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ON THE UNIQUENESS OF THE POSITION OPERATOR
FOR RELATIVISTIC ELEMENTARY SYSTEMS

A. Galindo ^{*)†)}

CERN, Geneva

ABSTRACT

The regularity condition first introduced by Newton and Wigner is here re-examined in some detail for the case of a relativistic elementary system $\underline{m} \neq 0, s$. Three different types of normalizable sequences $\{\psi_n\}$ approaching some localized state ψ_a are considered, namely, (A) : $\psi_n(p) \in \mathcal{D}(R^3) \times \mathcal{V}_{2s+1}$ and $\psi_n(p) \rightarrow \psi_a(p)$ in the topology of $L_2^{loc}(R^3) \times \mathcal{V}_{2s+1}$; (B) : $\psi_n(p) = \psi_a(p) \eta_n(p)$, where $\eta_n(p) \in \mathcal{S}(R^3)$ and $\eta_n(p) \rightarrow 1$ uniformly on any compact set; and (C) : ψ_n are characteristic functions in q -space with compact supports such that $\text{supp}_q \psi_n \supset \text{supp}_q \psi_{n+1} \supset \{a\}$, $\mu(\text{supp}_q \psi_n) \rightarrow 0$. It is then proved that even the ω -order regularity condition is satisfied by all possible position operators $\underline{Q}$ if (A) sequences are allowed, while the restriction to (B) or (C) sequences renders $\underline{Q}$ unique as soon as the 2nd order regularity condition is imposed. Finally a different but related abstract characterization is provided for the standard $Q^{(0)}$ as being that (unique) position operator for which Q^2 is essentially selfadjoint when restricted to the domain

$$D_{Q^2} \cap D_{M_{\mu\nu}} \cap D_{M_{\mu\nu_1} M_{\mu\nu_2}} \cap D_{M_{\mu\nu_1} M_{\mu\nu_2} V_1}$$

*) On leave of absence from the Junta de Energia Nuclear, Madrid.

†) Present Address: Facultad de Ciencias, Universidad de Zaragoza, Zaragoza.

INTRODUCTION

The localizability of relativistic elementary systems has been dealt with in two fundamental papers by Newton-Wigner ¹⁾ and Wightman ²⁾. Apart from the natural requirements that the position operator $\underline{Q}$ must be canonically conjugate to $\underline{P}$ and transform properly under rotations, parity and time inversion, these authors are forced to impose an extra mathematical condition, namely, the regularity condition (hereafter RC), in order to get uniqueness for $\underline{Q}$. This condition states explicitly that if ψ_a is an eigenfunction of $\underline{Q}$ with eigenvalue a , and if $\{\psi_n\}$ is a sequence of normalizable wave functions approaching ψ_a , then

$$\max_n \frac{\|M_{\mu\nu}\psi_n\|}{\|\psi_n\|} < \infty \quad (1)$$

where $M_{\mu\nu}$ are the infinitesimal generators of the homogeneous proper Lorentz group. Since $M_{\mu\nu}$ contain essentially first-order derivatives with respect to $\underline{p}$, the conclusion is drawn ²⁾ from the requirement (1) that $\psi_a(\underline{p})$ must be a continuous (vector-valued) function and consequently that $\underline{Q}$ is unique. However, in order to derive unambiguously which restrictions does (1) impose on $\underline{Q}$, i.e., which are those $\underline{Q}$ compatible with (1), both the class of allowed sequences $\{\psi_n\}$ and the topology in which $\psi_n \rightarrow \psi_a$ must be specified. Among all possible choices only three cases will be here analyzed which are intended to show how the consequences of (1) as to the uniqueness of $\underline{Q}$ do actually depend on the meaning of $\psi_n \rightarrow \psi_a$. Let us introduce briefly our notation so that we can present the results of this analysis :

Only $\underline{m} \neq 0, \underline{s} \geq \frac{1}{2}$, type $(++)$ relativistic elementary systems will be considered here. They will be described ³⁾ by $(2s+1)$ component wave functions $\psi(\underline{p})$ with squared norm

$$\|\psi\|^2 \equiv \int \psi^\dagger(\underline{p}) \psi(\underline{p}) \frac{d^3 p}{p_0} < \infty \quad (2)$$

2.

where $p_0 \equiv (m^2 + p^2)^{\frac{1}{2}}$, and $\Psi^+ \Psi \equiv \sum_{-S \leq m \leq S} \Psi_m^* \Psi_m$. The infinitesimal generators of the Poincaré group are given by

$$P^\mu = p^\mu, \quad \underline{M} = i \underline{p} \times \underline{\nabla}_p + \underline{S}, \quad \underline{N} = i p_0 \underline{\nabla}_p - (p_0 + m) \underline{S} \times \underline{p} \quad (3)$$

Finally, parity Π and time-inversion Θ are represented by

$$(\Pi \Psi)(\underline{p}) \equiv \Psi(-\underline{p}) \quad (4)$$

$$(\Theta \Psi)_m(\underline{p}) \equiv (-1)^{s+m} \Psi_{-m}^*(-\underline{p}) \quad (5)$$

Let $\tilde{\mathcal{H}}$ denote the Hilbert space of the wave functions Ψ satisfying (2); $\mathcal{V}_{2s+1}$ will stand for the $(2s+1)$ -dimensional Hilbert space consisting of $(2s+1)$ -component vectors v with norm $\|v\| \equiv (v^+ v)^{\frac{1}{2}}$; and finally, $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ will stand for the usual test-function spaces in the theory of distributions ⁴⁾.

The three aforementioned types of approaching normalizable sequences which we will analyze here are the following ones :

(A) : $\Psi_n(\underline{p}) \in \mathcal{D}(\mathbb{R}^3) \times \mathcal{V}_{2s+1}$ and $\Psi_n(\underline{p}) \rightarrow \Psi_a(\underline{p})$ in the topology of

$$L_2^{\text{loc}}(\mathbb{R}^3) \times \mathcal{V}_{2s+1}.$$

(B) : $\psi_n(p) = \psi_a(p) \eta_n(p)$, $\eta_n(p) \in \mathcal{F}(R^3)$ and $\eta_n(p) \rightarrow 1$ uniformly on any compact set.

(C) : $\psi_n(p) = c_n f_n(p) e^{ip \cdot a} \psi_a(p)$, where $f_n(p)$ is the Fourier transform of the characteristic function $\phi_n(q)$ of a compact set in R^3 , c_n are suitable normalization constants, and $\text{supp } \phi_n \supset \text{supp } \phi_{n+1} \supset \{a\}$, $\mu(\text{supp } \phi_n) \neq 0 \rightarrow 0$, where μ denotes here the Lebesgue measure in R^3 . c_n can be taken as

$$c_n = \left(\int \phi_n(q) d^3 q \right)^{-1}.$$

Sequences chosen according to any of these criteria will be called (A), (B), or (C) sequences, respectively. And they will be said to satisfy the n -th order regularity condition (n -RC) if

$$\max_m \frac{\| M_{\mu\nu}^{(r)} \psi_m \|}{\| \psi_m \|} < \infty \quad (6)$$

for all $r \leq n$, where $M_{\mu\nu}^{(r)} \equiv M_{\mu\nu} M_{\mu\nu} \dots M_{\mu\nu}$ (r factors). The requirement that (6) be satisfied for every r will be called the ∞ -th order regularity condition (∞ -RC).

The following statements will be established :

(A1) : Every localized wave function $\psi_a(p)$ of any position operator ⁵⁾ Q can be approached by (A) sequences which satisfy ∞ -RC.

(BC1) : The eigenfunctions $\psi_a(p)$ of Q can be approached by (B) or (C) sequences satisfying 1-RC if and only if Q belongs to a finite set $\mathcal{R}$ consisting of all those position operators whose eigenfunctions are continuous ⁶⁾ everywhere outside the origin $p = 0$.

(BC2) : There exists a unique position operator, namely,

$$Q^{(c)} \equiv i p_0^{1/2} \nabla_p p_0^{-1/2}$$

whose eigenfunctions can be approached by (B) or (C) sequences satisfying 2-RC.

These statements clearly show

- i) that the restrictions imposed on $\underline{Q}$ by the regularity condition depend strongly on the topological nature of the limit $\Psi_n \rightarrow \Psi_a$; and
- ii) that 2-RC, but not 1-RC, renders $\underline{Q}$ unique if (B) or (C) sequences are chosen. No explicit choice of sequences is made in ¹⁾ and ²⁾, even though the analysis in ²⁾ might suggest the use of (C) sequences. As to that order of RC which ensures uniqueness for $\underline{Q}$, it is parenthetically remarked ⁷⁾ in ¹⁾ that (6), with $r = 2$, must be imposed when $s > 0$ to eliminate the possibility of discontinuities of $\Psi_a(p)$ at $p = 0$, whereas only 1-RC is required in ²⁾ to make $\underline{Q}$ unique ⁸⁾⁹⁾. This rather confusing situation will be clarified in Section 1, where (A1), (BC1) and (BC2) will be proved. In Section 2 an alternative intrinsic characterization of $\underline{Q}^{(0)}$ will be given; more specifically, we will single out the relevant features of Section 1 by proving the following statements :

- (D1) : $\underline{Q} \in \mathcal{R}$ iff $Q^2 \equiv \underline{Q} \cdot \underline{Q}$ is essentially selfadjoint when restricted to the domain $D_1 \equiv D_{Q^2} \bigcap_{(\mu, \nu)} D_{M_{\mu\nu}}$.
- (D2) : $\underline{Q} = \underline{Q}^{(0)}$ iff Q^2 is essentially selfadjoint when restricted to

$$D_2 \equiv D_{Q^2} \bigcap_{(\mu, \nu)} D_{M_{\mu\nu}} \bigcap_{(\mu_1, \nu_1), (\mu_2, \nu_2)} D_{M_{\mu_1, \nu_1} M_{\mu_2, \nu_2}}$$

1. In this section, statements (A1), (BC1) and (BC2) will be established. Before doing this, however, we shall give explicitly the general expression for any position operator $\underline{Q}$ whose eigenfunctions satisfy the Newton-Wigner ¹⁾ postulates (a), (b) and (c), within the realization we adopted above for the $\underline{m} \neq 0$, $\underline{s}$, type (++) , elementary systems :

Let $\underline{Q}^{(0)} \equiv i p_0 \nabla_p^{-1/2}$ denote the standard Newton-Wigner position operator ³⁾. It is well known ²⁾³⁾ that any other position operator ⁵⁾ $\underline{Q}$ is a unitary transform of $\underline{Q}^{(0)}$

$$\underline{Q} = B \underline{Q}^{(0)} B^{-1} \quad (7)$$

where B is a unitary operator which commutes ²⁾ with the representation $U(a, A)$ of the Euclidean group $\mathcal{E}_3$ $[A \in SU(2)]$. Therefore B is given by a unitary matrix $B(\underline{p})$ such that ¹⁰⁾

$$B(\underline{p}) \mathcal{D}(A) = \mathcal{D}(A) B(A^{-1} \underline{p}) \quad (8)$$

where $\mathcal{D}(A)$ is the unitary irreducible representation of $SU(2)$ associated with S . From (8) and from the requirements

$$\pi \underline{Q} \pi^{-1} = -\underline{Q} \quad (9)$$

$$\theta \underline{Q} \theta^{-1} = \underline{Q} \quad (10)$$

it follows that B is completely determined by its values $B(\underline{r})$ along the 3rd-axis ($r_1 = r_2 = 0$) and that $B(\underline{r})$ can always be taken, without any loss of generality, as a real diagonal matrix ⁽¹¹⁾ such that its diagonal elements satisfy

$$B_{mm}(\underline{r}) = B_{-m,-m}(\underline{r}) \quad \text{if } S \text{ is integer} \quad (11)$$

$$B_{mm}(\underline{r}) = \epsilon B_{-m,-m}(\underline{r}) \quad \text{if } S \text{ is half-integer} \quad (12)$$

where $\epsilon = \pm 1$ and independent of the index value m . $\overline{(11)}$ and (12) do not involve summation over $\underline{m}$. It is now quite simple to write down the most general form for $B(\underline{r})$ in terms of S_3 (third component of S) and then use (8) to obtain $B(\underline{p})$. The general result is this :

(1.1) : $B(\underline{p})$ defines through (7) a position operator whenever

$$B(\underline{p}) = e^{i\pi\alpha(\underline{p})} e^{i\pi\beta(\underline{p})} \quad (13)$$

where

- i) $\alpha(|\underline{p}|)$ is any measurable function satisfying $\alpha^2 = 1$, and
- ii) $\beta(\underline{p})$ is any linear combination, with coefficients 0 or 1, of the following Hermitean operators :

$$\beta(p) \equiv \prod_{\substack{m' \neq m \\ m' \geq 0}} \frac{1}{m'^2 - m^2} \left[m'^2 - \left(\frac{S \cdot p}{|p|} \right)^2 \right], \quad 1 \leq m \leq S \quad (\text{case } S \text{ integer})$$

$$\beta_{\frac{1}{2}}(p) \equiv \frac{S \cdot p}{|p|} - \frac{1}{2}, \quad \beta(p) \equiv \prod_{\substack{m' \neq m \\ m' \geq \frac{1}{2}}} \frac{1}{m'^2 - m^2} \left[m'^2 - \left(\frac{S \cdot p}{|p|} \right)^2 \right], \quad \frac{3}{2} \leq m \leq S \quad (\text{case } S \text{ half-integer})$$

(1.2) : Conversely, any position operator $\underline{Q}$ can be obtained from $\underline{Q}^{(0)}$ through (7) by means of some $\underline{B}(p)$ of the form (13) where α and β satisfy i) and ii) of (1.1). Given B_1 and B_2 , as given by (1.1),

$$B_1(p) = e^{i\pi\alpha_1(|p|)} e^{i\pi\beta_1(p)}, \quad B_2(p) = e^{i\pi\alpha_2(|p|)} e^{i\pi\beta_2(p)}$$

in order for B_1 and B_2 to lead to the same $\underline{Q}$ through (7) it is necessary and sufficient that $\beta_1 \equiv \beta_2$ and that $\alpha_1 \equiv \pm \alpha_2$.

Consequently, we can say that, up to the real arbitrary phase-factor unitary transformation $e^{i\pi\alpha(|p|)}$, there exist 2^S different position operators for integer s , and $2^{S+\frac{1}{2}}$ for the case of half-integer s . Since the eigenfunctions ψ_a of $\underline{Q}$ are the B-transforms of those $\psi_a^{(0)}$ of $\underline{Q}^{(0)}$, i.e., $\psi_a(p) = B(p)\psi_a^{(0)}(p)$, with $\psi_a^{(0)}(p) \sim e^{-ip \cdot a \frac{1}{p_0^2 v}}$, $v \in \mathbb{V}_{2s+1}$, the discontinuities of $\psi_a(p)$ have a two-fold origin :

- possible jumps in $\alpha(|p|)$;
- discontinuity (indeterminacy) of $\beta(p)$ at $p = 0$ whenever $\beta \neq 0$.

Therefore the family $\mathcal{R}$ defined in (BC1) consists ⁽¹²⁾ of those 2^s (or 2^{s+1}) according to the value of the spin) position operators obtained from (7), (13), when $\alpha \equiv \pm 1$. For instance, for $s = \frac{1}{2}$, $\mathcal{R}$ consists of 2 position operators obtained from (7) with $B_1 \equiv 1$, and $B_2(p) = e^{i\pi/\beta_2(p)} = (\text{for } s = \frac{1}{2}) = 2 \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{p}|}$ respectively. The same is true for $s = 1$, with $B_1 \equiv 1$, and $B_2(p) = e^{i\pi/\beta_2(p)} = (\text{for } s = 1) = \cos \pi \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{p}|}$.

And now, let us prove (A1), (BC1) and (BC2) :

Proof of (A1) - It follows from the previous considerations that, given any position operator $\tilde{Q}$, all its eigenfunctions $\psi_a(p)$ belong to $L_2^{loc}(R^3) \times \tilde{V}_2$. Since $\mathcal{D}(O) \times \tilde{V}_{2s+1}$ is dense in $L_2(O) \times \tilde{V}_{2s+1}$ for any open set $O \subset R^3$, there exists some sequence $\{\varphi_n\}$ such that $\varphi_n(p) \in \mathcal{D}(R^3) \times \tilde{V}_{2s+1}$ and $\varphi_n \rightarrow \psi_a$ in $L_2^{loc}(R^3) \times \tilde{V}_{2s+1}$. Let $\rho(x) \neq 0 \in \mathcal{D}(R^1)$ and $\text{supp } \rho \subset [1, 2]$; define $\psi_n(p) \equiv \varphi_n(p) + \rho(\lambda_n^{-1} |p|) u \equiv \varphi_n + \rho_n$ with $u \neq 0 \in \tilde{V}_{2s+1}$, and

$$\Lambda_n^2 \geq \|P_1 \varphi_n\|^2 + \dots + \|P_n \varphi_n\|^2 + \lambda_n^2 + n^2 \quad (14)$$

where λ_n is the radius of any (fixed) sphere centered at $p = 0$ and containing $\text{supp } \varphi_n$, and where $\{P_j\}$ represents the ordered sequence of operators $M_{\mu\nu}^{(r)}$ ($\mu, \nu = 0, 1, 2, 3; r = 1, 2, \dots$). It is clear that $\psi_n \in \mathcal{D}(R^3) \times \tilde{V}_{2s+1}$ (in particular, $\psi_n \in D_{M_{\mu\nu}}^{(r)}$ $\equiv$ the domain of definition of $M_{\mu\nu}^{(r)}$) and that

- i) $\psi_n \rightarrow \psi_a$ in $L_2^{loc}(R^3) \times \tilde{V}_{2s+1}$;
- ii) $\|\psi_n\|^2 = \|\varphi_n\|^2 + \|\rho_n\|^2$, $\|P_j \psi_n\|^2 = \|P_j \varphi_n\|^2 + \|P_j \rho_n\|^2$
- iii) $\max_n \frac{\|P_j \varphi_n\|}{\|\rho_n\|} < \infty$ for all j ;
- iv) $\max_n \frac{\|P_j \rho_n\|}{\|\rho_n\|} < \infty$ for all j .

i) comes out from the fact that $\text{supp } \rho_n$ is outside the sphere $|\underline{p}| < \Lambda_n$, and $\Lambda_n \rightarrow \infty$; ii) is an obvious consequence of $\text{supp } \rho_n \cap \text{supp } \varphi_n = \emptyset$; iii) follows from (14), since $\Lambda_n^{-1} \|\rho_n\| \rightarrow \text{constant} \neq 0$ as $n \rightarrow \infty$; finally, iv) can be simply proved by noting that P_j becomes asymptotically a homogeneous operator of degree 0 in $\underline{p}$ as $|\underline{p}| \rightarrow \infty$.

It is now an obvious consequence of i) ... iv) that ψ_n is an (A) sequence approaching ψ_a and satisfying ∞ -RC, Q.E.D. ¹³⁾.

Proof of (BC1) - Let us first prove the "if" part : given $Q \in \mathcal{Q}$, its eigenfunctions ψ_a are of the form

$$\psi_a(\underline{p}) = e^{i\pi\beta(\underline{p})} \psi_a^{(o)}(\underline{p})$$

where $\beta(\underline{p})$ has the form given in (1.1) ii), and $\psi_a^{(o)}(\underline{p})$ are eigenfunctions of $Q^{(o)}$ with the same eigenvalue a . Suppose that $\eta_n(\underline{p}) \in \mathcal{S}(R^3)$ and tend to 1 uniformly on any compact set as $n \rightarrow \infty$. Suppose, in addition, that $\psi_n^{(o)}(\underline{p}) \equiv \psi_a^{(o)}(\underline{p}) \eta_n(\underline{p})$ (which gives clearly a (B) sequence approaching $\psi_a^{(o)}$) satisfies 1-RC ¹⁴⁾. We claim that also $\{\psi_n \equiv \psi_a \eta_n\}$ satisfies 1-RC. Indeed, since $e^{i\pi\beta(\underline{p})}$ is a polynomial of finite degree in $\underline{S} \cdot \underline{p}/|\underline{p}|$, and

$$\frac{\partial}{\partial p_j} \left(\frac{\underline{S} \cdot \underline{p}}{|\underline{p}|} \right)^r = r \left(\frac{\underline{S} \cdot \underline{p}}{|\underline{p}|} \right)^{r-1} \left[S_j \frac{1}{|\underline{p}|} - \left(\frac{\underline{S} \cdot \underline{p}}{|\underline{p}|} \right) \frac{p_j}{|\underline{p}|^3} \right]$$

in the sense of the theory of distributions, we have that $\psi_n \in D_{M_{\mu\nu}}$ and that

$$(M_{\mu\nu} \psi_n)(\underline{p}) = e^{i\pi\beta(\underline{p})} (M_{\mu\nu} \psi_n^{(o)})(\underline{p}) + X_{\mu\nu}(\underline{p}) \psi_n^{(o)}(\underline{p}) \quad (15)$$

where $X_{\mu\nu}(\underline{p})$ is a $(2s+1) \times (2s+1)$ matrix such that all its elements belong to $L_2^{loc}(R^3)$ and are bounded outside the origin (i.e., for $|\underline{p}| > \varepsilon > 0$, ε arbitrarily fixed). Therefore, as the sequence $\{\psi_n^{(0)}\}$ is uniformly bounded in any compact neighbourhood of $\underline{p} = 0$ and satisfies, by hypothesis, 1-RC, it follows from (15) that ψ_n is also a (B) sequence (approaching obviously ψ_a) which satisfies 1-RC. Similarly, suppose that ψ_n is any (C) sequence approaching ψ_a ; then $\psi_n(\underline{p}) = e^{i\pi\beta(\underline{p})} \psi_n^{(0)}(\underline{p})$ where $\psi_n^{(0)}$ is also a (C) sequence for $\psi_a^{(0)}$. The previous argument can be repeated for in this case $\{\psi_n^{(0)}(\underline{p})\}$ is again uniformly bounded in any compact neighbourhood of $\underline{p} = 0$; therefore, $\{\psi_n\}$ satisfies 1-RC whenever $\{\psi_n^{(0)}\}$ does it ⁽¹⁴⁾.

Let us finally prove the "only if" part of (BC1): for $\{\psi_n\}$ to satisfy 1-RC, it is first necessary that ψ_n belong to $D_{M_{\mu\nu}}^{(2)}$. In particular [use (3)], $\psi_n(\underline{p})$ must be such that $\partial/\partial p_j \psi_n(\underline{p})$, as a distribution, belongs to $L_2^{loc}(R^3) \times \sqrt{2s+1}$. Consequently, $\psi_n(\underline{p})$ must be (Lebesgue) equivalent to a (vector-valued) function absolutely continuous in P_j along all the parallels to the j axis ⁽¹⁵⁾. In particular, $\psi_n(\underline{p})$ cannot be discontinuous with respect to P_j all over some spherical surface $|\underline{p}| = b$ ($b > 0$). Therefore, if $\{\psi_n\}$ is a (B) sequence for ψ_a which satisfies 1-RC, it follows that $\alpha \equiv \pm 1$, since $\psi_n(\underline{p})$ will not vanish on $|\underline{p}| = b$ for n sufficiently large (b representing the radius of a discontinuity surface of $\alpha(|\underline{p}|)$ which may appear as soon as $\alpha \neq \pm 1$); similarly, if $\{\psi_n\}$ is a (C) sequence for ψ_a satisfying 1-RC, the same conclusion holds, for $B^{-1}(\underline{p}) \psi_n(\underline{p})$ is analytic and cannot therefore vanish almost everywhere on $|\underline{p}| = b$, Q.E.D.

Proof of (BC2) - Let $\{\psi_n\}$ represent a (B) or (C) sequence approaching ψ_a . In order for $\{\psi_n\}$ to satisfy 2-RC it is first necessary [from (BC1)] that $\underline{Q} \in \mathcal{R}$ and secondly, that $\psi_n \in D_{M_{\mu\nu}}^{(2)}$. As $M_{\mu\nu}^{(2)}$ is a differential operator with co-differentiable coefficients, $\psi_n \in D_{M_{\mu\nu}}^{(2)}$ implies that $M_{\mu\nu}^{(2)} e^{i\pi\beta(\underline{p})} \psi_n^{(0)}(\underline{p})$ must, as a distribution, belong to $L_2^{loc}(R^3) \times \sqrt{2s+1}$. But a simple algebraic computation shows that $M_{\mu\nu}^{(2)} e^{i\pi\beta(\underline{p})} \psi_n^{(0)}(\underline{p})$ has a singularity of the type $1/|\underline{p}|^2$ at $\underline{p} = 0$ whenever $\psi_n^{(0)}(0) \neq 0$ (which is always our case) and as soon as $\beta \neq 0$. This proves (BC2).

Remark : We could get at a similar conclusion in a simpler way if mixed 2-RC is required, i.e., if one imposes 1-RC as well as

$$\max_n \frac{\| M_{\mu_1, \nu_1} M_{\mu_2, \nu_2} \psi_n \|}{\| \psi_n \|} < \infty$$

In fact, the fulfilment of these two requirements will imply that all the 2nd-order derivatives of ψ_n (as a distribution) belong to $L_2^{\text{loc}}(\mathbb{R}^3) \times \mathcal{V}_{2s+1}$, and therefore ¹⁶⁾, that ψ_n must be (Lebesgue-equivalent to) a continuous vector-valued function.

2. In this section we will establish (D1) and (D2) :

Proof of (D1) - Let $\psi \in D_1$. As $D_{Q^2} = BD_{Q(o)^2}$, it follows that $\psi^{(o)}(p) \equiv B^{-1}(p)\psi(p)$ must be equivalent to a continuous vector-valued function ¹⁷⁾. On the other hand, $\psi \in \bigcap_{(\mu, \nu)} D_{M_{\mu\nu}}$ implies ¹⁵⁾ that ψ must be (equivalent to) a function absolutely continuous in p_j along all parallels to the j axis ($j = 1, 2, 3$). Consequently, $\psi^{(o)}$ must vanish on all spherical surfaces $|p| = b > 0$ of discontinuity of $B(p)$, i.e., of $\alpha(|p|)$; but it is well known that :

a) $Q(o)^2$ is not essentially selfadjoint when restricted to any subdomain of $D_{Q(o)^2}$ consisting of (continuous) functions vanishing at a certain fixed point ¹⁸⁾. Therefore, whenever $\alpha \neq \pm 1$, $Q^2|_{D_1}$ is not essentially selfadjoint.

Suppose now that $Q \in \mathcal{R}$; then $B(p)$ has at most one discontinuity point, $p = 0$; as $Q(o)^2|_{\mathcal{S}(R^3) \times \sqrt{2s+1}}$ is essentially selfadjoint and $B[\mathcal{S}(R^3) \times \sqrt{2s+1}] \subset D_{M_{\mu\nu}}$ if $\alpha \equiv \pm 1$, it follows clearly that $Q^2|_{D_1}$ is essentially selfadjoint when $Q \in \mathcal{R}$, Q.E.D.

Proof of (D2) - Suppose first that $Q = Q(o)$; then $\mathcal{S}(R^3) \times \sqrt{2s+1} \subset D_2$ and as $Q(o)^2|_{\mathcal{S}(R^3) \times \sqrt{2s+1}}$ is essentially selfadjoint, the "if" part has been proved.

Conversely, suppose that $Q^2|_{D_2}$ is essentially selfadjoint; as $D_2 \subset D_1$, it follows from (D1) that $Q \in \mathcal{R}$. But $\psi \in \bigcap_{(\mu, \nu)} D_{M_{\mu\nu}} \bigcap_{(\mu_1, \nu_1, \mu_2, \nu_2)} D_{M_{\mu_1 \nu_1, \mu_2 \nu_2}} \bigcap_{(\mu_1, \nu_1, \mu_2, \nu_2)} L_2^{loc}(R^3) \times \sqrt{2s+1}$ implies that all the 2nd-order derivatives of ψ belong to $L_2^{loc}(R^3) \times \sqrt{2s+1}$, whence it follows that $\psi(p)$ must be continuous in p . On the other hand, $\psi \in D_{Q^2}$ implies that $\psi^{(o)}(p) \equiv B^{-1}(p)\psi(p)$ must also be a continuous function of p . Therefore, $\psi(o) = 0$ whenever $B \neq 0$, and consequently, also $\psi^{(o)}(o) = 0$. Once again (D1) a) can be applied, and we get at a contradiction with our initial assumption that $Q^2|_{D_2}$ was essentially selfadjoint. The proof is thus complete.

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- 2) A.S. Wightman, Rev. Mod. Phys. 34, 845 (1962).
- 3) We use the same notation as in W.Weidlich and A.K. Mitra, Nuovo Cimento 30, 385 (1963). See this reference for further details.
- 4) L. Schwartz, Théorie des Distributions, I, II, Hermann et Cie, Paris.
N. Dunford and J.T. Schwartz, Linear Operators, I, II, Interscience, New York.
- 5) By position operator we will mean any operator Q satisfying the Newton-Wigner postulates (a), (b), (c), given in ¹⁾.
- 6) It should be noted that the set $\mathcal{R}$ is here defined in a fixed realization of $[\bar{m}, \bar{s}]$; since continuity of ψ_a is not an intrinsic (i.e., realization-independent) property, $\mathcal{R}$ will be defined by unitary equivalence in other realizations. An intrinsic characterization of $\mathcal{R}$ will be given, however, in Section 2.
- 7) See ¹⁾, page 404:
- 8) This contradicts (BC1) for $s > 0$.
- 9) ³⁾ the authors claim that Q is unique independently of any RC. This result is in contradiction with those in ^{1), 2)} and with the present paper.
- 10) Actually, it is shown in ²⁾ that (8) holds for all p and $A \in \text{SU}(2)$ except at most for a fixed null-set in $|p|$. Such technical points can be easily incorporated in our analysis, but we will simply ignore them for brevity's sake.

16.

- 11) As usual, we take S_3 diagonal.
- 12) If (9) is not imposed, $\mathcal{R}$ becomes larger : it consists of 2^{2s} different position operators for both integer and half-integer spin s . Explicit expressions for β in this case can also be given.
- 13) There are of course (A) sequences approaching ψ_a which do not satisfy even the simple 1-RC. But this also occurs for $\tilde{Q}^{(0)}$, due to the local character of the unbounded operators $M_{\mu\nu}$, i.e., because $M_{\mu\nu} \mathcal{D}(0) \times \mathcal{V}_{2s+1} \subset \mathcal{D}(0) \times \mathcal{V}_{2s+1}$.
- 14) Numerous examples of such $\psi_n^{(0)}$ approaching $\psi_a^{(0)}$ and satisfying 1-RC can be easily constructed.
- 15) See L. Schwartz, loc. cit., Livre I, Theor. V, page 57.
- 16) This is a consequence of the following statement : if $T \in \mathcal{D}'(R^n)$ and if all the derivatives of T of order $[\frac{n}{2}] + 1$ belong to L_2^{loc} , then T is a continuous function. (See L. Schwartz, loc. cit., Livre II, pages 37, 38, 44).
- 17) Note that in the 3-dimensional space every function in the domain of the Laplacian operator is equivalent to a continuous function.
- 18) It suffices to show this for the Laplacian operator Δ in R^3 , i.e., we must find some $f \in L_2(R^3)$ such that $\overline{(\Delta + Z)g, f} = 0$ for some complex Z (with $\text{Im} Z \neq 0$) and all $g \in D$, D being a subdomain of D_Δ consisting of functions vanishing at some point, say $g(\underline{x}_0) = 0$. By Fourier-transforming the problem, one immediately realizes that
- $$f(\underline{x}) \equiv \int \frac{1}{-\xi^2 + Z} e^{-i \underline{\xi} \cdot (\underline{x} - \underline{x}_0)} d^3 \xi$$
- is the desired function.