p_2}{2}, \quad q = p_1 + p_2$$

Moreover, in order to have a practically tractable problem we limit ourselves to the case $q = 0$. In so doing, once the mass of the composite particle is fixed the parameter to be quantized is the coupling constant g . It is important to remember that the quantity Γ we are considering has to be understood as the renormalized vertex function between the pair of ψ particles, interacting through the ϕ field, and a bound state of zero momentum, mass and angular momentum. (In the present situation the bound state and the ϕ particle are identical so that, at this stage, Γ can be also considered as the true vertex function between ψ , ϕ .) In addition we are taking the bound state on the mass shell while the component particles are virtual.

Now following the lines of the Wick's ⁴⁾ reasoning, it is possible to rotate the integration contour going to an Euclidean metric (this amounts practically to assume analyticity of Γ in the lower complex p^2 plane and good behaviour at the infinity). After integration on the angles we arrive to the one-dimensional integral equation

$$\Gamma(u) = Z + \lambda \int_0^\infty du' K(u, u') \Gamma(u') \quad (2.2)$$

where

$$u = p^2 \quad (u=1), \quad \lambda = q^2/4\pi^2$$

and

$$K(u, u') = \frac{1}{4(u'+1)^2} \left\{ \theta(u'-u) + \frac{u'}{u} \theta(u-u') \right\} \quad (2.3)$$

We notice the fact, proper to the super-renormalizable theories, that the kernel is of the Fredholm type. By differentiation we get from Eq. (2.2)

$$\left\{ \frac{d^2}{du^2} + \frac{2}{u} \frac{d}{du} + \frac{\lambda}{4u(u+1)^2} \right\} \Gamma(u) = 0 \quad (2.4)$$

while the boundary conditions implied by the starting integral equation are

$$\begin{aligned} \lim_{u \rightarrow 0} u^2 \Gamma'(u) &= 0 \\ \lim_{u \rightarrow \infty} [\Gamma(u) + u \Gamma'(u)] &= Z \end{aligned} \quad (2.4')$$

Equation (2.4) can be reduced to a standard hypergeometric form and the solution regular at the origin, as implied by the first boundary condition, is

$$\begin{aligned} \Gamma(u) &= A {}_2F_1(a, b, 2, u/(u+1)) \\ a, b &= \frac{1}{2} (1 \pm \sqrt{1+\lambda}) \end{aligned} \quad (2.5)$$

From the second boundary condition we get

$$Z = \lim_{u \rightarrow \infty} \Gamma(u) = A {}_2F_1(a, b, 2, 1) = A Z_0(\lambda) \quad (2.6)$$

4.

$$Z_0(\lambda) = \frac{1}{\Gamma(a+1)\Gamma(b+1)} z - \frac{4}{\pi\lambda} \cos \left[\frac{\pi}{2} \sqrt{1+\lambda} \right]$$

We are now faced with a problem of normalization : following the usual convention we have to normalize Γ on the mass shell. To do so we have to continue Eq. (2.2) and Γ to negative values of u and this can be done without difficulties. Moreover, there is an infra-red problem due to the zero mass of the exchanged quantum (in other words as $u \rightarrow -1$, $p^2 \rightarrow -m^2$, Γ becomes logarithmically divergent). The standard way to get rid of this difficulty is to ascribe a small mass to the scalar photon ϕ , or equivalently to add to $-m^2$ an infinitesimal part $m^2 \eta$ and to consider the limit $u+1 \rightarrow \eta$. With this device we can factorize the dependence on the infra-red cut-off, introducing it in the Z function. We have

$$\begin{aligned} \Gamma(-1) &= \frac{Z}{Z_0} \left[{}_2F_1(a, b, 2, u/u+1) \right]_{u=-1} \approx \\ &\approx \frac{Z}{Z_0} (\eta)^b \frac{\Gamma(a-b)}{a \Gamma^2(a)} \end{aligned} \quad (2.7)$$

$$Z = Z_0 (\eta)^{-b} \frac{a \Gamma^2(a)}{\Gamma(a-b)}$$

We remark from the previous expressions that ${}_2F_1$ and Z_0 (owing to the symmetry with respect to the exchange $a \leftrightarrow b$) are entire functions of λ , while Z presents branch points : we have not to worry about this because it is related to the fact that the limit $\eta \rightarrow 0$ is a singular one (as it happens usually with any cut-off procedure) and in this limit some analyticity properties can be lost.

Now following our philosophy in order to get the bound states we have to look for the zeros of Z . We observe that from the point of view of the vanishing Z and Z_0 are equivalent because the coefficient of the η part is always different from zero. This was expected because the cut-off procedure is not unique and changing it we can introduce new constants in the expression of Z .

The point is that this arbitrariness cannot affect the physical properties of the theory and as significant quantity we must use Z_0^*). After this we find that the zeros of Z_0 give the eigenvalue condition

$$\lambda_m = 4(m+1)(m+2) \quad m = 0, 1, \dots \quad (2.8)$$

which is a well-known result ⁵⁾.

b.) We want to remark here that Z_0 is coincident with the Fredholm determinant of Eq. (2.1) (and consequently it is an entire function in λ). This is easily shown if we use the expansion of $K(u, u')$ in terms of biorthonormal eigenfunctions $\{\phi_n, \psi_n\}$

$$K(u, u') = \sum_m \phi_m(u) \psi_m^*(u') / \lambda_m$$

Then we can write

$$D(\lambda) = \det(1 - \lambda K) = \exp[\text{Tr} \ln(1 - \lambda K)] =$$

$$= \exp[\text{Tr} \sum_m \phi_m(u) \psi_m^*(u) \ln(1 - \lambda/\lambda_m)] =$$

$$= \prod_{m=0}^{\infty} (1 - \lambda/\lambda_m) = \prod_{m=0}^{\infty} (1 - \lambda/4(m+1)(m+2))$$

Conversely using the well-known expression of a trigonometric function as an infinite product

$$\cos \frac{\pi}{2} \sqrt{1+\lambda} = - \frac{\pi\lambda}{4} \prod_{m=0}^{\infty} (1 - \lambda/4(m+1)(m+2))$$

*) This amounts practically to normalize $\Gamma(u)$ to one at $u = 0$.

we verify by direct comparison that

$$D(\lambda) \equiv Z_0(\lambda) \quad (2.9)$$

Perhaps it is not without interest to recall the analogy with potential scattering where the equivalent of Γ is $(k^2 - E)\psi_E(u)$ (ψ is the wave function). In the bound state region (s-wave) we define the quantities

$$\begin{aligned} N_0 &= \lim_{k^2 \rightarrow \infty} \Gamma_B(k^2) \\ N &= \lim_{k^2 \rightarrow -B} \Gamma_B(k^2) \end{aligned} \quad (2.10)$$

which represent the coefficient of the singular part of the wave function at small distances and the coefficient of asymptotic normalization, respectively. On the other hand it is well known that if we try to evaluate the coefficient of asymptotic normalization (at least as above defined) in the Coulomb case, this is divergent owing to the long range force. In fact

$$Z = \frac{N_0}{N} \quad (2.11)$$

and Z coincides with the Jost function $Z^{\pm}(k) = f^{\mp}(k)$ ³⁾. In addition, the Jost and Pais ⁶⁾ theorem holds

$$\bar{Z}^{(+)}(k) \equiv D(k) \quad (2.12)$$

where $D(k)$ is the Fredholm determinant. Thus our last result could be thought as a sort of verification of this property also in the case of the covariant treatment, in ladder approximation, of the two-body problem.

c.) The previous considerations can be easily generalized to the case of bound states of higher spin. To this end we consider the vertex function between a spin "s" particle and the two scalar fields ψ . We remember then that a spin "s" particle can be represented by a totally symmetric tensor of rank s, with vanishing traces and which obeys some auxiliary conditions. Thus the general form of the vertex function will be

$$\Gamma_{\mu\nu\dots\lambda}(p) = S_{\mu\nu\dots\lambda}(p) \Gamma^{(s)}(p^2) \quad (2.13)$$

where $S_{\mu\nu\dots\lambda}$ is the totally symmetric, traceless tensor we can form from $p_\mu, p_\nu, \dots, p_\lambda$. On the other hand the more general entities with these properties are the harmonic polynomials in the four dimensional space, that is we can write (Euclidean metric is clearly understood)

$$\Gamma_{\mu\nu\dots\lambda}(p) = p^s Y_{\ell m}(\theta, \varphi) (\sin \theta_z)^{\ell} G_{s-\ell}^{\ell+1}(\cos \theta_z) \Gamma^{(s)}(p^2) \quad (2.14)$$

where the G functions are normalized Gegenbauer polynomials ⁷⁾ and the other symbols have the standard meaning. Thus the equation for the vertex part becomes

$$G_s^1(\cos \theta_p) \Gamma^{(s)}(p^2) = Z^{(s)} G_s^1(\cos \theta_p) + \frac{\lambda}{4} \int_0^\infty \frac{u' du'}{(u'+1)^2} \Gamma^{(s)}(u') \left(\frac{u'}{u}\right)^{s/2} \int_0^\pi \frac{\sin^2 \theta_{p'} G_s^1(\cos \theta_{p'}) d\theta_{p'}}{p^2 + p'^2 - 2pp' \cos \theta_{pp'}} \quad (2.15)$$

The integration on the angles can be easily done by developing the denominator in Gegenbauer polynomials and we quote the final result

$$\Gamma^{(s)}(u) = Z^{(s)} + \lambda \int_0^\infty du' K_s(u, u') \Gamma^{(s)}(u') \quad (2.16)$$

8.

with

$$K_S(u, u') = \frac{1}{4(S+1)(u'+1)^2} \left\{ \Theta(u'-u) + \Theta(u-u') \left(\frac{u'}{u} \right)^{S+1} \right\} \quad (2.17)$$

With the same procedure as before we derive the differential equation

$$\left\{ \frac{d^2}{du^2} + \frac{S+2}{u} \frac{d}{du} + \frac{\lambda}{4u(u+1)^2} \right\} \Gamma^{(S)}(u) = 0 \quad (2.18)$$

and the boundary conditions

$$\begin{aligned} \lim_{u \rightarrow 0} u^{S+2} \Gamma^{(S)}(u) &= 0 \\ \lim_{u \rightarrow \infty} \left[\Gamma^{(S)}(u) + \frac{u}{S+1} \Gamma^{(S)'}(u) \right] &= Z^{(S)} \end{aligned} \quad (2.19)$$

Along the lines of the previous discussion we find that

$$\begin{aligned} Z_0^{(S)} &= \frac{\Gamma(S+2) \Gamma(S+1)}{\Gamma(S+1+a) \Gamma(S+2-a)} = \\ &= - \frac{4^{S+1} \Gamma(S+2) \Gamma(S+1) \cos \left[\frac{\pi}{2} \sqrt{1+\lambda} \right]}{\pi \lambda (8-\lambda)(24-\lambda) \dots [(2S+1)^2 - \lambda]} \end{aligned} \quad (2.20)$$

and from its vanishing the eigenvalue condition

$$\lambda_m = 4(m+S+1)(m+S+2) \quad m=0, 1, \dots \quad (2.21)$$

Correspondingly the bound states wave functions are

$$\psi_m^{(s)}(p) = p^s G_s'(\cos\theta) (p^2 + u^2)^{-\frac{s+1}{2}} {}_2F_1\left(n+s+2, s+1-m, s+2, \frac{u}{u+1}\right) \quad (2.22)$$

It can be amusing to compare the discussed results with the corresponding situation in non-relativistic theory for a Coulomb potential. If we take care of the correct behaviour of the wave function at large distances, the well-known expression for the Jost function can be derived

$$f_l(k) = \frac{\Gamma(l+1)}{\Gamma(l+1-i\gamma)} = - \frac{\Gamma(l+1+i\gamma) \pi \sin(\pi i\gamma) \Gamma(l+1)}{i\gamma \cdot \gamma^2 \dots [(l-1)^2 + \gamma^2] [l^2 + \gamma^2]}$$

$$\gamma = - \frac{ze^2}{2k}$$

and its zeros lying on the imaginary negative axis give the hydrogen series

$$k_m = - \frac{1}{2} \frac{ze^2}{(m+l+1)}$$

3. RENORMALIZABLE THEORIES

One of the properties which characterize, in this kind of problems, the renormalizable interactions is that the kernel of the integral equation is no longer of the Fredholm type. This has as a consequence the fact that every iteration (for the vertex function) is divergent. The reasons of this difficulty and the way to overcome it, at least in the bound state problem, have been discussed in detail elsewhere⁸⁾. On the other hand the standard procedure, which visualizes this difficulty consists in the introduction of a cut-off and we already have discussed the way to understand the bound state condition in this case.

In this section we will follow, in different examples, two approaches. Namely in one case we will use the procedure by Bastai et al. to pick up a finite Z function (or, if someone prefers, the finite part of it); in another, formally simpler, case we will explicitly introduce the cut-off and discuss some peculiar features of the problem.

Let us start by considering the case of two scalar particles ψ interacting through the exchange of one and of a pair of massless mesons ϕ , that is with an interaction

$$L_I = g_1 \psi^2 \phi + g_2 \psi^2 \phi^2 \quad (3.1)$$

and the g_2 part of L_I gives rise to a renormalizable interaction. The equation for the vertex relative to a bound state of zero momentum is, in ladder approximation,

$$\begin{aligned} \Gamma(p) = Z + \frac{g_1^2}{(2\pi)^4} \int \frac{dp' \Gamma(p')}{(p'^2 + m^2)^2 (p-p')^2} + \\ + \frac{g_2^2}{(2\pi)^8} \int \frac{dp' \Gamma(p')}{(p'^2 + m^2)^2} \int \frac{dk}{k^2 [k+p'-p]^2} \end{aligned} \quad (3.2)$$

In this case it is very useful to perform as an intermediate step the transition to the co-ordinate space. Using the relation

$$\int \frac{e^{iqx}}{q^2} dq = \frac{4\pi^2}{x^2} \quad (3.3)$$

equation (3.2) becomes

$$(\Delta - m^2)^2 F(x) = Z \delta(x) + \lambda_1 \frac{F(x)}{x^2} + \lambda_2 \frac{F(x)}{x^4} \quad (3.4)$$

where

$$F(x) = \frac{1}{(2\pi)^4} \int \frac{dp T(p)}{(p^2 + m^2)^2} e^{ipx} \quad (3.5)$$

$$\lambda_1 = g_1^2/4\pi^2, \quad \lambda_2 = g_2^2/(4\pi^2)^2$$

We can go to polar co-ordinates writing Eq. (3.4) as

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - m^2 \right) \varphi(r) = Z \frac{\delta(r)}{r^2} + \frac{\lambda_1}{r^2} \varphi + \frac{\lambda_2}{r^4} \varphi \quad (3.6)$$

$$F(x) = \varphi(r)/r$$

As is well known, the troubles come from the $1/r^4$ part of the potential but it is possible to apply to Eq. (3.6) the procedure discussed in Ref. ⁸⁾ which enables us, taking carefully into account the small distance behaviour, to reduce the problem to a less singular one. For a complete discussion we send back to the above cited papers and use the fact that Eq. (3.6) can be put in the form

$$r^{-\beta} \bar{O} r^{\beta} \bar{O} \bar{\varphi}(r) = Z \frac{\delta(r)}{r^2} + \frac{\lambda}{r^2} \bar{\varphi}(r) \quad (3.7)$$

where the symbols have the following meaning

$$\bar{O} = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\bar{V}^2}{r^2} - 1 \quad (w=1)$$

$$\beta = 2 - \left\{ 4 - 2\sqrt{-\lambda_2} \right\}^{1/2}; \quad \bar{V}^2 = 1 + \beta - \beta^2/4 = 1 + \frac{1}{2}\sqrt{-\lambda_2} \quad (3.8)$$

$$\lambda = \lambda_1 - \beta(\beta-1) \quad ; \quad \bar{\varphi} = r^{-\beta/2} \varphi.$$

We observe that the choice of the above parameters ensures us that as $\lambda_2 \rightarrow 0$ Eq. (3.7) goes back to the super-renormalizable theory case. Moreover, complete reality conditions are satisfied only if $-4 < \lambda_2 < 0$. The advantage of the version (3.7) of our problem is that in this way, the more singular part of the potential is collected in the "free" operator on the left-hand side. Then we can go to a momentum space putting

$$\bar{\varphi}(r) = \int_0^\infty J_{\bar{V}}(\sqrt{u}r) \varphi(u) \frac{du}{u+1} \quad (3.9)$$

and after some manipulations we arrive to the integral equation

$$\varphi(u) = \bar{Z} H(\bar{\gamma}, \beta, u) + \frac{\lambda}{4} \int_0^\infty du' \varphi(u') K(\bar{\gamma}, \beta, u, u') \quad (3.10)$$

where the kernel $K(\bar{\gamma}, \beta, uu')$ has been already calculated in Ref. ⁸⁾ (part II, App. B, Eq. 13). Moreover

$$H(\bar{\gamma}, \beta, u) = \frac{1}{2\bar{\gamma}} \int_0^\infty \frac{du'}{u'+1} (u')^{\bar{\gamma}/2} \int_0^\infty s^{1-\beta} ds J_{\bar{\gamma}}(\sqrt{u'} s) J_{\bar{\gamma}}(\sqrt{u} s) \quad (3.11)$$

and

$$\bar{Z} = \lim_{R_0 \rightarrow 0} Z R_0^{\bar{\gamma} + \beta/2 - 1} \quad (3.12)$$

In order to understand the meaning of the definition (3.12), we remember that Z can also be defined as the coefficient in co-ordinate space of the singular part of the wave function ^{*)}. However, this definition is completely meaningful (Z finite) only for super-renormalizable theories where the potential (λ_1/r^2 in our example) does not determine the small r behaviour. On the contrary for the renormalizable theories the potential (λ_2/r^4) directly enters into the small distances behaviour so that Z , as usually defined, is infinite.

*) To be more precise since we have a fourth order differential equation there are two regular and two irregular solutions. Here Z is defined as the coefficient of the less singular solution $\sqrt{\text{Eq. (3.12)}}$.

In spite of our careless treatment it is still possible to extract from Z a finite part which contains all the information of physical interest. In our problem this term corresponds to $\bar{Z}$ that in fact, as we will verify later, is finite. It is in complete agreement with the spirit of the treatment we have outlined here, to get automatically this finite $\bar{Z}$ so that we ascertain, once more, that a proper treatment at small distances necessarily has to give finite answers.

We now have to evaluate the quantity $H(\bar{\nu}, \beta, u)$. We outline only the procedure which is analogous to that used in Ref. ⁸⁾ (part II) to calculate K . We start by calculating the $\oint$ integral which introduces an hypergeometric function. Using its integral representation we can next perform the u' integration which gives a Legendre function as a final result. We quote the result :

$$H(\bar{\nu}, \beta, u) = 2^{1-\bar{\nu}-\beta} \Gamma(1-\beta/2) \frac{\Gamma(\bar{\nu}+1-\beta/2)}{(u+1)^{1-\beta/2}} P_{-\beta/2}^{-\bar{\nu}}\left(\frac{1-u}{1+u}\right) \quad (3.13)$$

It is convenient to introduce a new function

$$\varphi(u) = \frac{\varphi(u) (u)^{-\bar{\nu}/2}}{4(u+1)^{\beta/2-1}} \quad (3.14)$$

which obeys the integral equation

$$\varphi(u) = \bar{Z} C_1 u^{-\bar{\nu}/2} P_{-\beta/2}^{-\bar{\nu}}\left(\frac{1-u}{1+u}\right) + \lambda \int_0^\infty \frac{\varphi(u') R(\bar{\nu}, \beta, u, u') du'}{(u+1)^2} \quad (3.15)$$

where C_1 is the appropriate constant and the expression for $R(\sqrt{\beta}uu')$ is

$$R(\sqrt{\beta}, u, u') = \frac{1}{4} (\sqrt{\beta} - \beta/2) \Gamma(\sqrt{\beta} - \beta/2) \Gamma(\sqrt{\beta} + \beta/2) \left(\frac{u'}{u}\right)^{\sqrt{\beta}/2} \cdot \left\{ \Theta(u-u') P_{-\beta/2}^{-\sqrt{\beta}} \left(\frac{1-u'}{1+u'} \right) P_{-\beta/2}^{-\sqrt{\beta}} \left(\frac{u-1}{u+1} \right) + \Theta(u'-u) P_{-\beta/2}^{-\sqrt{\beta}} \left(\frac{1-u}{1+u} \right) P_{-\beta/2}^{-\sqrt{\beta}} \left(\frac{u'-1}{u'+1} \right) \right\} \quad (3.16)$$

It is not useless at this point to check the limit $\lambda_2 \rightarrow 0$: now $\beta \sim 0$, $\sqrt{\beta} \sim 1$, $\lambda \sim \lambda$, and remembering that

$$P_0^{-1} \left(\pm \frac{1-u}{1+u} \right) = u^{\pm 1/2}$$

equation (3.15) becomes

$$\psi_0(u) = z_0 + \lambda \int_0^\infty \frac{\psi_0(u') du'}{4(u'+1)^2} \left\{ \Theta(u-u') \frac{u'}{u} + \Theta(u'-u) \right\}$$

which is Eq. (2.2), as it should be.

We use now the fact that

$$\left\{ \frac{d^2}{du^2} + \frac{\sqrt{\beta}+1}{u} \frac{d}{du} + \frac{\beta/2(\beta/2-1)}{u(u+1)^2} \right\} u^{-\sqrt{\beta}/2} P_{-\beta/2}^{-\sqrt{\beta}} \left(\pm \frac{1-u}{1+u} \right) = 0$$

to deduce for $\psi(u)$ the differential equation

$$\left\{ \frac{d^2}{du^2} + \frac{\bar{\nu}+1}{u} \frac{d}{du} + \frac{\lambda_1}{4u(u+1)^2} \right\} \psi(u) = 0 \quad (3.17)$$

with the boundary conditions

$$\lim_{u \rightarrow 0} u^{\bar{\nu}+1} \psi'(u) = 0$$

$$\lim_{u \rightarrow \infty} [u \psi'(u) + \bar{\nu} \psi(u)] = \bar{Z} C; \quad C = \frac{\Gamma(\bar{\nu}+1) \Gamma(1-\beta/2)}{2^{\bar{\nu}+\beta-1} \Gamma(\bar{\nu}+\beta/2)} \quad (3.17')$$

This problem is completely analogous to the one studied in section 1 the only difference being $2 \rightarrow \bar{\nu}+1$. Thus at the end of this business the effect of the λ_2 part of the interaction is to produce a shift of a term (the centrifugal one) of the free Hamiltonian. So we get (with the same remarks as before on the mass shell normalization)

$$\psi(u) = {}_2F_1(a, b, c; u/u+1), \quad c = \bar{\nu}+1$$

$$a, b = \frac{1}{2} (1 \pm \sqrt{1+\lambda_1}) \quad (3.18)$$

$$\bar{Z} = \frac{2^{\bar{\nu}+\beta-1} \Gamma(\bar{\nu}+1) \Gamma(\bar{\nu}+\beta/2)}{\Gamma(1-\beta/2) \Gamma(\bar{\nu}+a) \Gamma(\bar{\nu}+1-a)}$$

$\bar{Z}$ coincides, as before, with the Fredholm determinant of the modified equation and is an entire function of λ_1 . Moreover the eigenvalues spectrum is

$$\lambda_{1m} = 4(m+\bar{\nu})(m+\bar{\nu}+1) \quad m=0, 1, \dots \quad (3.19)$$

as found in Ref. ⁸⁾. On the other hand the dependence on λ_2 shows the presence of critical values for the coupling constant. We have to discuss later this property of some renormalizable theories, anyway we remark that the condition $-4 < \lambda_2 < 0$ implies imaginary coupling constants. This unaesthetic feature is weakened if we consider higher spin bound states. With the same considerations as in Section 3 we arrive to the same results of Eq. (3.18) with the only difference in the definitions

$$\beta = 2 - \left\{ 2 \left[(s+1)^2 + 1 \right] - 2 \left[s^2 (s+2)^2 - \lambda_2 \right]^{1/2} \right\}^{1/2}$$

$$\bar{\gamma} = \left\{ \frac{1 + (s+1)^2 + \left[s^2 (s+2)^2 - \lambda_2 \right]^{1/2}}{2} \right\}^{1/2}$$

The allowed interval for λ_2 is now $-4(s+1)^2 < \lambda_2 < s^2(s+2)^2$ and real coupling constants lie in the interval $0 < \lambda_2 < s^2(s+2)^2$. Moreover, in the g_2 plane we have branch points at $\pm(s+2)s$, $\pm 2i(s+1)$.

4. RENORMALIZABLE INTERACTIONS (CONTINUED)

Let us turn now to the case of a theory $\psi(\gamma_5, 1)\psi$ (ψ massless) where the spinors ψ can form a pseudoscalar bound state of zero mass. Owing to this character of the composite particle $\Gamma(p)$ takes the simple form $\gamma_5 \Gamma(p^2)$ $\Gamma(p^2)$ is a scalar function and in the standard way we get

$$\Gamma(u) = Z - \lambda \varepsilon \int_0^u du' K(u, u') \Gamma(u') \quad (4.1)$$

where

$$K(u, u') = \frac{1}{4(u'+1)} \left\{ \theta(u-u') \frac{u'}{u} + \theta(u'-u) \right\} \quad (4.2)$$

and $\varepsilon = \pm 1$ according to the γ_5 or 1 term in the interaction. We denote the two cases as repulsive and attractive respectively.

We can derive the differential equation

$$\left\{ \frac{d^2}{du^2} + \frac{2}{u} \frac{d}{du} - \frac{\lambda \varepsilon}{4u(u+1)} \right\} \Gamma(u) = 0 \quad (4.3)$$

with the boundary conditions

$$\begin{aligned} \lim_{u \rightarrow 0} u^2 \Gamma'(u) &= 0 \\ \lim_{u \rightarrow \infty} [u \Gamma'(u) + \Gamma(u)] &= Z \end{aligned} \quad (4.4)$$

For this problem we will use the cut-off procedure and the simplest way is to ascribe a maximum radius L to our space so that the last condition (4.4) becomes

$$Z \approx Z(L) = \int_0^L [u \Gamma'(u) + \Gamma(u)] du \quad (4.4_L)$$

The solution of Eq. (4.3), regular at the origin, is an hypergeometric function

$$\Gamma(u) \sim {}_2F_1(a, b, c; -u) \quad ; \quad a, b = \frac{1}{2} (1 \pm \sqrt{1 + \lambda \varepsilon}) \quad (4.5)$$

$$c = 2$$

In addition, the particular properties of the δ_5 matrices make the problem free from infra-red divergences so that we can define $\Gamma(u)$ on the mass shell. Thus the normalized vertex is

$$\Gamma(u) = \frac{{}_2F_1(a, b, c; -u)}{{}_2F_1(a, b, c; 1)} = ab \Gamma(a) \Gamma(b) {}_2F_1(a, b, c; -u) \quad (4.6)$$

Then we evaluate $Z(L)$ with the result

$$Z(L) = ab \Gamma(a) \Gamma(b) \left\{ \frac{\Gamma(b-a)}{\Gamma^2(b)} L^{-a} (1 + O(1/L)) + \frac{\Gamma(a-b)}{\Gamma^2(a)} L^{-b} (1 + O(1/L)) \right\} \quad (4.7)$$

It is apparent that this quantity is still analytic in λ (symmetry with respect to the exchange $a \leftrightarrow b$) because a finite cut-off makes the problem still of the Fredholm type. Let us go to the limit of very large cut-off. It is necessary to distinguish, at this point, between the attractive and repulsive cases.

a) Repulsive case

$$a, b = \frac{1}{2} (1 \pm \sqrt{1+\lambda}) ; \quad a > 1, \quad b < 0$$

so that for Z we get

$$Z \simeq L^{-b} \frac{ab \Gamma(b) \Gamma(a-b)}{\Gamma(a)} \quad (4.8)$$

After Eq. (4.8) to say that $Z = \infty$ is not all the story because we have a factorization of the cut-off and beside this a finite part whose vanishing can give the bound states. In the particular example we are considering there are no eigenvalues for λ (beside the trivial solution $\lambda = 0$ ^{*)}). Moreover, we remark that the good analyticity properties in λ are lost in the limit $L \rightarrow \infty$, which stresses the singular character of this procedure.

b) Attractive case

$$a, b = \frac{1}{2} (1 \pm \sqrt{1-\lambda}) ; \quad 0 < a < 1, \quad 0 < b < \frac{1}{2} \quad \text{if} \quad 0 < \lambda < 1$$

otherwise a, b are complex conjugate.

We are here in the presence of critical values of the coupling constant, a fact strictly peculiar to renormalizable theories. For $\lambda < 1$, the result goes, in the limit $L \rightarrow \infty$, as in the previous case, namely

$$Z(L) \simeq L^{-b} \frac{ab \Gamma(b) \Gamma(a-b)}{\Gamma(a)} \quad (4.8')$$

*) This corresponds to the result discussed by Goldstein ⁹⁾ some time ago.

and the same comments are of value here. Moreover $Z \rightarrow 0$ (in the same sense as before) ^{*)}. On the contrary for $\lambda > 1$ both the terms in the expression (4.7) for $Z(L)$ are of the same order so that we can write

$$Z(L) \approx ab \Gamma(a) \Gamma(b) L^{-1/2} \left\{ \frac{\Gamma(b-a)}{\Gamma^2(b)} L^{-\frac{1}{2}\sqrt{\lambda-1}} + \frac{\Gamma(a-b)}{\Gamma^2(a)} L^{\frac{1}{2}\sqrt{\lambda-1}} \right\} \quad (4.9)$$

There is no complete factorization of the cut-off now and the bound state condition $Z = 0$ seems more difficult to understand because the part to be put to zero is not unique. In other words a vanishing of Z necessarily gives a spectrum dependent on L .

However, one can show that each eigenvalue $\lambda_n(L)$ decreases to 1 when L goes to infinity. Moreover the spacing of the eigenvalues simultaneously goes to zero, in such a way that one gets, in the limit, a continuum of eigenvalues from 1 to $+\infty$.

The situation we just reviewed is completely analogous to what happens in potential scattering with potentials of the form $1/r^2$ at the origin.

This can be made quite apparent if we go to co-ordinate space. The equivalent of Eq. (4.3) is now (see the following section for an explicit derivation)

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - u^2 \right) \psi(r) = \pm \frac{\lambda}{r^2} \psi(r) \quad (4.10)$$

with the boundary conditions

$$\begin{aligned} \lim_{r \rightarrow \infty} e^{-r} r^{1/2} (\psi' + \psi) &= 0 \\ -\frac{1}{2} \lim_{r \rightarrow 0} r (r \psi' - \psi) &= Z \end{aligned} \quad (4.11)$$

^{*)} Something like this seems to happen in Q.E.D. ¹⁰⁾ but there the problem is more complicated owing to the vector character of the photon.

Eq. (4.10) can be written

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\lambda^2}{r^2} - u^2 \right) \psi(r) = 0 \quad (4.12)$$

$$\lambda^2 = 1 \pm \lambda = (a-b)^2$$

For the repulsive case (and the attractive with $\lambda < 1$) the solution vanishing at the infinity (as required by the first boundary condition) is

$$\psi(r) = K_\lambda(r) \quad (4.13)$$

and

$$Z = -\frac{1}{2} (\lambda+1) \Gamma(-\lambda) \left(\frac{r_0}{2} \right)^{1-\lambda} \quad (r_0 \rightarrow 0) \quad (4.14)$$

where r_0 is the cut-off in co-ordinate space ^{*)}. For the attractive case $\lambda > 1$ the solution can be written

$$\psi(r) = A I_{\lambda}(r) + B I_{-\lambda}(r) \quad (4.15)$$

*) We see from here that Z is not only a function of a cut-off but it depends also on the regularization procedure itself. (Compare Eqs. (4.8) and (4.14). The important thing is that the physical quantities, such as Z itself or the spectrum, go to limits which are independent of the regularization procedure, when one removes it. For instance, the two Z 's above go to zero when $L \rightarrow \infty$ and $r_0 \rightarrow 0$. Moreover, the two spectra have the same limit: a continuum from 1 to ∞ .

The first boundary condition requires $A+B = 0$ and from the second one we get

$$Z = \cos t, \left\{ \frac{iv-1}{\Gamma(1+iv)} e^{iv\ell u r_0} + \frac{iv+1}{\Gamma(1-iv)} e^{-iv\ell u r_0} \right\} \pi_0 \quad (4.16)$$

analogous to Eq. (4.9) ($r_0^2 \sim L^{-1}$). The origin of these difficulties lies in the fact that it is no more possible to discriminate between the regular and irregular solution at the origin (or at infinity in the "u" space). Thus, Z being defined as the coefficient of the irregular part of the wave function, its vanishing loses the usual meaning of giving the regular solution. We can also remember the non-relativistic version of our problem which has been discussed by Case¹¹⁾. It is shown there that the above remarked ambiguity introduces in the solution an arbitrary phase factor φ so that the bound state spectrum of the energy depends on this quantity. In other words, we have a continuum of eigenvalues. Our Eq. (4.10) is the same as a Schrödinger equation with energy $k^2 = -m^2$ fixed, in presence of a λ/r^2 potential. Once the energy is fixed, the arbitrary phase is fixed so that there is no Case factor in our solution. We will come back to this point in the next section.

5. NON-RENORMALIZABLE INTERACTIONS

After the work by Feinberg and Pais ¹²⁾ with their peratization recipe and by Domokos et al. ¹³⁾ in the frame of the Bethe-Salpeter equation, non-renormalizable interactions no longer seem so terrible. Thus we conclude our look with a short discussion of this case, with the hope that a better understanding of some aspects of the matter can be gained. As a test theory between the non-renormalizable ones, we choose the interaction

$$L_I = g \bar{\Psi} (\gamma_5, 1) \Psi \phi^2 \quad (5.1)$$

where the non-renormalizable character of the problem is due to the $1/2$ spin of the Ψ fields while the ϕ are massless bosons. Clearly the first interaction is not parity conserving : as before, the two couplings $(\gamma_5, 1)$ will be denoted as repulsive and attractive respectively. The model is academic, however, it has all the difficulties of the more realistic ones and, in addition it can be solved exactly. If we consider a pseudoscalar bound state in the ladder approximation, the vertex function takes the form $\Gamma(p) = \gamma_5 \Gamma(p^2)$ and it satisfies the equation

$$\Gamma(p^2) = Z - \frac{\lambda \varepsilon}{(2\pi)^4} \int \frac{d^4 p' \Gamma(p'^2)}{p'^2 + m^2} \int \frac{dk}{k^2 (k + p' - p)^2} \quad (5.2)$$

$$\lambda = \frac{g^2}{(2\pi)^4 m^2}$$

where $\varepsilon = \pm 1$ according to the γ_5 or 1 coupling.

Now we go to a Euclidean metric ^{*)}, we perform a Fourier transform and defining

*) Really the delicate point is here represented by the transition to the Euclidean metric, because $\Gamma(p^2)$ has an essential singularity at infinity. Arguments have been put forward to justify this procedure. R.F. Sawyer ¹⁴⁾ starts directly from the equation in x-space and continues it to imaginary time values according to the Schwinger prescription ¹⁵⁾. On the other hand Domokos et al. ¹³⁾ take this procedure as a way to give a reasonable definition of the non-renormalizable Green functions.

$$\frac{\Gamma(p^2)}{p^2 + \omega^2} = \frac{1}{(2\pi)^2} \int e^{-ipx} F(x^2) dx. \quad (5.3)$$

we find that the radial part $\psi(r) = rF(x^2)$ satisfies the integral equation

$$\psi(r) = Z \kappa_1(r) - \lambda \varepsilon \int_0^\infty \frac{dr'}{r'^3} \psi(r') \left\{ \Theta(r-r') \kappa_1(r) \right. \\ \left. I_1(r') + \Theta(r'-r) \kappa_1(r') I_1(r) \right\} \quad (5.4)$$

$$\omega = 1.$$

From this we arrive to the differential equation and boundary conditions

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} - 1 \right) \psi(r) = \varepsilon \lambda \psi(r) / r^4 \\ \lim_{r \rightarrow \infty} e^{-r} r^{1/2} (\psi + \psi') = 0 \quad (5.5)$$

$$\lim_{r \rightarrow 0} r (\psi' - \psi) = -2Z.$$

Equation (5.5) is analogous to a Schrödinger equation (aside the $-m^2$ in place of k^2), and with the particular potential we chose, it can be solved exactly in terms of Mathieu functions. In order to avoid the mathematical developments masking the physics of the problem, we quote here only the results, leaving the derivation for the Appendix.

In the repulsive case we find ($m = 1$)

$$Z = \sqrt{\frac{\pi}{8}} (\lambda)^{1/4} e^{\mu r/2} B r_0^{1/2} e^{\sqrt{\lambda} r/r_0}, \quad r_0 \rightarrow 0 \quad (5.6)$$

where for λ not too large ($\lambda < \lambda_0 \sim 1300$)

$$B = \frac{\sin(2\varphi)}{\sinh(\mu\pi)}$$

μ is the so-called characteristic parameter of the equation and φ is the phase of a rather involved expression which is called D_+ in the Appendix. We do not worry about the existence of bound states, because the repulsive character of the interaction makes plausible that there are no solutions of the condition $B = 0$ ^{*)}. Rather we observe that the cut-off dependence is still factorized, as it also can be learned directly by studying the small r behaviour of the solutions of Eq. (5.6). We want to make a remark here: we remember that in the elastic region the S-matrix is given by

$$S(k^2) = \frac{Z(k^2 - i\varepsilon)}{Z(k^2 + i\varepsilon)} \quad (5.7)$$

Really in the case we are considering $k = 0$, so that we cannot calculate the S-matrix. However, the small distance behaviour is independent of k so that the factorization of the cut-off is a quite general property. This shows that, at least for ladder graphs, the S-matrix necessarily has to be finite. In other words, we are sure that if we introduce a cut-off to regularize the graphs of the

*) A more realistic situation, where probably there are bound states, could be obtained by adding a $1/r^2$ (renormalizable) interaction.

S-matrix perturbative expansion, we sum up and finally go to the limit of large (in p space) cut-off, the result must be cut-off independent and finite. This is a way to foresee the results derived by the authors cited at the beginning.

If we turn now to the attractive case, we find

$$Z = e^{i\pi/4} \sqrt{\frac{\pi}{8}} (\lambda)^{1/4} (\mu_0)^{1/2} \left\{ iA e^{-\mu^{1/2}} e^{i\sqrt{\lambda}/\mu_0} + B e^{\mu^{1/2}} e^{-i\sqrt{\lambda}/\mu_0} \right\} \quad (5.8)$$

where A , B are given in Appendix. Like for the renormalizable theories ^{*)}, in this case there is not a complete factorization of the cut-off dependence.

As a consequence the bound state condition gives a spectrum which depends on " r_0 ". Of course the same thing happens in non-relativistic theory and as a manifestation of this difficulty Case is forced to introduce an arbitrary phase factor in the solution. As already discussed this gives rise to a continuum spectrum.

Physical intuition does not disagree with the remark that very singular, attractive potentials give an ill-defined bound state problem. However, the same difficulty is present for the scattering. This can be understood for instance through Eq. (5.8) : if there is no complete factorization of the cut-off, this probably does not disappear in the ratio and we are finally left with a cut-off dependent S-matrix. The more immediate way to overcome this annoying point would be to ascribe a small imaginary part to the coupling constant ^{**)} : This allows to discriminate between regular and irregular solutions. At the same time it gives rise to a complex potential and, therefore to a strong absorption. In fact it is well verified that the scattering length is, in the attractive case, complex ¹³⁾.

*) The presence for renormalizable interactions of critical constants is related to the fact that there the repulsive, centrifugal part has the same $1/r^2$ singularity at the origin, so that there is a competition between centrifugal and external potentials.

***) We thank Professor Fubini for a clarifying discussion on this point.

6. CONCLUDING REMARKS

As a conclusion we want to emphasize the more relevant points which result from the above considered models.

In the case of super-renormalizable interactions Z is finite (if we do not worry about the infra-red divergences) and we have established its relation with the Fredholm determinant of the integral equation for the vertex part : a "Jost and Pais theorem" is still valid in the covariant example we have studied.

We have then considered the theories where the introduction of a cut-off seems, at least at present, the only way to give a meaning to otherwise divergent quantities. Really this makes the bound state condition cut-off dependent and therefore difficult to understand. However, for renormalizable theories, just their defining property (which after all means that physical results are cut-off independent) ensures us that the bound state condition is still meaningful owing to a factorization of the cut-off dependence²⁾. So we are left with a finite part to put to zero. Actually, this part has to be defined in an unambiguous way and this rests on the possibility to isolate, in the limit of very large cut-off, a single dominant term with its coefficient. In ladder approximation, for renormalizable theories, Z becomes a power function of the cut-off. When this cut-off goes to infinity (in momentum space) Z is infinite if the potential is repulsive and zero if the potential is attractive. In this last case it appears that the cut-off dependence cannot be factorized when the coupling constant is sufficiently large. Therefore the spectrum, given by the condition $Z = 0$, is still cut-off dependent but we have seen that, when removing the regularization, it tends towards a well-defined limit that is a continuum spectrum.

For non-renormalizable theories nothing so general can be said about a possible factorization of the cut-off. However, we have seen that for the sum of a particular class of graphs (the ladder ones) this property is still verified.

More precisely this happens, in the sense discussed above, only for repulsive potentials and not for attractive ones. Thus, in this respect, the situation for renormalizable and non-renormalizable theories is not very different, aside the form of the dependence of Z on the cut-off. Clearly we have verified this common feature only in ladder approximation and it is not given to foresee if this remains true also in the general case.

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RIASSUNTO

Vengono considerati diversi esempi di teorie di campo, super-renormalizzabili, rinormalizzabili e non rinormalizzabili, per calcolare la costante Z che rinormalizza il vertice tra uno stato legato e le particelle componenti. In approssimazione a gradino, vengono esaminate alcune proprietà della quantità Z , in particolare per quel che riguarda la possibilità di fattorizzare il fattore di taglio.

A P P E N D I X

We discuss briefly here the solutions of Eq. (5.5) and the consequent determination of the Z 's. We start from the equation

$$\left\{ \frac{d^2}{d\zeta^2} + \frac{1}{\zeta} \frac{d}{d\zeta} - \left(1 + \frac{1}{\zeta^2} + \frac{\lambda \varepsilon}{\zeta^4} \right) \right\} \psi = 0 \quad (\text{A.1})$$

where $\zeta = m r$. Now we define two constants ν and q by

$$\frac{1}{\nu^2} = \lambda \varepsilon \nu^2 = -q \quad (\text{A.2})$$

and make the change of variables

$$\begin{aligned} \nu \zeta &= e^{-z} & \text{if } \nu \zeta \text{ real} < 1 \\ \nu \zeta &= e^z & \text{if } \nu \zeta \text{ real} > 1 \end{aligned} \quad (\text{A.3})$$

Then $\psi(\zeta) = W(z)$ satisfies the modified Mathieu equation

$$W'' - (1 - 2q \cosh 2z) W = 0 \quad (\text{A.4})$$

The general solution of (A.4) can be written as an infinite series of cylindrical functions. It is convenient to choose as two linearly independent solutions

$$M_{-i\mu}^{(3,4)}(z, \sqrt{q}) = \left(\sum_{m=-\infty}^{+\infty} c_m \right)^{-1} \sum_{m=-\infty}^{+\infty} (-1)^m c_m H_{2m-i\mu}^{(1,2)}(2\sqrt{q} \cosh z) \quad (\text{A.5})$$

where μ is a complicated function of q , which can be computed by means of a Taylor expansion in q and the c_n are solutions of the system

$$q (C_{m+1} + C_{m-1}) = C_m [1 + (\mu + 2mi)^2] \quad (\text{A.6})$$

We must notice that the expansions (A.5) converge in two disconnected domains, the one containing the origin, the other the point at infinity. It is then necessary to use other expansions of the solutions of Eq. (A.4) which converge in domains connecting the regions of convergence of $M_{-i\mu}^{(3,4)}$. For further details see Refs. ¹⁶⁾ and ¹⁷⁾.

In the repulsive case one has ($\varepsilon = 1$)

$$v = \lambda^{-1/4}, \quad q = i(\lambda)^{1/4} \quad (\text{A.7})$$

so that the argument of the Hankel functions is

$$2\sqrt{q} \cosh z = i(\mu r + \sqrt{\lambda}/\mu r) \quad (\text{A.8})$$

In the attractive case ($\varepsilon = -1$)

$$v = e^{i\pi/4} (\lambda)^{-1/4}, \quad q = e^{i\pi/4} (\lambda)^{1/4} \quad (\text{A.7'})$$

$$2\sqrt{q} \cosh z = i\mu r + \sqrt{\lambda}/\mu r$$

When $r \rightarrow \infty$, $2\sqrt{q} \cosh z$ goes like $i\mu r$ in the two cases, but when $r \rightarrow 0$, $2\sqrt{q} \cosh z$ goes like $i\sqrt{\lambda}/\mu r$ or $\sqrt{\lambda}/\mu r$ respectively and this gives two completely different behaviours at the origin.

The asymptotic behaviour of $M_{-i\mu}^{(3,4)}$ at infinity is easily found to be

$$M_{-i\mu}^{(3,4)}(z, \sqrt{q}) \simeq (\mu r)^{-1/2} e^{\mp \mu r} \quad (\text{A.9})$$

with the upper sign for $M^{(3)}$ and the lower one for $M^{(4)}$. Now the first boundary condition excludes $M^{(4)}$, so that in the convergence domain around the point $r = \infty$, the solution reads

$$\omega(z) = A M_{-i\mu}^{(3)}(z, \sqrt{q}) = \left(\sum c_n\right)^{-1} \sum_m (-1)^m c_m H_{2m-i\mu}^{(1)}(2\sqrt{q}) \chi(z)$$

Moreover to fix its normalization, we will normalize $\psi(\zeta)$ in such a way that

$$\psi(\zeta) = \kappa_1(\zeta) \quad \zeta \rightarrow \infty$$

Near the origin $W(z)$ is

$$\omega(z) = A M_{-i\mu}^{(3)}(z, \sqrt{q}) + B M_{-i\mu}^{(4)}(z, \sqrt{q}) \quad (\text{A.10})$$

and using the method just outlined above we get

$$\begin{aligned} A &= \frac{1}{2\sinh\mu\pi} \left(D_-/D_+ - D_+/D_- \right) \\ B &= \frac{1}{2\sinh\mu\pi} \left(D_-/D_+ - e^{-2\mu\pi} D_+/D_- \right) \end{aligned} \quad (\text{A.11})$$

where

$$D_{\pm} = \left(\sum c_n\right)^{-1} \sum_m (-1)^m c_m J_{2m-i\mu}(2\sqrt{q}) \quad (\text{A.12})$$

When $r \rightarrow 0$, W is, in the repulsive case, the sum of two functions exponentially increasing and decreasing, and in the attractive one the sum of two infinitely oscillating functions. More precisely

$$W_r(z) \sim \left(\frac{\pi \mu r}{2\sqrt{\lambda}} \right)^{1/2} e^{\mu\pi/2} \left\{ A e^{-\sqrt{\lambda}/\mu r} e^{-\mu\pi/2} + iB e^{\sqrt{\lambda}/\mu r} e^{\mu\pi/2} \right\} \quad (A.13)$$

$$W_a(z) = \left(\frac{\pi \mu r}{2\sqrt{\lambda}} \right)^{1/2} e^{\mu\pi/2} e^{-i\pi/4} \left\{ iA e^{i\sqrt{\lambda}/\mu r} e^{-\mu\pi/2} + B e^{-i\sqrt{\lambda}/\mu r} e^{\mu\pi/2} \right\}$$

Using these expressions we can evaluate Z from the second boundary condition with the result given in the text.

As in the previous sections one can study the pseudotensor bound states of spin s . We quote here only the final results. For the repulsive case one has

$$Z_r^{(s)} = \frac{1}{(2\mu)^{s+2} (s+1)!} \sqrt{\frac{\pi}{2}} i (\lambda)^{1/4} e^{\mu_s \pi/2} B_s (\mu r_0)^{s+1/2} e^{\sqrt{\lambda}/\mu r_0}$$

$$r_0 \rightarrow 0$$

and for the attractive one

$$Z_a^{(s)} = \frac{e^{i\pi/4}}{(2\mu)^{s+2} (s+1)!} \sqrt{\frac{\pi}{2}} (\lambda)^{1/4} e^{\mu_s \pi/2} \left\{ iA_s e^{-\mu_s \pi/2} e^{i\sqrt{\lambda}/\mu r_0} + B_s e^{-i\sqrt{\lambda}/\mu r_0} e^{\mu_s \pi/2} \right\}$$

where A_s and B_s are still given by Eq. (A.11), and μ_s depends now on s .

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