CM-P00056946

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RELATIVISTIC FIELD THEORIES WITH SYMMETRY BREAKING SOLUTIONS

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ABSTRACT

Following some recent developments in many-body theory, we suggest a functional approach to relativistic field theories particularly suitable to treat the case of symmetry breaking solutions. The method is characterized by the introduction of a functional of the vacuum expectation value of the field variables which has the property of being stationary around the values produced by the solutions of the theory. To illustrate the formalism we discuss the Goldstone theorem whose proof becomes almost trivial in this language. Finally, we discuss a stability or degeneracy property of the solutions of a field theory expressed by the persistence of the stationary conditions if one performs certain infinitesimal variations of the vacuum expectation values of the field. When one has symmetry breaking solutions, the admissible variations correspond to the superposition of zero mass classical waves of infinitesimal amplitude.

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The possibility that a relativistic field theory defined by a Lagrangian invariant under a certain transformation group might possess unsymmetrical solutions has been widely discussed in recent years ¹⁾. It is generally believed that, if this is the case, the "anomalous" solutions must be degenerate in some sense and, furthermore, if the symmetry group of the Lagrangian is continuous, massless particles of dynamical origin should appear ²⁾. The latter statement is actually only a possible interpretation of a result which follows quite generally once the existence of the unsymmetrical solution has been assumed. Stated in its most general form, this result indicates that a Green's function suitably defined and which has the character of a boson propagator must be singular at $p = 0$ when transformed into momentum space. This is the content of the so-called Goldstone theorem and there are indeed many examples of field theoretic models in which the singular behaviour described above is associated with massless particles. This happens at least with the approximate solutions that one is able to construct. However, the general validity of this rule has been questioned ³⁾. One is forced to recognize that, in spite of the progress made by various authors, the structure of asymmetrical solutions has not been yet satisfactorily clarified. This is not very surprising considering the general status of relativistic field theories. However, one feels that an additional difficulty is caused by the lack of a sufficiently general and flexible formulation of the problem. In fact, all the results obtained so far are largely model dependent. Substantial progress, on the other hand, has been made towards a much better treatment of this type of questions in the non-relativistic many-body problem. We refer specifically to the work of Martin, Hohenberg and De Dominicis ⁴⁾. These authors have been concerned with the construction of a suitable formalism allowing, in principle, a unified treatment in terms of Green's functions of both normal and asymmetrical problems, e.g., liquid ⁴He above and below the critical temperature, superconductors, etc.. While nobody expects that the classical difficulties of field theories might be overcome by simply using a different language, one may still hope to obtain some insight into the problems by giving them the most appropriate formulation. The new insight can take, for example, the form of general theorems that would be difficult to state in a different language.

The work of the above-mentioned authors is in the line of classical and quantum statistical thermodynamics and makes an extensive use of the functional calculus which allows to reduce the relevant problems to variational problems. In this note, which is mainly pedagogical in character, we shall make a preliminary attempt to extend these techniques to relativistic field theories with the aim of discussing from a more general point of view and possibly generalizing the results so far obtained in the study of relativistic field theories with symmetry breaking solutions. However, as it will be realized soon, the techniques proposed appear to have a rather general applicability and their possibilities will be exploited in more detail in a second paper.

The starting point of our discussion will be the analogy known since a long time between the partition function in statistical mechanics and the vacuum expectation value of the S matrix in relativistic field theories ⁵⁾. The essential point is that they both allow to construct functionals which generate the time ordered Green's functions. In the first part of the paper, we shall reduce to a variational problem the conditions for the existence of solutions in a relativistic field theory possessing a certain basic invariance property. To this end, an appropriate functional of the vacuum expectation values of the field variables is introduced. An invariance property of this functional which is easily established makes it possible to prove and give a precise meaning to the degeneracy of the anomalous solutions.

Alternative very simple proofs of the Goldstone theorem will be then presented. One of them is the relativistic counterpart of the proof of Hugenholtz and Pines theorem for a system of interacting bosons given by Hohenberg ⁶⁾. The second proof is more in the spirit of the present treatment and is perhaps even more direct.

Finally, we shall discuss a result which comes out most clearly in this type of formalism. This is a stability property common to both the normal and the symmetry breaking solutions. This property is exhibited by the invariance of the stationary conditions of the system under infinitesimal transformations which are essentially gauge transformations of the second kind. This invariance property appears to be a characteristic feature of relativistic theories and is strictly connected with the singularities of the propagator.

I. In order to make our discussion definite, we shall consider the model theory already studied by Bludman and Klein ²⁾ and defined by a set of N Hermitian boson fields $\bar{\Phi}_i$ transforming according to the fundamental representation of the orthogonal group and satisfying equations of motion of the form

$$(\square - \mu^2)\bar{\Phi}_i = j_i \quad (1)$$

Let us add to the corresponding Lagrangian $\mathcal{L}(x)$ which is invariant under the group, a symmetry breaking term of the form $\mathcal{L}'(x) = \sum_i \lambda_i(x) \bar{\Phi}_i(x)$ where the λ_i describe external sources, and construct the vacuum functional

$$S_0(\lambda) = \langle 0 | T \exp i \int d^4x (\mathcal{L}(x) + \mathcal{L}'(x)) | 0 \rangle \quad (2)$$

$|0\rangle$ is the bare vacuum. We define next the time ordered function generating functional $Z(\lambda)$

$$Z(\lambda) = -i \ln S_0(\lambda) \quad (3)$$

We have

$$\begin{aligned} \frac{\delta Z}{\delta \lambda_i(x)} &= \langle \bar{\Phi}_i(x) \rangle = \varphi_i(x) \\ \frac{\delta^2 Z}{\delta \lambda_i(x) \delta \lambda_j(y)} &= i \left[\langle T(\bar{\Phi}_i(x) \bar{\Phi}_j(y)) \rangle - \varphi_i(x) \varphi_j(y) \right] \\ &= \Delta_{ij}(x, y) \end{aligned} \quad (4)$$

The "normal" solution of the theory is the one for which

$$\lim_{(\lambda_i) \rightarrow 0} \frac{\delta Z}{\delta \lambda_j} = 0 \quad j = 1, 2, \dots, N$$

while for the asymmetrical solution this limit is different from zero ⁷⁾. It is now convenient to perform a Legendre transformation and introduce the functional of the natural variables φ_i

$$W(\varphi) = Z - \sum_i \int d^4x \lambda_i(x) \varphi_i(x) \quad (5)$$

The relevant property of this functional is that it is stationary when $(\lambda_i) \rightarrow 0$. We have in fact

$$\frac{\delta W}{\delta \varphi_i(x)} = -\lambda_i(x) \quad (6)$$

To clarify further the structure of W let us take the functional derivative of the first of the equations (4) with respect to φ_i

$$\begin{aligned} \delta_{ij} \delta(x-y) &= \frac{\delta^2 Z}{\delta \varphi_i(x) \delta \lambda_j(y)} = \sum_k \int d^4\xi \frac{\delta^2 Z}{\delta \lambda_j(y) \delta \lambda_k(\xi)} \frac{\delta \lambda_k(\xi)}{\delta \varphi_i(x)} \\ &= - \sum_k \int d^4\xi \Delta_{jk}(y, \xi) \frac{\delta^2 W}{\delta \varphi_k(\xi) \delta \varphi_i(x)} \end{aligned} \quad (7)$$

From (7) we learn that the second functional derivatives of W generate the inverse propagators. We can now proceed further by taking the functional derivative of (7) and find after some manipulations

$$\begin{aligned} \frac{\delta^3 W}{\delta \varphi_i(x) \delta \varphi_j(y) \delta \varphi_k(z)} &= \\ &= - \sum_{\ell, m, n} \iiint d^4\xi d^4\eta d^4\theta \frac{\delta^2 W}{\delta \varphi_i(x) \delta \varphi_\ell(\xi)} \frac{\delta^3 Z}{\delta \lambda_\ell(\xi) \delta \lambda_m(\eta) \delta \lambda_n(\theta)} \frac{\delta^2 W}{\delta \varphi_m(\eta) \delta \varphi_j(y)} \frac{\delta^2 W}{\delta \varphi_n(\theta) \delta \varphi_k(z)} \end{aligned} \quad (8)$$

This is the ordinary irreducible Γ vertex. From (8) all the higher derivatives can now be obtained explicitly in terms of time ordered functions by successive functional differentiation. The n -th order derivative gives the irreducible part of the n -point Green's function.

We want now to prove an invariance property of the W functional that will be crucial for the following. Let U be the unitary operator which induces on the fields the transformations of the invariance group according to the rule

$$U \Phi_i(x) U^{-1} = \sum_j T_{ij} \Phi_j(x) \quad (9)$$

where T is an $N \times N$ orthogonal matrix. It is now very easy to verify, using (3), (4), (9) and the assumed invariance property of the Lagrangian, that the transformation

$$\lambda_i \rightarrow \lambda'_i = \sum_j T_{ij} \lambda_j \quad (10)$$

is such that

$$Z(\lambda') = Z(\lambda) \quad (11a)$$

$$\varphi_i(x; \lambda') = \sum_j T_{ij}^{-1} \varphi_j(x; \lambda) \quad (11b)$$

From (11b) it follows that the substitution

$$\varphi_i \rightarrow \sum_j T_{ij} \varphi_j \quad (12)$$

induces the transformation

$$\lambda_i \rightarrow \sum_j \lambda_j T_{ji}^{-1} \quad (13)$$

on the source function considered as a functional of φ .

W is then invariant under the transformation (12). Furthermore, the unsymmetrical solution, if it exists, is determined up to the transformation (12). It is, therefore, degenerate.

II. We are now in a position to supply two very simple proofs of the Goldstone theorem. The first proof uses the Z functional, the second one the W functional.

First Proof

Let us consider an infinitesimal transformation of the group $T_{ij} = \delta_{ij} + t_{ij}$. Using (12), (13) and the second of the equations (4) we find

$$\begin{aligned} \delta \varphi_i(x) &= \sum_j t_{ij} \varphi_j(x) = \sum_j \int \frac{\delta^2 Z}{\delta \lambda_j(x) \delta \lambda_j(z)} \delta \lambda_j(z) d^4 z = \\ &= - \sum_{j \neq i} \int d^4 z \Delta_{ij}(x, z) \tilde{t}_{jk} \lambda_k(z) \quad \tilde{t}_{jk} = t_{kj} \end{aligned} \quad (14)$$

We take then the Fourier transform and invert Eq. (14) considered as a matrix equation. Finally, we go to the limit $(\lambda_i) \rightarrow 0$. This yields after remembering that because of translation invariance $\varphi_i(x, \lambda) \xrightarrow{(\lambda_i) \rightarrow 0} \varphi_i = \text{constant}$ ⁸⁾

$$\delta(p) \sum_{j \neq i} \tilde{t}_{ij}^{-1} \Delta_{jk}^{-1}(p) t_{ke} \varphi_e = 0 \quad (15)$$

In order that the matrix $\tilde{t}^{-1} \Delta^{-1}(0) t$ may have a non-zero eigenvector corresponding to a zero eigenvalue we must have

$$\text{Det}(\tilde{t}^{-1} \Delta^{-1}(0) t) = \text{Det}(\Delta^{-1}(0)) = 0 \quad (16)$$

In the particular case considered here all the φ_i but one (say φ_1) can be set equal to zero and we find for all $j \neq 1$ $\Delta_{jj}^{-1}(0) = 0$. However, in more general cases, all that one can prove is an equation of the form (16).

Second Proof

From the invariance property of W it follows that

$$\delta W + \delta^2 W + \dots =$$

$$= \sum_i \int d^4x \frac{\delta W}{\delta \varphi_i(x)} \delta \varphi_i(x) + \sum_{ij} \iint d^4x d^4y \frac{\delta^2 W}{\delta \varphi_i(x) \delta \varphi_j(y)} \delta \varphi_i(x) \delta \varphi_j(y) + \dots = 0 \quad (17)$$

when $\delta \varphi_i$ corresponds to a transformation of the form (12). In the limit $(\lambda_i) \rightarrow 0$ (17) implies (16).

III. In the previous Section we have seen that the eigenvalue problem

$$\sum_j \int d^4\xi \frac{\delta^2 W}{\delta \varphi_i(x) \delta \varphi_j(\xi)} \delta \varphi_j(\xi) = 0 \quad (18)$$

in the limit $(\lambda_i) \rightarrow 0$ has the solution

$$\delta \varphi_i = \sum_j t_{ij} \varphi_j$$

We want to investigate now whether there exist other possible eigenvectors of the system (18). It is clear that if we shall be able to find such variations that satisfy (18), the equations $\delta W / \delta \varphi_i = 0$ will hold for the displaced vacuum expectation values $\varphi_i + \delta \varphi_i(x)$. We now admit that the matrix $\Delta(p)$ singular at $p = 0$, is singular also for every other $p^2 = 0$ as it depends on p only through the invariant p^2 .

This being the case, a non-trivial eigenvector of (18) will be

$$\delta \varphi_i = \sum_j t_{ij} \varphi_j f(x) \quad (19)$$

where $f(x)$ is of the form $\int d^4p \tilde{f}(p) \delta(p^2) e^{ipx}$.

A similar stability property of the solution holds also in the normal case with

$$\delta \varphi_i = f_i(x) \quad (\square - \mu^2) f_i(x) = 0$$

μ is the renormalized mass. We can perhaps get a physical picture of this phenomenon already in the free field case. We consider a single field obeying $(\square - \mu^2) \Phi = 0$ and the generator F of gauge transformations such that $\Phi \rightarrow \Phi + f(x)$ with $(\square - \mu^2) f = 0$. If we consider then the modified vacuum

$$|0'\rangle \simeq (1 + iF) |0\rangle$$

we have $\langle 0' | \bar{\Phi}(x) | 0' \rangle = f(x)$ but there is no first order change in the vacuum expectation value of the Hamiltonian H_0 i.e., $\langle 0' | H_0 | 0' \rangle \cong 0$. The same happens for the expectation value in many particle states as well as for the other observables of the theory.

From the above analysis we conclude that oscillations of the vacuum corresponding to zero mass particles can be associated with symmetry breaking solutions. At this point, however, it may be objected that we have not considered the possibility of "spurious" states such as those mentioned in Ref. ³⁾. As far as we can understand this question, it seems to us that in the pure Green's function approach we have been describing spurious states would manifest themselves with the existence of several solutions for our variational problem. In this case stability conditions should play a role in discussing their meaning ⁹⁾. It is clear that a decision on this problem requires a much more elaborate analysis that goes beyond the general ideas considered here.

In conclusion we have presented an alternative formal approach to the problem of symmetry breaking solutions in relativistic field theories and previously known results have been obtained in a simpler way. A number of questions are open. Among others, it is of prominent importance to investigate the nature of the stationary points of the W functional. A more systematic development of these ideas and rules for calculations will be given elsewhere.

We would like to thank Professor N. Cabibbo for a very useful conversation.

REFERENCES AND FOOTNOTES

- 1) For an extensive discussion of the whole problem, the reader is referred to "Proceedings of seminar on unified theories of elementary particles" edited by D. Iurié and N. Mukunda - URPA - 11 July, 1963.
- 2) See, for example, J. Goldstone, *Nuovo Cimento* 19, 154 (1961) ;
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- 7) N.N. Bogoliubov, Physica 26 (Supplement) S1 (1960).
- 8) We do not consider, for the moment, the possibility of metastable non-Lorentz invariant solutions.
- 9) In this connection the analogy with the many-body problem emphasized by our approach may be particularly useful. However, arguments indicating a basic difference between relativistic and non-relativistic problems have been advanced by W. Gilbert, Phys. Rev. Letters 12, 713 (1964). We are indebted to Professor Y. Nambu for calling our attention to this paper.