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SPECTRAL REPRESENTATION AND MASS FORMULA FOR SPIN $3/2$ PARTICLES

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A B S T R A C T

The canonical quantization of a spin $3/2$ field is considered. Using Lehmann's representation, sum rules are derived connecting the spectral functions with the mass and renormalized constant, assuming that the interaction Lagrangian does not violate the Rarita-Schwinger subsidiary conditions and the current commutes with the field. Considering the inverse Green's function a $1/m$ mass formula is obtained. The result is applied to the decuplet of spin $3/2$ resonances.

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1. INTRODUCTION

With the discovery of new particles and resonances and the attempts to understand them in terms of a basic underlying symmetry structure, the mass formulae have been used with considerable success both as a test of the symmetry scheme and to predict the existence of new particles.

All these formulae are, in general, derived using group theoretical arguments for the internal symmetry space ¹⁾. However, it is also necessary to use some space-time aspects to decide, for example, whether the Gell-Mann - Okubo mass formula must be written in terms of the square of the mass, linearly on the mass, etc. .

Until recently one used a linear mass formula for half-integer spin particles or resonances and a quadratic mass formula for integer spin-one particles. More recently some authors have noticed that for vector systems the quadratical mass formula must be replaced by the one over m square.

For this reason it is not clear what mass formulae must be applied in the decuplet of spin $3/2^+$ particles, although, experimentally, the linear mass formula has a striking success. Our aim is to investigate this question from the general field theoretical point of view.

To clarify our method we shall study briefly, in Section 2, the case of vector particles. We shall derive the sum rules obtained by Johnson ²⁾, using however a slightly different approach which seems to us more compact and easier to generalize and apply in the next case. Furthermore, we shall demonstrate how the one over m square mass formula is obtained as a result of conservation of the current. In Section 3 we shall consider the case of spin $3/2$ and obtain the sum rules and mass formula. Finally, the results are used to obtain mass splitting between the members of an SU_3 decuplet broken by the customary assumed medium-strong interactions and the results are compared with the N^* (1236), Y^* (1382), Ξ^* (1529) and Ω^- (1675) mass splittings.

2.

2. THE VECTOR BOSON CASE

A massive vector field coupled to a conserved current is described by the following Lagrangian density

$$\mathcal{L} = -\frac{1}{2} A^\mu [(p^2 - m_0^2) g_{\mu\nu} - p_\mu p_\nu] A^\nu + j_\mu A^\mu$$

where the first two terms describe the free Lagrangian and the last one is the interaction Lagrangian. j_μ is the conserved current, i.e.,

$$p^\mu j_\mu = 0$$

From this Lagrangian we get the following equations of motion

$$(p^2 - m_0^2) A_\mu = j_\mu$$

$$p_\mu A^\mu = 0$$

As it is well known the subsidiary condition, $p_\mu A^\mu = 0$, eliminates the spin zero part of the vector field. Since the field contains redundant components (as expressed in the subsidiary condition) we shall introduce new fields $\phi_\mu^{(\alpha)}$ (where α runs from 1 to 3 and labels the three independent fields, for example, the helicity states) which are transversal, and therefore do not contain redundant components, using the projection operators

$$\phi_\mu^{(\alpha)} = P_{\mu\nu}^{(\alpha)}(1) A^\nu$$

where

$$\sum_{\alpha=1}^3 P_{\mu\nu}^{(\alpha)} = P_{\mu\nu}(1) = \eta_{\mu\nu} \quad \eta_{\mu\nu} \equiv g_{\mu\nu} - z_\mu z_\nu$$

where $P_{\mu\nu}(1)$ is the spin one projection operator and where $z_\mu = p_\mu / \sqrt{p^2}$ (we use the metric $g_{ii} = -g_{00} = -1$).

In terms of the new field the Lagrangian can be written in the following way,

$$L = -\frac{1}{2} \phi^\mu (\Box - m_0^2) \phi_\mu + j_\mu \phi^\mu$$

The spin zero field, which does not interact due to the current conservation, has been eliminated in an obvious way.

The equal time commutation relations for the free fields are

$$[\phi_\mu^{(\alpha)}(x), \phi_\nu^{(\beta)}(y)] = i \delta_{\alpha\beta} P_{\mu\nu}^{(\alpha)}(1) \Delta(x-y; m_0^2) \Big|_{x_0=y_0} \quad (1)$$

We remark that in the above formula $P_{\mu\nu}(1)$ is essentially the identity operator since we are in the subspace of transversal fields. If the current, j_μ , commutes with the field ϕ_μ , the Eq. (1) is still true for the coupled fields.

Now let us consider the vacuum expectation value of the commutation relations. In the Lehmann's representation we have

$$\langle 0 | [\phi_\mu^{(\alpha)}(x), \phi_\nu^{(\beta)}(y)]_- | 0 \rangle = i \delta_{\alpha\beta} \int_0^\infty d\lambda^2 \rho(\lambda) P_{\mu\nu}^{(\alpha)}(1) \Delta(x-y; \lambda^2) \quad \rho(\lambda) \geq 0$$

4.

From this and Eq. (1) we get the following sum rules

$$Z^{-1} = 1 + \int_M^{+\infty} d\lambda \varphi_R(\lambda)$$

$$\frac{Z^{-1}}{m_0^2} = \frac{1}{m^2} + \int_M^{+\infty} d\lambda \frac{1}{\lambda^2} \varphi_R(\lambda)$$

where $\varphi_R(\lambda)$ is the renormalized spectral function (the pole term has been subtracted) and $M > m$.

Let us now consider the Green's function. Using it, we obtain the mass formula. The Green's function for the transversal vector field has the following Lehmann's representation

$$G_{\mu\nu}(p) = \frac{1}{p^2 - m^2} \left(g_{\mu\nu} - \frac{1}{m^2} p_\mu p_\nu \right) + p^2 \underline{P}_{\mu\nu}(1) \int_M^\infty \frac{\varphi_R(\lambda)}{p^2 - \lambda^2 + i\epsilon} d\lambda^2 \quad (2)$$

To find the mass formula we must look at the zeros of the inverse Green's function. We have

$$G_{\mu\nu}^{-1}(p) = Z^{-1} \left[(p^2 - m_0^2) g_{\mu\nu} - p_\mu p_\nu \right] + p^2 \underline{P}_{\mu\nu}(1) \int_M^\infty \frac{\sigma_R(\lambda)}{p^2 - \lambda^2 + i\epsilon} d\lambda^2 \quad (3)$$

where

$$G_{\lambda\mu} \underline{P}^{\mu\nu}(1) \sigma_R(\lambda) G_{\nu\tau}^+ = \underline{P}_{\lambda\tau}(1) \varphi_R(\lambda)$$

All the tensor terms of the type $p_\mu p_\nu$ in Eq. (3) are unimportant for determining the mass formula since when $G_{\mu\nu}^{-1}$ operates in the transversal field, the $p_\mu p_\nu$ terms give zero. Therefore, the physical mass is given by the zero of the $g_{\mu\nu}$ coefficient, i.e., the roots of

$$p^2 - m_0^2 + p^2 \psi(p^2) = 0$$

$$\psi(p^2) \equiv Z \int_M^\infty \frac{\sigma_R(\lambda)}{p^2 - \lambda^2} d\lambda^2$$

The factor p^2 in front of $\psi(p^2)$ is very important and it gives rise, as we shall see soon, to an inverse square mass formula. We should remark that this factor is not artificially written in front of $\psi(p^2)$ for the sake of definition, but it is a genuine result of the current conservation (or equivalently the transversality of the field). Indeed, using the sum rules and assuming a reasonable spectral function ρ_R [when the coupling is not too strong we can assume that ρ_R does not contain any pole since we have subtracted the physical pole out of ρ_R in Eq. (2)], one sees easily that $\psi(p^2)$ does not have a pole at $p^2 = 0$.

This equation can now be applied in a perturbative way and yields, what is called usually, a mass formula. For this we expand $\psi(p^2)$ around m_0^2 and write $\psi(p^2) = \psi(m_0^2) + (p^2 - m_0^2) \psi'(\theta_p)$ where θ_p is a number between p^2 and m_0^2 . Assuming that $(m^2 - m_0^2) \psi'(\theta_m)$ is small compared with $\psi(m_0^2)$ (notice that this has nothing to do with a perturbation theory in the coupling constant), we can write $\psi(m^2) = \psi(m_0^2)$ and the physical mass is

$$\frac{1}{m^2} = \frac{1}{m_0^2} (1 + \psi(m_0^2)) \quad (4)$$

Let us apply this result to an SU_3 multiplet. If only the very strong SU_3 invariant interactions are present, the $\psi(m_0^2)$ are the same for all members of a multiplet and therefore their physical mass, $m(I, Y)$, is the same for all of them. Let us now turn on the medium strong interaction with the usual assumption that the symmetry breaking Lagrangian transforms like the $I = 0, Y = 0$ component of an 8 representation. When the medium strong interactions are introduced, the current conservation is broken. Therefore, the Eq. (4) is no longer true. However, if we restrict to the Born approximation with respect to the medium strong interactions this equation still holds. In this case $\psi(m_0^2)$ transforms like the interaction Lagrangian and we get the usual formula

$$\frac{1}{m^2(I, Y)} = a + bY + c \left[I(I+1) - \frac{Y^2}{4} \right] \quad (5)$$

If the coupling constant is very small the inverse square mass formula is, from the practical point of view, equivalent to the square mass formula, but we shall keep the formula (5) since sometimes the Born approximation turns out to be better than what the magnitude of the coupling constant may suggest. (We have in mind such cases where the Born approximation is a very good approximation such as in one-pion exchange models in nucleon-nucleon scattering where the coupling constant is very big or turns out to be exact as in Thomson scattering. For this reason, we want to keep the Eq. (5) - keeping in mind that it is only a Born approximation, but leaving open the possibility that this formula might be better than what the magnitude of the coupling constant may suggest.)

3. THE SPIN 3/2 PARTICLES

Let us consider now the case of a particle of spin 3/2. In the Rarita-Schwinger formalism the Lagrangian density is given by

$$L = L_0 + L_I$$

where L_0 is the free Lagrangian ³⁾

$$L_0 = - \bar{\psi}_\mu \Lambda^{\mu\nu} \psi_\nu$$

where

$$\Lambda_{\mu\nu} = (\not{x} - m_0) g_{\mu\nu} - \frac{1}{3} (\gamma_\mu \not{p}_\nu + \gamma_\nu \not{p}_\mu) + \frac{1}{3} \gamma_\mu (\not{x} + m_0) \gamma_\nu$$

and L_I is the interaction Lagrangian.

Let us consider the free field case. Then the equations of motion are

$$\Lambda_{\mu\nu} \psi^\nu = 0$$

which, as it is well known, implies the equations of motion

$$(\not{x} - m_0) \psi_\nu = 0$$

and the subsidiary conditions

$$\gamma^\mu \psi_\mu = 0$$

$$\not{p} \psi_\mu = 0$$

(6)

8.

As it is exhibited by the subsidiary conditions, which eliminate the two particles of spin 1/2, the field ψ_μ contains very many redundant variables. As before, using projection operators we introduce new fields $\phi_\mu^{(\alpha)}$ which automatically satisfy the subsidiary conditions. In complete analogy with the vector case we define

$$\phi_\mu^{(\alpha)} = P_{\mu\nu}^{(\alpha)}(3/2) \psi^\nu$$

where α runs from one to four, labelling the four independent states, for example, the helicity states. We have

$$\sum_{\alpha=1}^4 P_{\mu\nu}^{(\alpha)}(3/2) = P_{\mu\nu}(3/2)$$

where 4)

$$P_{\mu\nu}(3/2) = \frac{1}{2} (1 + \not{\epsilon}) \left[\eta_{\mu\nu} - \frac{1}{3} \gamma_\mu \not{\epsilon} \gamma^\rho \gamma_\rho \gamma_\nu \right]$$

As before, we can deduce that, for free fields, the $\phi_\mu^{(\alpha)}$ satisfy the following equal time commutation relations

$$[\phi_\mu^{(\alpha)}(x), \bar{\phi}_\nu^{(\beta)}(y)]_+ = i \delta_{\alpha\beta} P_{\mu\nu}^{(\alpha)}(3/2) \Delta(x-y; m_0) \Big|_{x_0=y_0} \quad (7)$$

Let us consider the case of coupled fields. We assume that L_I is such that for the coupled 3/2 field subsidiary conditions (6) still hold, (i.e., the virtual field has spin 3/2) if furthermore the interaction Lagrangian commutes with ϕ_μ , then the Eq. (7) is still true for the interacting field.

To obtain the sum rules we consider the Lehmann's representation of the vacuum expectation value of the anticommutator :

$$\langle 0 | [\phi_f^{(\alpha)}(x), \bar{\phi}_v^{(\beta)}(y)]_+ | 0 \rangle = i \delta_{\alpha\beta} \int_{-\infty}^{+\infty} d\lambda \rho(\lambda) P_{\mu\nu}^{(\alpha)}(3/2) \Delta(x-y; \lambda^2) \quad \rho(\lambda) \geq 0 \quad (8)$$

From Eqs. (7) and (8) we obtain the following sum rules

$$Z^{-1} m_0^i = m^i + \int_{-\infty}^{+\infty} d\lambda \lambda^i \rho_R(\lambda) \quad i = 1, 0, -1, -2$$

where $\rho_R(\lambda)$ is the renormalized spectral function (the pole term has been subtracted).

The Green's function is given by

$$G_{\mu\nu} = \frac{1}{p^2 - m^2} d_{\mu\nu}(p, m) + p P_{\mu\nu}(3/2) \int_{-\infty}^{+\infty} \frac{\rho_R(\lambda)}{p^2 - \lambda^2 + i\epsilon} d\lambda$$

where

$$d_{\mu\nu} = (\not{p} + m) \left[g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{1}{3m} (\gamma_\mu p_\nu - \gamma_\nu p_\mu) - \frac{2}{3m^2} p_\mu p_\nu \right] \\ + \frac{2}{3m^2} (p^2 - m^2) \left[(\gamma_\mu p_\nu - \gamma_\nu p_\mu) + (\not{p} + m) \gamma_\mu \gamma_\nu \right]$$

To write this formula we have used the fact that the interaction Lagrangian is such that both subsidiary conditions still hold for the coupled fields.

To derive the mass formula we consider the inverse Green's function

$$G_{\mu\nu}^{-1} = Z^{-1} \Lambda_{\mu\nu}(p, m_0) + p \mathcal{P}_{\mu\nu}(3/2) \int_{-\infty}^{+\infty} \frac{\sigma_R(\lambda)}{p^2 - \lambda^2 + i\epsilon} d\lambda$$

where

$$G_{\mu\lambda} \mathcal{P}^{\lambda\rho}(3/2) \sigma_R(\lambda) G_{\rho\nu}^+ = \mathcal{P}_{\mu\nu}(3/2) \rho_R(\lambda)$$

From here and as in the vector case, we get the following equation

$$\left[(\not{p} - m_0) + \not{p} \frac{\psi_1(p) - \psi_2(p)}{2} - p \frac{\psi_1(p) + \psi_2(p)}{2} \right] = 0$$

where

$$\psi_1(p) = -Z \int_0^\infty \frac{\sigma_R(\lambda)}{p^2 - \lambda^2} d\lambda \quad \psi_2(p) = -Z \int_0^\infty \frac{\sigma_R(-\lambda)}{p^2 - \lambda^2} d\lambda$$

The physical mass, m , is given by the solution of the following equation

$$p - m_0 + p \psi_1(p) = 0 \quad (9)$$

As before, the factor p in front of $\psi_1(p)$ is a genuine factor and therefore the mass formula is

$$\frac{1}{m} = \frac{1}{m_0} (1 + \psi(m_0)) \quad (10)$$

Let us now apply this result to a spin $3/2$ SU_3 multiplet. In a similar way to the vector case, we find now that the masses of the different members of the SU_3 multiplet, when the medium strong interactions are turned on, are given by

$$\frac{1}{m(I, Y)} = a + bY + c \left[I(I+1) - \frac{Y^2}{4} \right] \quad (11)$$

It is well known that for an SU_3 decuplet the isotopic spin and the hypercharge are related through the equation $I = \frac{Y}{2} + 1$. This equation can be used to eliminate I from (11) and we obtain

$$\frac{1}{m(Y)} = A + B Y \quad (12)$$

Therefore, if the masses of two particles of the multiplet (with different Y) are known, we can compute the mass of the other particles.

Let us now assume that the $J^P = 3/2^+$ particles N^* , Y^* , Ξ^* , Ω^- , with experimental masses (6) (1236 ± 2) , (1382.1 ± 0.9) , (1529.1 ± 1.0) , (1675 ± 3) MeV, respectively, form an SU_3 decuplet. Let us now assume that we know the masses of m_{Y^*} and m_{Ξ^*} (the ones with smaller errors) and try, using Eq. (12) to determine the other masses. We obtain

$$m_{N^*} = (1261 \pm 3) \text{ Mev} \quad m_{\Omega^-} = (1711 \pm 4) \text{ Mev}$$

i.e., the masses are predicted with errors of approximately 2%. Comparing this numbers with those obtained using a linear mass formula ($m_{N^*} = 1235.1 \pm 2.8$, $m_{\Omega^-} = 1676.1 \pm 2.9$) we see that the last ones are in better agreement (error less than 1%) with the experimental data. However, the very good agreement of the linear, in m , mass formula might be accidental and does not imply that the linear mass formula is the correct one.

Let us consider now possible effects of second order Born approximation. The group theoretical structure of the mass formula is now $A' + B'Y + C'Y^2$, i.e., we have a term proportional to Y^2 coming from the 27 irreducible representation of the 8×8 representation. If, for some special reason, Eq. (10) is still true in second order Born approximation, then the masses of the members of the SU_3 decuplet are given by

$$\frac{1}{m(Y)} = A' + B'Y + C'Y^2$$

Eliminating the constants A' , B' , C' we get

$$\frac{1}{m(1)} - \frac{1}{m(-2)} = 3 \left(\frac{1}{m(0)} - \frac{1}{m(-1)} \right) \quad (13)$$

Considering the case of the N^* , Y^* , Ξ^* , Ω^- decuplet we can use this equation to compute m_{Ω^-} if we assume the other three masses known. We get

$$m_{\Omega^-} = (1666 \pm 11) \text{ MeV}$$

in close agreement with the experimental data (error $\simeq 5\%$). The value of m_{Ω^-} computed using the equation obtained from Eq. (13) by the substitution $\frac{1}{m} \rightarrow m$ is $m_{\Omega^-} = (1677 \pm 8) \text{ MeV}$ (error $\simeq 2\%$).

Finally, we want to point out the fact that the mass splitting between the members of the SU_3 decuplet considered here is of the order of 10% of the average mass of the multiplet. Then we expect that the second order equation (13) will fit the experimental data, better than the first order equation (12) by a factor of 10 approximately. That is so in the case of the $1/m$ mass formula. For the linear mass formula both equations fit equally well the experimental data but we consider this as accidental. If the linear mass formula is true it will be necessary to find a mechanism to explain why the second order term does not give any contribution to the mass formula.