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LARGE-ANGLE CROSS-SECTIONS $p+p \rightarrow A+B$ AND $\pi+p \rightarrow A+B$
AT HIGH ENERGIES PREDICTED BY THE STATISTICAL MODEL

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A. INTRODUCTION

Two years ago, Fast, Hagedorn and Jones ^{1),2)} proposed to describe large angle elastic scattering by means of the statistical model. The predictions ²⁾ have been confirmed experimentally ^{3),4)} for pp collisions and there is little doubt that for πp collisions the comparison will turn out favourably too. We shall extend here the arguments given in Ref. ²⁾ - from now on quoted as FH - to a more definite classification of central collisions and to include more general scattering processes. Since statistical calculations have been made only for the initial states p+p and π +p, we are limited to these two. The final states in both cases may contain any two particles A+B which can be produced in accordance with the selection rules from the initial state by strong interactions. Thus, A and B may be pions, kaons, nucleons, hyperons and resonances thereof.

Our units are throughout $\hbar = c = M_{\text{nucleon}} = 1$.

B. THE STATISTICAL MODEL APPROACH

1. The essential idea of the statistical model

The sketch given here is a brief resumé of articles by the author ⁵⁾ which may be consulted for details.

	<u>Channel label</u>
$p+p \rightarrow p+p$	0
$p+p+\pi^0$	1
$n+p+\pi^+$	2
$\vdots$	$\vdots$
$\vdots$	j
$\vdots$	$\vdots$
$N+N+n\pi$	$\vdots$
$N+Y+K$	$\vdots$
$\vdots$	$\vdots$
$N+Y+K+n\pi$	$\vdots$
$N+N+\bar{N}+N+\dots$	$\vdots$

The probability for any one of the many possible channels (labelled by $j=0,1,2,\dots$; $0 \equiv$ elastic channel); is given - up to a common constant α - by the expression, valid in the centre-of-momentum frame :

$$P_j = \int |\langle f_j | S | i \rangle|^2 d^3 p_1 \dots d^3 p_n = \quad (P)$$

$$= F(T, \tau_1, \dots, \tau_n, \dots) \prod_{i=1}^n \Omega_i \underbrace{\int d^3 p_1 \dots d^3 p_n \delta(\sum \vec{p}_i) \delta(E - \sum \epsilon_i)}_{\text{Phase space factor } \rho(E, n, \dots)}$$

Here use has been made of four-momentum conservation : $\vec{p}_1 \dots \vec{p}_n$ are the momenta of the n particles produced in channel j ; $F(T, \tau_1, \dots, \tau_n)$ takes into account isospin and statistics, whereas the product of the "interaction volumes" Ω_i is the average of the squared matrix element with respect to the integration over the phase space. From arguments considering the inverse process, Ω_i should be of the order of $(\frac{1}{2\pi})^3 \cdot \frac{4\pi}{3} (\frac{1}{m_\pi})^3$ for nucleons, pions, etc., and somewhat smaller for strange particles. Taking them independent, or slowly dependent, of the primary energy means shifting everything to the phase space factor and in fact is equivalent to the assumption that all centre-of-momentum energy E is available for the production of particles. This is certainly not true and has to be corrected - how to do it is an additional assumption which will be specified soon.

2. The situation of large angle elastic p-p scattering dealt with by the statistical model; comparison with recent experiments

Large angle elastic and exchange scattering ^{*)} is very probably the one and only process at high energy which can be treated by the statistical theory. Why? Because here, by the experimental arrangement, events are recorded which come from central collisions and because -- on the other hand -- the statistical theory can be applied only to central collisions.

The statistical model yields non-normalized probabilities P_j for every possible channel j and if we designate by the subscript zero the elastic channel, then there are certainly relations between the "compound-elastic" (that is "non-diffractive") elastic cross-section $\sigma_{c,el}$ and P_0 on the one hand, and between the total inelastic cross-section σ_{inel} and the sum of all channel probabilities, namely $\sum P_j$, on the other hand. These relations were not very well specified in Ref. ²⁾ and the predictions left open a factor β of the order of 0.1. We wish to replace this factor by a formula based on a very rough model which was proposed by the author already several years ago ⁵⁾.

The point is this: large angle elastic scattering comes from central collisions; the statistical weight P_0 , on the other hand, is calculated from the statistical model under the supposition of a central collision; hence if we write

$$\sigma_{c,el} = \alpha P_0 \quad (1)$$

then α relates "central to central". If, however, we would write a similar equation between σ_{inel} and $\sum P_j$ with the same α , then this would be wrong since here σ_{inel} comes largely from non-central collisions and consequently α would relate "non-central to central", which it cannot do. We may, however, write

*) We call $a+b \rightarrow A+B$ exchange scattering if A and B differ from a and b .

4.

$$\sigma_{\text{centr}} = \alpha \sum P_j \quad (2)$$

and

$$\sigma_{\text{inel}} \equiv \frac{\sigma_{\text{inel}}}{\sigma_{\text{centr}}} \cdot \sigma_{\text{centr}} = \frac{\sigma_{\text{inel}}}{\sigma_{\text{centr}}} \cdot \alpha \sum P_j \quad (3)$$

Then we obtain - assuming isotropy for simplicity (which does not follow from the statistical theory ⁵⁾) and adding a factor two in the case of pp scattering

$$\left(\frac{d\sigma_{\text{el}}}{d\omega} \right)_{\text{pp, large angles}} = \frac{2}{4\pi} \sigma_{\text{inel}} \left(\frac{\sigma_{\text{centr.}}}{\sigma_{\text{inel}}} \right) \cdot \left(\frac{P_0}{\sum P_j} \right)_{\text{pp}} \quad (4)$$

The ratio $\sigma_{\text{centr}}/\sigma_{\text{inel}} = \beta$ was left undetermined in Ref. ²⁾. To fix it up, we adopt the model of a nucleon without hard core and ask under which conditions a collision can be considered "central", that is, possibly leading to thermal equilibrium. From Fig. 1 we notice :

Fig. 1

in the shaded region equilibrium may be established without difficulty. In order to establish it over the whole of both particles, we require that the interaction travels z times from the overlap region to the end of the nucleons and back during the time Δt of contact. Since everything goes with velocity of light ($c=1$) we have the condition (which defines ρ_0)

$$\rho \leq \rho_0 = \frac{\Delta t}{2z} = \frac{r}{\gamma z}$$

where z (of the order of 1) says how often the interaction travels to and back, and where r is the radius of the nucleon; γ is the c.m. Lorentz factor. We put now - and this is the additional assumption ^{*)} not contained in the statistical model -

$$\frac{\sigma_{\text{centr}}}{\sigma_{\text{inel}}} = \frac{\pi \rho_0^2}{\pi r^2} = \frac{1}{z^2 \gamma^2} \quad (5)$$

and obtain finally by dividing (1) by (3) :

$$\left(\frac{d\sigma_{\text{c,el}}}{d\omega} \right)_{\text{pp, large angles}} = \frac{\sigma_{\text{inel}}}{2\pi} \frac{1}{z^2 \gamma^2} \left(\frac{P_0}{\sum P_j} \right)_{\text{pp}} \quad (6)$$

Taking the ratio $P_0/\sum P_j$ from Fast and Hagedorn ²⁾ and expressing it in a formula, it reads

^{*)} Such a factor agrees also well with the observation that in 25 GeV pp collisions roughly 10 times less antinucleons are produced than one expects from statistical model calculations : pair production at this energy requires a rather central collision. Angular momenta for central collisions will then be confined to $l \lesssim K \rho_0 \approx \gamma m_p \cdot \rho_0 = \frac{m_p r_p}{z} \approx \frac{5+7}{z}$, independent of $E_{\text{c.m.}}$.

$$\left(\frac{P_0}{\sum P_j} \right)_{pp} = e^{-3.10(E-2)} \quad (7)$$

where E is the centre-of-momentum total energy in nucleon masses (the difference between E in GeV and E in nucleon masses is significant in such a formula!).

The statistical theory calculations leading to Eq. (7) contained two parameters: the interaction volume for non-strange particles, $\Omega_0 = \frac{4\pi}{3} \left(\frac{1}{2\pi m_\pi} \right)^3$, and for strange particles, $\Omega_1 \cong 0.7 \Omega_0$, fitted from $K\bar{K}$ production in $N\bar{N}$ annihilation at rest. [Lorentz contracted: $\Omega_{\pi,K} = \frac{\Omega_{0,1}}{\delta}$ was used.] These two seem very natural and in so far as $(P_0/\sum P_j)$ comes from material which has accumulated in statistical calculations over several years in hundreds of hours of computer time, everything is fixed now and no free parameter (except $z \approx 1$) remains at our disposal. The comparison of Eqs. (6) and (7) with recent data ³⁾ on large angle pp scattering (around 90°) is shown on Fig. 2. There we have plotted formula (6) with $\sigma_{inel} = 3 \times 10^{-26} \text{ cm}^2$ and used Eq. (7); the value of z , which should lie near 1, had to be chosen 1.18 to give good over-all agreement.

It should be remarked that the behaviour (6) does not contradict the lower bound formula for the scattering amplitude around 90° given by Cerulus and Martin ⁶⁾, which reads

$$\left(\frac{d\sigma_{el}}{d\omega} \right)_{90^\circ} \geq a s^{-b\sqrt{s}} = a e^{-b\sqrt{s} \lg s} ; \quad s \rightarrow \infty \quad (8)$$

Since our formula can be written (with $\gamma^2 = \frac{E^2}{4m^2} = \frac{s}{4m^2}$ for pp scattering)

$$\left(\frac{d\sigma_{el}}{dw}\right)_{\approx 90^\circ} = \frac{A}{s} e^{-B\sqrt{s}} = A e^{-B\sqrt{s} - lq s} \quad (9)$$

it is seen by division that

$$1 \geq \frac{a}{A} e^{-\sqrt{s}(lq s - B) + lq s} \rightarrow 0 \text{ if } s \rightarrow \infty.$$

3. Generalization to $\pi + p \rightarrow A+B$

We have considered so far only pp elastic scattering; the generalization to πp elastic scattering is obvious since in Ref. 2) also a curve for $(P_0/\sum P_j) \pi p$ was given. The high-energy part (above $E_{c.m.} > 2.5 M_p$) is again an exponential, though with a somewhat different slope. Numerically, it is given by (E = total c.m. energy in M_p)

$$\left(\frac{P_0}{\sum P_j}\right)_{\pi p} = e^{-2.8(E - 1.4)} \quad (10)$$

[in Ref. 1) the numerical values in the exponent were slightly wrong]. Since no calculations at rather high energies were done (see FH, Fig. 3) in the πp case, one should not expect the values 2.8 and 1.4 to be too precise. The numerical calculations yielding Eq. (10) have been made for a $\pi^- p$ initial state and the contributions of isospin $\frac{1}{2}$ and $\frac{3}{2}$ have been added with weight $2/3$ and $1/3$, respectively. In fact, the values of (10) calculated separately for $T = \frac{1}{2}$ and $T = \frac{3}{2}$ differ only by a few per cent and thus Eq. (10) may be used for $\pi^- p$ in the initial state as well as for $\pi^+ p$.

Let us consider now the reaction $\pi + p \rightarrow A + B$. We expand the states $|AB\rangle$ and $|\pi p\rangle$ into isospin states $|\frac{1}{2}\rangle$ and $|\frac{3}{2}\rangle$, omitting all other quantum numbers :

$$\left. \begin{aligned} |AB\rangle &= \alpha |\frac{1}{2}\rangle + \beta |\frac{3}{2}\rangle + \dots \\ |\pi p\rangle &= a |\frac{1}{2}\rangle + b |\frac{3}{2}\rangle \end{aligned} \right\} \quad (11)$$

a, b, α, β are the Clebsch-Gordan coefficients corresponding to what A, B and π mean in any particular case. The transition amplitude is then of the form

$$\langle AB | M | \pi p \rangle = a \alpha M_{\frac{1}{2}} + b \beta M_{\frac{3}{2}} \quad (12)$$

and the differential cross-section

$$\frac{d\sigma}{d\omega}(\pi p \rightarrow AB) = a^2 \alpha^2 |M_{\frac{1}{2}}|^2 + b^2 \beta^2 |M_{\frac{3}{2}}|^2 + 2a\alpha b\beta \operatorname{Re}[M_{\frac{1}{2}} \cdot M_{\frac{3}{2}}] \quad (13)$$

As we consider high energy processes, we may make the assumption $|M_{\frac{1}{2}}| \approx |M_{\frac{3}{2}}|$ (other quantum numbers being equal). Furthermore, the M 's - as they stand - are still functions of the total energy and the scattering angle; in any actual experiment $d\sigma/d\omega$ is in fact averaged over an energy interval and an angular interval. Assuming then $M_{\frac{1}{2}}$ and $M_{\frac{3}{2}}$ independent of each other, we have a fair chance that

$$\frac{1}{\Delta s \Delta \Omega} \int \operatorname{Re}[M_{\frac{1}{2}} \cdot M_{\frac{3}{2}}] ds d\omega$$

becomes small compared to the first two terms - of course we cannot prove that. We shall, however, assume it to be true and neglect the interference term altogether. We thus are left with

$$\frac{d\sigma}{d\omega}(\pi p \rightarrow AB) = (a^2 \alpha^2 + b^2 \beta^2) \cdot M^2(E, m_A m_B, \Theta) \quad (14)$$

Here $M^2(E, m_A m_B, \Theta)$ contains the dependence on E and Θ . In the forward direction anything may happen; but for large scattering angle Θ we expect the same behaviour as in purely elastic scattering: weak angular dependence - which we approximate by isotropy (see p. 4) - and transition from the initial to the final state via a more or less established statistical equilibrium. For the latter the conditions have been stated in Eq. (5); the statistical probability P_{AB} of the channel A, B can be calculated from Eq. (P).

The squared amplitude $M^2(E, m_A m_B)$ equals the cross-section for the case that both states $|\pi p\rangle$ and $|AB\rangle$ are pure isospin states ($\frac{3}{2}$, say). Thus, $M^2(E, m_A m_B)$ is given by the same arguments as used to obtain Eq. (6) [without the factor 2 for equal particles]

$$M^2(E, m_A m_B) = \frac{\sigma_{\pi p, \text{inel}}}{4\pi} \frac{1}{Z^2 \gamma^2} \left(\frac{P_{AB}}{\sum P_j} \right) \pi p \quad (15)$$

Hence, with A and B being any two particles or resonances which can be produced by πp in strong interaction:

$$\left[\frac{d\sigma}{d\omega}(\pi p \rightarrow AB) \right]_{\text{large angles}} = (a^2 \alpha^2 + b^2 \beta^2) \frac{\sigma_{\pi p, \text{inel}}}{4\pi} \cdot \frac{P_{AB}}{P_0} \cdot \frac{1}{Z^2} \left(\frac{1}{\gamma^2} \frac{P_0}{\sum P_j} \right)_{\pi p} \quad (16)$$

(symbols explained below)

The symbols of this formula have still to be expressed in an explicit form :

i) according to Eq. (11), if

$$\pi = \pi^+, \text{ then } a^2 = 0, \quad b^2 = 1$$

$$\pi = \pi^-, \text{ then } a^2 = \frac{2}{3}, \quad b^2 = \frac{1}{3}$$

whereas α^2 and β^2 follow from Eq. (11) once A and B are specified;

ii) $\sigma_{\pi p, \text{inel}}$ is to be taken from experiments;

iii) $z \approx 1$ is reasonable. In view of the good agreement of the pp cross-section with the experiment for $z = 1.18$, this value might be taken for πp scattering too;

iv)

$$\gamma = \frac{E_{c.m.}^2 - m_\pi^2 + 1}{2E_{c.m.}}$$

[remember : $\gamma_p = 1$] is the Lorentz factor for the motion of the πp centre-of-momentum frame seen from the lab.;

v) $\frac{P_{AB}}{P_0}$ is the ratio of the probabilities [see Eq. (P)] as calculated from the statistical model ⁵⁾. Using the non-covariant phase space $\mathcal{P}$ [for consistency: the numerical forms (7) and (10) were found from calculations using the non-covariant phase space] one has

$$\frac{P_{AB}}{P_0} = \frac{(2S_A+1)(2S_B+1)}{2} \cdot \frac{\Omega_A \Omega_B}{\Omega_0^2} \cdot \frac{\rho_2(E, m_A m_B)}{\rho_2(E, m_\pi m_p)} \quad (17)$$

with

$$\frac{\rho_2(E, m_A m_B)}{\rho_2(E, m_1 m_2)} = \frac{E^4 - (m_A^2 - m_B^2)}{E^4 - (m_1^2 - m_2^2)} \sqrt{\frac{E^4 - 2E^2(m_A^2 + m_B^2) + (m_A^2 - m_B^2)^2}{E^4 - 2E^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}} \quad (18)$$

[remember : $m_p = 1$]

$$\Omega_0 = \frac{4\pi}{3} \left(\frac{1}{2\pi m_\pi} \right)^3 \quad (19)$$

Ω_A and Ω_B are of the same order, for strange particles $\Omega_{\text{strange}} = 0.7 \Omega_0$ is a good value. For resonances nothing is known for the Ω 's, but if one gives Ω_0 to the non-strange and $0.7 \Omega_0$ to the strange resonances, one will have the right order.

Finally, S_A and S_B are the spins of A and B.

- vi) $(P_0 / \sum P_j) \pi_p$ is given in numerical form by Eq. (10).
In Fig. 3 we plot

$$W_1 \equiv \left(\frac{1}{\gamma^2} \frac{P_0}{\sum P_j} \right) \pi_p$$

as a function of E = centre-of-mass energy in proton masses.

4. Generalization to $p+p \rightarrow A+B$

This is almost the same as the previous case, we will only point out the differences. Expanding $|AB\rangle = c|0\rangle + d|1\rangle + \dots$ we see that only the $|1\rangle$ state contributes, since $|pp\rangle = |1\rangle$. Secondly,

in the actual measurements, an elastic cross-section appears twice as large when $A \equiv B$ as when A and B can be distinguished experimentally; therefore a factor $1 + \delta_{AB}$ must be added. Hence with A and B being two particles or resonances which can be produced from pp by strong interactions :

$$\left[\frac{d\sigma}{dw} (pp \rightarrow AB) \right]_{\text{large angles}} = d^2 \frac{\sigma_{pp, \text{inel}}}{4\pi} (1 + \delta_{AB}) \cdot \frac{P_{AB}}{P_0} \cdot \frac{1}{z^2} \left(\frac{1}{\delta^2} \frac{P_0}{\sum_j P_j} \right)_{pp} \quad (20)$$

(symbols explained below)

The symbols mean [when different from those explained under Eq. (16)] :

- i) d^2 is the squared Clebsch-Gordan coefficient of $|AB\rangle = c|0\rangle + d|1\rangle + \dots$;
- ii) $\sigma_{pp, \text{inel}}$ must be taken from experiments;
- iii) $\delta_{AB} = 1$ if $A \equiv B$; otherwise 0.
- iv)

$$\frac{P_{AB}}{P_0} = \frac{(2S_A+1)(2S_B+1)}{2 \cdot 2} \cdot \frac{2!}{(1+\delta_{AB})!} \cdot \frac{\Omega_A \Omega_B}{\Omega_0^2} \frac{\rho_2(E, m_A m_B)}{\rho_2(E, m_p m_p)} \quad (21)$$

where the ratio of the ρ 's follows from Eq. (18);

- v) $z = 1.18$ [fits $p+p \rightarrow p+p$ well (Fig. 2)]; $\chi = \frac{E}{2m_p} = E/2$;
 $[m_p = 1]$;

vi) $(P_o/\sum P_j)_{pp}$ is given in numerical form by Eq. (7).

In Fig. 3 we plot

$$W_2 \equiv \left(\frac{1}{8^2} \frac{P_o}{\sum P_j} \right)_{pp}$$

as a function of E = centre-of-mass energy in proton masses.

C. CONCLUDING REMARKS

Even if the statistical approach is correct [for further discussion see H.A. Bethe ⁷⁾ and G. Cocconi ⁸⁾], it cannot be expected that the simple exponential behaviour of $P_o/\sum P_j$ goes on indefinitely when the energy becomes much larger. Our exponential formulae came from numerical calculations which took into account the different masses, isospin, etc., and the $N_{\frac{3}{2}\frac{3}{2}}^*$. Presently, many resonant states are known which - for consistency - should be included just as well. What the effect would be is difficult to say, $P_o/\sum P_j$ might remain almost the same, it might also change by a factor 2 to 3. Since the excitation of more and more resonant states becomes possible with E growing, this alone might result in a slightly different behaviour.

On the other hand, an analytical discussion ⁹⁾ of the asymptotic behaviour for $E \rightarrow \infty$ of $P_o/\sum P_j$ in the simple case of one single kind of particles, as well as thermodynamical considerations ¹⁰⁾, show a behaviour of the form

$$P_o/\sum P_j \sim f(E) e^{-aE^\alpha}$$

where $f(E)$ is slowly varying (in powers of E) and α is 1/2, 2/3, 3/4 or 1, according to what one assumes about how the interaction volume Ω depends on E and whether or not the particles are considered indistinguishable. [In pp collisions below 30 GeV primary energy the number

of different particles appearing in the average final state is so large that each occurs hardly more than once or at most twice; therefore one can consider them to be indistinguishable. If then an asymptotic model for $E \rightarrow \infty$ and particle number $\rightarrow \infty$ with only one kind of particle is to be extrapolated down to 20-30 GeV primary energy, it would be logical to assume the particles to be distinguishable, i.e., to omit $1/n!$ which otherwise should multiply the phase space.]

Another point is that for symmetry reasons the pp elastic scattering has the largest angle 90° and there the cross-section must be a minimum and flat. The statistical model suggests for different final particles $A \neq B$, that the differential cross-section stays rather constant beyond 90° (apart from a possible peak at 180°). This should hold in particular for πp scattering where, e.g., 180° and 0° are clearly distinguished (even when $\pi + p \rightarrow A+B$, since the scattering angle is $\theta_{\pi A}$ if A has nucleonic number zero).

A trivial consequence of our formulae (16) and (20) is that for large angles the branching ratios for different two-body channels are given by

$$\frac{\frac{d\sigma}{d\omega}(\pi p \rightarrow AB)}{\frac{d\sigma}{d\omega}(\pi' p \rightarrow A'B')} = \frac{a^2 \alpha^2 + b^2 \beta^2}{a'^2 \alpha'^2 + b'^2 \beta'^2} \cdot \frac{P_{AB}}{P_{A'B'}} \quad (22)$$

$$\frac{\frac{d\sigma}{d\omega}(pp \rightarrow AB)}{\frac{d\sigma}{d\omega}(pp \rightarrow A'B')} = \frac{d^2(1 + \delta_{AB})}{d'^2(1 + \delta_{A'B'})} \cdot \frac{P_{AB}}{P_{A'B'}} \quad (23)$$

where P_{AB} and $P_{A'B'}$ follow from Eqs. (17), (21) and (18).

R E F E R E N C E S

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FIG. 1 Illustration to the definition
of 'central' collisions

FIG.2 Comparison of eqs. (6/7) with experiment Ref. 3

FIG. 3 THE FUNCTIONS $W_1(E)$ AND $W_2(E)$ NEEDED FOR THE EVALUATION OF EQU.(16) AND (20)