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MUON ABSORPTION IN LIQUID HYDROGEN^{*)}

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Only in hydrogen can one study the basic μ -capture interaction free of complications arising from nuclear physics. Because of the small probability of absorption, experiments thus far ¹⁾ have all been performed in liquid hydrogen where the absorption rate depends sensitively on the particular atomic state from which the absorption takes place. The principal uncertainty in the interpretation of the results has so far been the lack of a precise estimate of the muon density at the proton for the states of interest. We present here the result of calculations which remove this uncertainty to an estimated accuracy of one percent.

There are strong theoretical arguments ²⁾, supported by experimental evidence ³⁾, that in isotopically pure liquid hydrogen absorption occurs mainly from the lowest ortho P state of the $(p\mu p)$ system, which is in fact the only bound ortho state. Furthermore, it is believed that this state is formed from a (μp) atom in its 1S_0 ground state and a proton through a spin-independent interaction (E1 Auger emission of a neighbouring electron), consequently it is produced with a total spin $\underline{S}$ of $\frac{1}{2}$. The amplitudes of states with $\underline{S} = 3/2$, which may develop as a result of spin-orbit coupling ^{2),4)}, have been calculated and found to be extremely small ⁵⁾. In an ortho state, the protons have a spin value $\underline{S}_p = 1$, hence

$$\underline{S}^2 = 3/4 = (\underline{S}_p + \underline{S}_\mu)^2 = 2 + 3/4 + 2 \langle \underline{S}_p \cdot \underline{S}_\mu \rangle$$

and

$$\langle \underline{S}_p \cdot \underline{S}_\mu \rangle = \frac{1}{4} [\langle \underline{\sigma}_{p1} \cdot \underline{\sigma}_\mu \rangle + \langle \underline{\sigma}_{p2} \cdot \underline{\sigma}_\mu \rangle] = -1.$$

The muon absorption rate in the ortho $(p\mu p)$ ion is therefore given by

$$\Gamma_{\text{mol}} = 2\gamma_0 C \left[a + \frac{1}{2} \langle \underline{\sigma}_{p1} \cdot \underline{\sigma}_{\mu} + \underline{\sigma}_{p2} \cdot \underline{\sigma}_{\mu} \rangle b \right] = 2\gamma_0 C (a - 2b) \quad (1)$$

where γ_0 is the ratio of the muon density at either proton in the $(p\mu p)$ ion to the muon density at the proton in a (μp) atom in the ground state, and the absorption rate in the (μp) atom has been taken as

$$\Gamma_{\text{atom}} = C \left[a + \langle \underline{\sigma}_{\mu} \cdot \underline{\sigma}_p \rangle b \right] \quad (2)$$

The factor γ_0 requires detailed knowledge of the wave function of the $(p\mu p)$ system. The velocity of the muon, which is the most rapidly moving particle, is $\sim \alpha$ so the system is quite non-relativistic. It is therefore a very good approximation to use the non-relativistic Schrödinger equation for the system, with the Hamiltonian, using units $\hbar = c = 1$,

$$H = \frac{1}{2M} (\mathbf{p}_1^2 + \mathbf{p}_2^2) + \frac{1}{2\mu} \mathbf{p}_3^2 + e^2 \left(\frac{1}{r_{12}} - \frac{1}{r_{13}} - \frac{1}{r_{23}} \right)$$

In the centre-of-mass system, the wave function depends on six co-ordinates, which we may choose as any three co-ordinates which specify the shape and size of the triangle formed by the three particles, and three angles which specify the orientation of the triangle in space. We choose for these angles the polar angles θ , ϕ of the interproton axis referred to a space-fixed polar axis, and the

azimuthal angle χ of the muon measured around the interproton axis. The state we are considering has unit angular momentum, odd parity, and must be antisymmetric under exchange of the proton co-ordinates. Furthermore, by choosing a state whose angular momentum has zero projection on the space-fixed polar axis, we can eliminate the dependence of the wave function on ϕ . The general form of such a wave function is given by ⁶⁾

$$\Psi = 1/\pi \sqrt{3/8} (G \cos \theta + F \sin \theta \cos \chi) \quad (3)$$

where G and F are functions of the triangular co-ordinates only, symmetric and antisymmetric respectively under proton exchange, i.e., under $r_{13} \leftrightarrow r_{23}$. Although the angular dependence can be given very simply, the Schrödinger equation still comprises coupled differential equations in three variables; we must therefore resort to the Ritz variational method to find an approximate eigenfunction. We expand G in the form suggested by Hylleraas ⁷⁾

$$G = e^{-ks} \sum_{lmn} C_{lmn} k^{l+2m+n} s^l t^{2m} u^n \quad (4)$$

where $s = r_{13} + r_{23}$, $t = r_{13} - r_{23}$, $u = r_{12}$, and k is a scale parameter whose optimum value is determined by the virial theorem. For F we must be more careful, since the adoption of a form (4) with $2m$ replaced by $2m+1$, leads in general to divergent integrals in evaluating $(\Psi, H\Psi)$. This is due to the fact that the term proportional to $\underline{F}$ in Ψ corresponds to the situation where the angular momentum of the molecule has a non-vanishing component along the molecular (interproton) axis, which can only arise from the revolution of the muon about that axis. In that case, the centrifugal barrier prevents the muon from ever reaching the molecular axis and, if we use trial wave functions which do not

incorporate this condition, we shall in general obtain infinities from the divergence of the muon centrifugal potential on that line. Since, however, $\underline{F}$ must go to zero whenever the muon is on the axis, it necessarily vanishes when the muon is at either of the proton positions. Therefore, the part of the wave function corresponding to $\underline{F}$ contributes nothing to γ_0 except in the negative sense that it adds to the normalization integral. Furthermore, it may be easily verified that $\underline{F}$ is of order μ/M with respect to $\underline{G}$, hence its contribution to the normalization is only of relative order $(\mu/M)^2$. Therefore, it is sufficient to ensure that $\underline{F}$ goes to zero on the interproton axis without worrying about its specific form in that region, since we are only interested in the norm of $\underline{F}$, which is expected to be quite small. To have a form as similar to (4) as possible, we choose

$$F = e^{-ks} t(s-u)(u-t) \sum_{\lambda\mu\nu} d_{\lambda\mu\nu} k^{\lambda+2\mu+\nu+3} s^{\lambda} t^{2\mu} u^{\nu} \quad (5)$$

With these approximations for $\underline{G}$ and $\underline{F}$, the variational integrals could all be evaluated exactly so that $(\underline{\Psi}, H\underline{\Psi})$ and $(\underline{\Psi}, \underline{\Psi})$ became quadratic forms in the c_{lmn} and $d_{\lambda\mu\nu}$. The minimisation of $E = (\underline{\Psi}, H\underline{\Psi})/(\underline{\Psi}, \underline{\Psi})$ could then be carried out by standard methods⁸⁾, and γ_0 was evaluated with the corresponding wave function. This procedure was followed with several different choices for the terms to be included in the series appearing in (4) and (5) and some results are shown in the table. No effort was made to find the optimum combination of terms, for a given number of parameters, to secure the lowest possible energy since that was not our concern here. The object was, rather, to find as dissimilar wave functions as possible, subject to the condition that they gave reasonably good energies, so as to study the possible range of variation of γ_0 . It is found that γ_0 varies only slightly between different variational

wave functions. This is not entirely surprising since the muon experiences the greatest attraction at the proton positions, consequently any wave function which gives a reasonable value of the energy, and does not artificially exclude the muon from the neighbourhood of the protons, is likely to contain a good description of the true eigenfunction in those regions of configuration space where the muon is near either proton. This interpretation is supported by the observation that, in almost all cases, a small change of the parameters from their optimum values led to a simultaneous decrease of γ_0 and of the binding energy. Wessel and Phillipson⁹⁾ have made similar calculations for the $(p\mu p)$ ortho state in an approximation equivalent to setting $\underline{F} = 0$, and obtained a result of $\gamma_0 = 0.500$ with a wave function containing 27 terms and yielding an energy of -0.46756 in our units. Our results, which show the $\underline{F}$ part of the wave function to contain much less than one percent of the normalization, thus provide an a posteriori justification of the Wessel-Phillipson approximation. Following this, we also made calculations with $\underline{F} = 0$, including expansions with negative powers as suggested by Kinoshita¹⁰⁾. The results are included in the table, from which we may finally conclude that the true value of $2\gamma_0$ lies in the range

$$2\gamma_0 = 1.01 \pm 0.01 \quad (6)$$

The most accurate possible prediction for the rate of muon absorption in the $(p\mu p)$ molecular state making use of our result (6) would require the evaluation of the other factors in (1) to at least the same accuracy. The usually employed Primakoff approximation¹¹⁾, which takes account of nucleon motion only to first order in k/M where k is the neutron momentum in the (μp) rest system, is already subject to corrections of this order. Furthermore, electromagnetic radiative corrections and the modification of the muon wave function due to the finite proton size are also expected to contribute at this level. However, since present experimental results are still an order of magnitude removed from the precision required to distinguish such small effects, we use the Primakoff approximations to make a preliminary comparison with the data. Then

6.

$$C = \frac{k^2}{(2\pi^2)} \left[\mu \alpha / (1 + \mu/M) \right]^3 \left[1 + k/(k^2 + M^2)^{1/2} \right]^{-1}$$

$$a = G_V^2 + G_P^2 + \frac{2}{3} G_A (3G_A - 2G_P)$$

$$b = 2 \left[G_A (G_V - G_A) - \frac{1}{3} G_P (G_V - 2G_A) \right]$$

where the effective coupling constants

$$G_V = g_\mu (1 + k/(2M)) F_1(q^2)$$

$$G_A = g_\mu \left[F_A(q^2) - (F_1(q^2) + (\kappa_P - \kappa_n) F_2(q^2)) k/(2M) \right]$$

$$G_P = g_\mu \left[F_P(q^2) - F_A(q^2) - F_1(q^2) - (\kappa_P - \kappa_n) F_2(q^2) \right] k/(2M)$$

where κ_P and κ_n are the anomalous magnetic moments of proton and neutron, in units $e/2M$, are derived from the assumed matrix element for μ capture

$$\mathcal{M} = g_\mu / \sqrt{2} (\bar{u}_\nu \gamma_\lambda (1 + \gamma_5) u_\mu) (\bar{u}_n (\gamma_\lambda + A_\lambda) u_p)$$

$$V_\lambda = \gamma_\lambda F_1(q^2) + i \sigma_{\lambda\tau} q_\tau (\kappa_P - \kappa_n) F_2(q^2) / (2M)$$

$$A_\lambda = \gamma_\lambda \gamma_5 F_A(q^2) + i q_\lambda \gamma_5 F_P(q^2) / \mu$$

which is the most general form arising from a bare V-A interaction if second-class interactions are excluded ¹²⁾. According to the Conserved Vector Current hypothesis ¹³⁾, g_μ may be set equal to the ¹⁴⁾ β -decay constant ¹⁴⁾, $g_\beta^V = (1.0034 \pm 0.0016) \cdot 10^{-5}/m_p^2$, and the form factors F_1 and F_2 identified with the corresponding isovector electromagnetic form factors of the nucleon; at the low "momentum transfer" $q_1^2 = -\mu^2(1+\mu/M)^{-1}$ of μ capture, they may in fact be set equal within experimental error: $F_1(q_1^2) = 0.97$. One is less certain about F_A and F_P . F_A is expected to vary little from $q^2 = 0$ to $q^2 = q_1^2$, so if we assume μ -e equivalence for the axial interaction also, we may set $F_A = g_\beta^A/g_\beta^V = -1.15 \pm 0.02$ ¹⁵⁾. Furthermore, if one assumes that one may neglect all contributions to the induced pseudoscalar term F_P other than that arising from single pion exchange and that the π -decay amplitude is given exactly by the Goldberger-Treiman relation ¹⁶⁾, one obtains

$$F_P/F_A = g_P/g_A = 2M\mu(m_\pi^2 - q_1^2)^{-1} = 6.7$$

With these values of the coupling constants and our result for γ_0 , one obtains a theoretical value of $\Gamma_{\text{mol,th}} = 477 \pm 20 \text{ s}^{-1}$, in satisfactory agreement with the experimental result of Rothberg et al.

$$\Gamma_{\text{mol,exp}} = 464 \pm 42 \text{ s}^{-1} \quad (7)$$

Since there is some uncertainty about the values of the axial vector and induced pseudoscalar coupling constants, an alternative and perhaps preferable manner of comparing the experimental result (7) with theoretical predictions is as shown in

the figure, where, following Rothberg et al., we take the vector coupling constant to have the value determined from β decay, and then use the experimental result and (1) to establish a relation between F_P and F_A .

While this note was being written, a paper appeared by Halpern¹⁷⁾ reporting the result of calculations of γ_0 by a method very similar to the present one. Halpern uses several non-linear parameters and up to 90° terms in his trial wave functions, and he has studied many possible combinations of terms. It is gratifying that his result for $2\gamma_0$ is in perfect accord with ours.

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A complete account of this work will be published elsewhere.

T A B L E

Number of terms in wave function	Number of terms in F	Contribution of F terms to nor- malization in percent	-E in units $\alpha^2\mu$	$2Y_0$
20	4	$1.93 \cdot 10^{-2}$	0.46609	1.009
31	6	$2.26 \cdot 10^{-2}$	0.46776	1.011
38	10	$3.10 \cdot 10^{-2}$	0.46801	1.010
30	0	-	0.46774	1.007
32 (including 6 with negative powers)	0	-	0.46768	1.007
36 (10 with negative powers)	0	-	0.46780	1.000

The mass ratio μ/M was taken as 0.112612

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FIGURE CAPTION

Permitted values of the coupling constants for μ capture corresponding to the measured rate of absorption in the (p μ p) orthomolecular state. The "canonical values are indicated by arrows.