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ON THE "MINIMAL INTERNAL COUPLING"

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ABSTRACT

We examine under what conditions the "minimal internal coupling" between the inhomogeneous Lorentz group and an internal symmetry group, which was introduced in an earlier paper ²⁾, arises naturally. We also answer the question whether the momenta in an irreducible representation can be diagonalized when the internal symmetry group is compact.

I. INTRODUCTION

Since the paper by McGlinn ¹⁾ on the coupling between an internal symmetry group S and the inhomogeneous Lorentz group (IHLG), there have appeared several related papers of which a few ^{2),3)} have dealt with situations where a non-trivial coupling is possible. The coupling is supposed to give rise to a fundamental group G which is not merely the direct product of S and the IHLG. The group G should be capable of explaining the multiplet structure and mass relations among the elementary particles.

Most authors (see, however, Ref. ³⁾) have considered the case when G is the product (but not the direct product) of the IHLG and S . This means that the Lie algebra of G is the sum of the Lie algebras of the IHLG and S^*). In Ref. ²⁾, where this assumption was used, it was shown that a non-trivial coupling is possible. The result was derived under the assumption that S is semi-simple and that there exists at least one generator H_m of S , which has a non-vanishing component in every simple part of the complex Lie algebra belonging to S , and which commutes with all generators of L ($=$ IHLG). This coupling was furthermore shown to have a very simple character. It consists essentially of the replacement

$$L_A \rightarrow \tilde{L}_A = L_A + \theta_A^F S_F$$

$$S_F \rightarrow S_F$$

of the generators L_A, S_F of L and S respectively in the direct product $L \otimes S$. After this replacement the group G is no longer the direct product of L and S but of $\tilde{L}$ and S , where $\tilde{L}$ is generated by the L_A . The numerical

*) All results in this paper are derived for the Lie algebras and to simplify notations we shall use the same symbol for the group and its Lie algebra.

constants b_A^p thus determine the coupling. This type of coupling was termed "minimal internal coupling" because of the similarity with the electromagnetic minimal interaction. Furthermore, in an example it was shown that it is not unreasonable to hope that such a coupling could explain the mass formulae and even constitute a natural frame for elementary particle physics.

In the physical application one should like to have S compact so as to allow finite dimensional unitary representations. In Ref. ²⁾ it was shown that under this assumption the operators L_A , which are the "physical" generators, cannot be Hermitean. Thus there arises the question whether they can be diagonalized at all.

In this paper we first make an investigation under what conditions the "minimal internal coupling" arises naturally. Secondly, we answer the question whether the operators L_A can be diagonalized when S is compact. Section II contains mainly mathematical theorems which are needed and Section III the physical conclusions which can be drawn. For a more extensive physical motivation of the problem the reader is referred to Ref. ²⁾.

II. SOME THEOREMS ON THE EXTENSION OF A LIE ALGEBRA BY ANOTHER

The problem of the coupling between a group L and a group S such that the resulting group G is the product of L and S can, as far as the local aspects are concerned, be formulated in terms of the corresponding Lie algebras as follows. Given two Lie algebras L and S with bases L_A and S_ρ , find a Lie algebra G which is spanned by L_A and S_ρ and such that not all L_A commute with all S_ρ . The problem is considerably simplified if one makes the assumption that S is an invariant subalgebra of G . To motivate this step, one can either make use of the general arguments given by Michel ⁴⁾ or notice that it follows rather immediately from the assumption that there exists an element of S (a charge) which commutes with all operators of L and which has a component in every simple part of the complex algebra ²⁾. Then the algebra G is an extension ⁵⁾ of L by S , i.e.,

$$L = G/S.$$

Furthermore, since L is not only a factor algebra of G but also a subalgebra the extension is of a type which mathematicians call inessential ⁵⁾. For such an extension we can prove the following theorem.

Theorem 1. Let L and S be two given arbitrary Lie algebras with bases L_A and S_ρ and let G be an extension of L by S such that L and S are subalgebras and $L = G/S$. Then the extension is non-trivial provided there exists a non-zero solution of the equations

$$b_A^\rho b_B^\sigma c_{\rho\sigma}^\tau = -c_{AB}^c b_c^\tau \quad (1)$$

for the constants b_A^P , C_{AB}^C and $C_{\rho\sigma}^\tau$ are the structure constants of L and S respectively. Furthermore G is the direct product $\tilde{L} \otimes S$ where $\tilde{L}$ is isomorphic to L and the generators of $\tilde{L}$ are

$$\tilde{L}_A = L_A + b_A^P S_P. \quad (2)$$

If the Lie algebra S is such that all derivations are inner derivations then this type of extension is the only possible one.

Proof. Since the extension is inessential it is in a one to one correspondence with a homomorphism from the Lie algebra L to the Lie algebra $\Delta(S)$ of derivations in S ⁵⁾. (A derivation in S is any linear transformation D in S which satisfies $D[s, t] = [Ds, t] + [s, Dt]$ where $s, t \in S$.) If one assumes that the derivation is inner it follows that⁵⁾

$$Ds = [s_D, s], \quad s_D \in S$$

and therefore, for every $l \in L$

$$[l, s] = [s_l, s]$$

for all $s \in S$, which means that the elements $\tilde{l} = l - s_l$ commute with all s of S . Because of the Jacobi identity

$$[[\tilde{l}, \tilde{m}], s] + [[s, \tilde{l}], \tilde{m}] + [[\tilde{m}, s], \tilde{l}] = 0$$

and assuming that $[\tilde{l}, \tilde{m}]$ does not have a component in the centre of S ,

we find that the elements $\tilde{l}$ generate a Lie algebra and therefore we have the direct product

$$G = \tilde{L} \otimes S \quad (3)$$

where the algebra $\tilde{L}$ is generated by all $\tilde{l}$. By putting

$$\tilde{L}_A = S_A = -b_A^P S_P \quad (4)$$

$$\tilde{L}_A = L_A - S_A \quad (5)$$

we find that

$$[\tilde{L}_A, \tilde{L}_B] = C_{AB}^C \tilde{L}_C$$

$$[S_A, S_B] = C_{AB}^C S_C$$

so that $\tilde{L}$ is isomorphic to L , and furthermore the second equation tells us that we have to find a subalgebra of S which is a representation of L . This requirement gives the equations

$$b_A^P b_B^Q C_{PQ}^T = -C_{AB}^C b_C^T \quad (1)$$

for the constants b_A^P . From the equation

$$[L_A, S_\sigma] = [\tilde{L}_A + S_A, S_\sigma] = -b_A^P C_{P\sigma}^\tau S_\tau$$

we get

$$C_{A\sigma}^\tau = -C_{P\sigma}^\tau b_A^P \quad (6)$$

so that the mixed structure constants are given by the constants b_A^P . Therefore, as soon as Eqs. (1) have a non-zero solution there exists a non-trivial extension G . Furthermore if the Lie algebra S is such that all derivations are inner (and all semi-simple algebras compact or not are of this kind) then it has no centre and this coupling is the only possible one. This completes the proof.

It should be remarked that an extension which satisfies the relation (3) is said to be mathematically trivial⁵⁾ but this does not mean that it is physically trivial.

For physical reasons one may wish to put further restrictions on the coupling between L and S . It is natural to require that one or more generators of S commute with those of L (the charges)²⁾. If we call such a generator S_m we see from Eq. (6) that we have the condition

$$C_{Am}^\tau = C_{Pm}^\tau b_A^P = 0. \quad (7)$$

This means that all (b_A^P) considered as vectors in the linear space of the algebra S lie in a plane orthogonal to the vector (C_{mP}^τ) , or the operators S_A must be chosen from that subalgebra of S which commutes with S_m . The extension problem is therefore simply reduced to that of L and the relevant subalgebra. In the case of a semi-simple algebra S one then obtains the Eqs. (3.24) and (3.25) of Ref. ²⁾.

In order that Eqs. (1) should have a solution in which the coefficients b_A^p , for A equal to a momentum index μ , are different from zero it is necessary that the lowest dimensional faithful representation of S has dimension five or higher. This is seen as follows. When b_μ^p are different from zero then the operators S_A form a Lie algebra isomorphic to the IHLG. But in the lowest dimensional representation we must also be able to find matrices which form a faithful representation of the IHLG. This is impossible unless the dimension is equal to or larger than five. Now compact simple Lie algebras have only faithful representations so in this case we must choose an algebra whose lowest dimension of its representations is larger than five.

In the physical application the operators L_A are interpreted as the generators of space-time motions for the real interacting particle ²⁾. It is therefore certainly interesting to know whether these operators and especially the momenta p_μ can be diagonalized. In Ref. ²⁾ it was found that they cannot be chosen Hermitean when S is compact. In fact, if L_A should be Hermitean, then S_A has to be Hermitean. But the operators S_A form a finite dimensional representation of the IHLG if S is compact and this is impossible for Hermitean matrices because of the non-compactness of the IHLG and its factor group. Now the operators L_A could still be diagonalizable, say normal, but the following theorem shows that this is not so.

Theorem 2. Let the coupling between the internal group S and the IHLG be of the type described in Theorem 1 and consider a representation in which the representation of S is finite dimensional. Then the momentum operators p_μ of the IHLG cannot be diagonalized unless b_μ^p is zero for all μ and p .

Proof. In the generators

$$L_A = \tilde{L}_A + S_A$$

the $\tilde{L}_A$ commutes with S_A . Therefore, in order to have L_A diagonal both $\tilde{L}_A$ and S_A must be chosen diagonal. In a finite dimensional

representation of S the operators S_A are finite matrices. Furthermore they form a representation of Lie algebra of the IHLG which is faithful if not all b_μ^P are zero. Now a faithful finite dimensional representation of the IHLG cannot be diagonalizable. More precisely, in a finite dimensional representation of the IHLG none of the momenta can be diagonalized.

Supposing

$$\{p'\}_{ij} = \lambda_i \delta_{ij}$$

where not all λ_i are equal. Then by taking the commutator with an angular momentum operator M we get another momentum

$$\{p''\}_{ij} = \{M\}_{ij} (\lambda_j - \lambda_i)$$

Now p'' must commute with p' and therefore

$$\{M\}_{ij} (\lambda_i - \lambda_j)^2 = 0$$

which means that $p'' = 0$ and that is impossible since we must be able to obtain p' from p'' by taking the commutator with an angular momentum operator. The theorem is thus proved since S_μ cannot be diagonalized and then p_μ cannot either.

It should be remarked that in a coupling where $b_\mu^P = 0$ the operators L_A can be diagonalized. In fact all finite dimensional representations of the homogeneous Lorentz group (HLG) are isomorphic to normal representations as can be seen from the well-known decomposition

$$\begin{aligned} M_j &= K_j + L_j \\ N_j &= i(K_j - L_j) \end{aligned}$$

of the Lie algebra of the HLG. The K_i and L_j are Hermitean commuting operators and thus M_i and N_j are normal operators which can always be diagonalized.

III. DISCUSSION

We have seen that the "minimal internal coupling" arises very naturally as soon as one makes the assumption that the fundamental group G is the product of the IHLG and the internal symmetry group S and that S is an invariant subgroup. On the other hand it is rather independent of the structure of the group S . The requirement is only that Eqs. (1) shall have a non-zero solution. From the mathematical point of view the coupling is rather trivial. For the physical application the simplicity is an advantage. We have already remarked that the operators L_A are interpreted as the operators of motion in space-time of a physical interacting particle. On the other hand the operators $\tilde{L}_A$ are supposed to be similar operators for bare non-interacting particles. Since G is the direct product $\tilde{L} \otimes S$ it is very easy to write down the unitary irreducible representations of G . In such a representation the operator

$$\tilde{m}^2 = \tilde{p}_\mu \tilde{p}^\mu = g^{\mu\nu} (p_\mu + b_\mu^P S_P)(p_\nu + b_\nu^P S_P)$$

has a fixed value. Since the mass is determined by the original translation operators

$$m^2 = p_\mu p^\mu$$

we see that in this way we obtain a mass splitting within the states of an irreducible representation²⁾. In the model discussed in Ref. 2) it was found that m^2 could not be diagonalized. Theorem 2 shows that the momentum operators can never be diagonalized when the representation of S is of finite dimension. Thus one either has to consider non-compact internal symmetry groups S or radically change the particle concept. In the first alternative one has to enter into the mathematical representation theory of non-compact groups which is far from simple and even not very complete. This way therefore hardly seems accessible at present. In the second case, when keeping a compact internal group, one encounters the problem

of how to define a physical particle state. Since the physical momenta cannot be diagonalized one cannot find eigenfunctions with given momenta nor with a given mass. Therefore it remains only to define a particle as a bare particle, i.e., as an eigenstate of say the "free momenta" $\tilde{p}_\mu$. To the bare particle states corresponds an energy-momentum matrix. The diagonal elements of these matrices may be the experimentally measured energy momentum and mass. Naturally such a scheme goes beyond the orthodox quantum mechanics according to which every observable is represented by a Hermitean operator.

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R E F E R E N C E S

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