Kodaira-Spencer deformation of complex structures
and Lagrangian field theory

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Abstract

In complete analogy with the Beltrami parametrization of complex structures on a (compact) Riemann surface, we use in this paper the Kodaira-Spencer deformation theory of complex structures on a (compact) complex manifold of higher dimension. According to the Newlander-Nirenberg theorem, a smooth change of local complex coordinates can be implemented with respect to an integrable complex structure parametrized by a Beltrami differential. The question of constructing a local field theory on a complex compact manifold is addressed and the action of smooth diffeomorphisms is studied in the BRS algebraic approach. The BRS cohomology for the diffeomorphisms gives generalized Gel’fand-Fuchs cocycles provided that the Kodaira-Spencer integrability condition is satisfied. The diffeomorphism anomaly is computed and turns out to be holomorphically split as in the bidimensional Lagrangian conformal models. Moreover, its algebraic structure is much more complicated than the one proposed in a recent paper [1].

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1 Introduction

A class of bidimensional conformal models, thanks to the equivalence between conformal and complex structures on a 2-d manifold, has been suitably incorporated in the framework of the local field theory upon using the Beltrami parametrisation of complex structures. Moreover, in this parametrisation the holomorphic factorization of the partition functions of these Lagrangian conformal models (free bosonic string, \(bc\) system possibly coupled to Yang-Mills gauge fields) has been proved \([2, 3, 4, 5, 6]\).

In real even dimensions larger than two, the equivalence between conformal and complex structures is no longer valid. However, complex structures on a complex manifold are still parametrized by a vector-valued \((0,1)\)-form fulfilling both an integrability and algebraic conditions. This is the Kodaira-Spencer deformation theory of complex structures on a manifold which generalizes the Beltrami parametrization to larger dimensions. While in two-(real) dimensions, the Beltrami differential has been used in the study of some bidimensional Lagrangian conformal models on a (compact) Riemann surface, the Kodaira-Spencer deformation of complex structures has occurred in theoretical physics in the context of mirror symmetry in string field models with values in some target Kähler manifold \([7, 8, 9, 10]\).

Quite recently, the holomorphicity of the partition function for chiral lagrangians of higher dimensional \(bc\) systems on a Kähler manifold has been considered in \([1]\), but only with respect to the so-called “chiral diffeomorphisms”. This restriction does not include the whole symmetry under any arbitrary reparametrization. Moreover, this \textit{a priori} holomorphicity requirement in the Beltrami differential defining an (integrable) complex structure is put by hand, and it is not straightforward to decide whether a higher dimensional analogue of the \(bc\) system actually enjoys the holomorphic factorization property. Finally, the proposed “chiral anomaly” in \([1]\) turns out to be much more complicated as it will be shown in the paper.

In the present work, we simply address the question whether this parametrization might give rise to interesting features for possible local field theories on a compact complex manifold, for instance, whether the Beltrami differential may play a role at the field theory level. In particular, we have in mind the four-(real) dimensional interesting case and possible links with the recent work \([11]\). For other noteworthy implications of holomorphicity in the Beltrami differential the reader is referred to \([1]\).

Following closely the bidimensional approach, the action of smooth diffeomorphisms (connected to the identity) is studied in the BRS formulation. In doing so, one has the advantage of dealing with the full invariance under reparametrizations instead of the “chiral diffeomorphisms” as treated in \([1]\). Furthermore, the BRS cohomology for diffeomorphisms is analysed along the line developed in \([12, 13, 14]\), and allows one to compute the local expression of a diffeomorphism anomaly. A few possible Lagrangian models will be also investigated. The first part of the present paper is devoted to some notation and, according to the book by K. Kodaira \([15]\), we will introduce the parametrisation of complex structures on a compact manifold by vector-valued \((0,1)\)-forms, the higher dimensional analogues of the Beltrami differentials. We shall be content into generalized matter fields considered as usual tensors.

The main part of the paper is based on the study of the diffeomorphism cohomology and
the finding of anomalies under the locality principle. The cohomology can be carried out for the BRS operator of the diffeomorphisms which is nilpotent under the restriction that the complex structure is integrable.

2 Basic idea for the perturbation of complex structures

Consider a complex compact manifold of complex dimension $n$ according to a finite complex analytic covering $\{(U_\alpha, z_\alpha)\}$ with $M = \cup_\alpha U_\alpha$ and $z_\alpha : U_\alpha \rightarrow \mathbb{C}^n$, the local complex coordinate mapping, $z \equiv (z^1, \ldots, z^n) = (z^k)$, with biholomorphic transition functions

$$in \ U_\alpha \cap U_\beta \neq \emptyset : z_\alpha = f_{\alpha\beta}(z_\beta),$$

for any point in the intersection. When the underlying smooth structure is considered (thus $M$ is viewed as a real smooth manifold of dimension $2n$), the complex conjugate coordinates $\bar{z} \equiv (z^1, \ldots, \bar{z}^n)$ is regarded as independent with respect to $z$.

Partial derivatives will be shortly denoted as $\partial / \partial z^k \equiv \partial_k$ and $\partial / \partial \bar{z}^j \equiv \partial_j$. We shall denote $T^1,0(M)$ as the holomorphic tangent bundle of $M$, and $\partial$ will stand for the natural frame on $T^1,0(M)$ associated to the local background complex coordinates $(z, \bar{z})$. Similarly, $dz$ will be the natural dual basis in the holomorphic cotangent space $(T^1,0(M))^*$, while we have for the external derivative (denoted here by a boldface symbol)

$$d = \partial + \partial = dz \cdot \partial + d\bar{z} \cdot \partial = dz^k \partial_k + d\bar{z}^j \partial_j, \quad d^2 = \partial^2 = \partial^2 = \{\partial, \partial\} = 0.$$ 

This system of local complex coordinates (which is fixed once and for all) will serve as background for the perturbation implemented by modifying transition functions, according to the Kodaira-Spencer’s approach, as follows. On any overlapping $U_\alpha \cap U_\beta \neq \emptyset$, of the finite covering, the background transition function $f_{\alpha\beta}(z_\beta)$ is replaced by another function $f_{\alpha\beta}(z_\beta, t)$, such that $f_{\alpha\beta}(z_\beta, 0) = f_{\alpha\beta}(z_\beta)$, and where $t$ is a complex parameter belonging to some domain of $\mathbb{C}^m$ containing the origin. One thus obtains a complex analytic family parametrized by $t$ of complex compact manifolds. It turns out that the infinitesimal deformation of $M$ can be represented by an element of the cohomology group $H^1(M, \Theta)$ of $M$ with coefficients in the sheaf $\Theta$ of germs of holomorphic vector fields on $M$. In most cases, the Kodaira-Spencer’s conjecture holds, so that $\dim H^1(M, \Theta) = m$ gives the dimension of the moduli space of $M$ when the number of moduli can be defined. A question of primary importance is to know whether it is always possible to deform a complex manifold. A partial answer is given for instance in [15]: It is always possible to deform the complex structure on any complex compact manifold. Furthermore, all the complex compact manifolds belonging to the same complex analytic family are all diffeomorphic since they are subordinated to the same smooth structure underlying the background complex structure. This
situation allows one to tackle the deformation problem by parametrizing the complex structures by a vector-valued (0, 1)-form on $M$ which extends to the higher dimensional case the well-known Beltrami parametrization of complex structures on a Riemann surface.

2.1 The Newlander-Nirenberg theorem and the Kodaira-Spencer integrability condition

Let $\mu$ be a vector-valued (0,1)-form, a smooth section of $T^{1,0\otimes}(T^{0,1})^*$ locally represented as

$$\mu(z, \overline{z}) = dz: \mu(z, \overline{z}) \partial = \mu^k_j(z, \overline{z}) d\overline{z}^j \otimes \partial_k, \quad \mu = (\mu^k_j). \quad (2.4)$$

This object generalizes the usual Beltrami differential but will be subject to an integrability condition as stated in the

**Theorem** (Newlander-Nirenberg [15]). For a given smooth vector-valued (0,1)-form, $\mu$, viewed as a differential operator of first order and locally defined on $M$ by eq.(2.4), consider the following first order partial differential operator,

$$\mathcal{L} = \overline{\partial} - \mu = d\overline{z} : (\overline{\partial} - \mu \cdot \partial) = d\overline{z} \mathcal{L}_\tau = d\overline{z} (\partial_k - \mu^k_j \partial) \cdot \mu = (\mu^k_j). \quad (2.5)$$

Next suppose that $\mu$ obeys the following two conditions

$$\det(I - \mu \overline{\mu}) \neq 0, \quad \text{and} \quad \mathcal{L} \mu = \overline{\partial} \mu - \mu^2 = \overline{\partial} \mu - \frac{1}{2} [\mu, \mu] = 0, \quad (2.6)$$

where the graded bracket reflects the bracket of vector fields.

Hence, the differential operators $\mathcal{L}_1, \ldots, \mathcal{L}_n, \overline{\mathcal{L}}_1, \ldots, \overline{\mathcal{L}}_n$ are linearly independent and they fulfill the condition

$$\mathcal{L}^2 = 0 \iff [\mathcal{L}_i, \mathcal{L}_j] = 0, \quad i, j = 1, \ldots, n. \quad (2.7)$$

Moreover, the system of $n$ PDE’s

$$\mathcal{L} f = 0 \iff \mathcal{L}_i f = 0, \quad i = 1, \ldots, n, \quad (2.8)$$

has locally $n$ linearly independent smooth solutions $Z = (Z^1(z, \overline{z}), \ldots, Z^n(z, \overline{z}))$, i.e. the Jacobian never vanishes,

$$\mathcal{J} = \frac{\partial (Z, \overline{Z})}{\partial (z, \overline{z})} = |\det \lambda|^2 \det(I - \mu \overline{\mu}) \neq 0 \implies \det \lambda \neq 0, \quad (2.9)$$

where $\lambda = (\partial_i Z^k)$ is the (half)-Jacobian matrix.

2.2 The smooth change of local complex coordinates

The solution $Z$ provides a new system of complex coordinates on $M$, that is a new integrable complex structure parametrized by $\mu$. The new complex structure $(Z, \overline{Z})$ on $M$ can be seen as a deformation of the background complex structure $(z, \overline{z})$, and each such $\mu$ determines an integrable complex structure on $M$. \pagebreak
Remark. The coordinates $Z$, local smooth solutions of the system (2.8), are defined up to an holomorphic function $F(Z)$. Also, when $\mu \equiv 0$ the change of local coordinates becomes holomorphic and belongs to the complex structure given by $z$. Moreover, one has $\det \lambda \neq 0$.

The Newlander-Nirenberg theorem infers a smooth change of local complex coordinates in $\mathbb{C}^n$, $(z, \bar{z}) \rightarrow (Z, \bar{Z})$. These complex coordinates $Z$ can be re-obtained by the following row-vector of 1-forms

$$dZ = \partial Z + \mu Z = (dz + d\bar{z} \cdot \mu) \cdot \lambda,$$

one accordingly gets by closure the two following integrability conditions, one on $\lambda = (\partial_i Z^k)$, $\det \lambda \neq 0$,

$$\overline{\partial} \partial Z + \partial (\mu Z) = 0 \iff \overline{\partial} \lambda - \partial (\mu \cdot \lambda) = 0,$$

and the other, the necessary integrability condition (2.6) on $\mu$. Therefore, the defining system of $n$ PDE’s (2.8) for $Z$, namely

$$LZ = (\overline{\partial} - \mu) Z = 0, \quad [\overline{\partial} \mu - \mu^2 = 0],$$

is the analogue of the Beltrami equation but with in addition an integrability condition. The latter looks like the vanishing of a curvature. One may also assume that $\det(I - \mu \overline{\mu}) > 0$ and $\det \lambda > 0$, when orientation-preserving coordinate transformations are in order.

According to definition (2.10) and its complex conjugate, one can write in matrix notation

$$\begin{pmatrix} dz, d\bar{z} \end{pmatrix} = \begin{pmatrix} I & \overline{\mu} \\ \mu & I \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix},$$

so that by duality

$$\begin{pmatrix} \partial Z \\ \bar{\partial} Z \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}^{-1} \begin{pmatrix} I & \overline{\mu} \\ \mu & I \end{pmatrix}^{-1} \begin{pmatrix} \partial_z \\ \bar{\partial}_{\bar{z}} \end{pmatrix}.$$

One explicitly finds

$$\begin{pmatrix} \partial Z \\ \bar{\partial} Z \end{pmatrix} = \begin{pmatrix} \lambda^{-1} \cdot (I - \overline{\mu} \cdot \mu)^{-1} \cdot (\partial_z - \overline{\mu} \cdot \partial_{\bar{z}}) \\ \overline{\lambda}^{-1} \cdot (I - \mu \cdot \overline{\mu})^{-1} \cdot (\partial_{\bar{z}} - \mu \cdot \partial_z) \end{pmatrix} \equiv \begin{pmatrix} \lambda^{-1} \cdot \nabla \\ \overline{\lambda}^{-1} \cdot \overline{\nabla} \end{pmatrix}.$$

3 The diffeomorphism action

By computing the pull-back of the 1-form (2.10) under any smooth diffeomorphism $\varphi$, $\varphi^* dZ = d\varphi^* Z = (d\varphi^* w + d\varphi^* \overline{w}) \cdot (\varphi^* \mu) \cdot (\varphi^* \lambda) = (dz + d\bar{z} \cdot (\mu \varphi)) \cdot (\lambda \varphi)$, where $(z, \bar{z})$ are coordinates.
in the source chart of $\varphi$ and $(w, \overline{w}) = \varphi(z, \overline{z})$ in the target chart, the finite diffeomorphism transformation of the matrices of basic geometric objects reads

$$\lambda^\varphi = \left( \partial \varphi + \partial \varphi \cdot (\mu \circ \varphi) \right) \cdot (\lambda \circ \varphi) , \quad (3.1)$$

$$\mu^\varphi = \left( \overline{\partial} \varphi + \overline{\partial} \varphi \cdot (\mu \circ \varphi) \right) \cdot \left( \partial \varphi + \partial \varphi \cdot (\mu \circ \varphi) \right)^{-1} .$$

Thus, as a vector-valued $(0,1)$-form,

$$\mu^\varphi = \d z \cdot \mu^\varphi \cdot \partial = \left( \overline{\partial} \varphi + \overline{\partial} \varphi \cdot (\mu \circ \varphi) \right) \cdot \left( \partial \varphi + \partial \varphi \cdot (\mu \circ \varphi) \right)^{-1} \cdot \partial . \quad (3.2)$$

We now consider the equivalence problem between integrable complex structures. Consider two copies of the compact complex manifold $M$ with local background complex coordinates $(z, \overline{z})$ and let $\varphi$ be any smooth diffeomorphism homotopic to the identity map. Given a vector-valued $(0,1)$-form, $\mu$, submitted to the condition $\overline{\partial} \mu - \mu^2 = 0$, let $(Z, \overline{Z})$ be the local complex coordinates at the point $(w, \overline{w}) = \varphi(z, \overline{z})$, by solving the system of $n$ PDE’s (2.12), one can show that the integrability condition is preserved by diffeomorphisms (as will be explicitly proved in Appendix A and also down below for the infinitesimal version)

$$\overline{\partial} \mu - \mu^2 = 0 \implies \overline{\partial} \mu^\varphi - (\mu^\varphi)^2 = 0 , \quad (3.3)$$

and thanks to the Newlander-Nirenberg theorem there are local complex coordinates $Z^\varphi$ pertaining to the integrable complex structure parametrized by $\mu^\varphi$; hence one has

$$(\overline{\partial} - \mu^\varphi) Z^\varphi = 0 . \quad (3.4)$$

The problem is to see whether the mapping $(Z^\varphi, \overline{Z}^\varphi) \rightarrow (Z, \overline{Z})$ is indeed bi-holomorphic. If it is so, then $\mu$ and $\mu^\varphi$ define equivalent integrable complex structures which correspond to the same point $t$ in the moduli space. The situation is the same as for Riemann surfaces, that is the mapping $(Z^\varphi, \overline{Z}^\varphi) \rightarrow (Z, \overline{Z})$ is actually bi-holomorphic, $\mu$ and $\mu^\varphi$ defining equivalent integrable complex structures. The reader is referred to Appendix B for the details.

The infinitesimal action given by the (graded) Lie derivative, $s \equiv L_C = i_c \partial - d i_c$ and, thanks to (2.10), writes locally in each open set $U$,

$$s Z = L_C Z = i_c \partial Z = (c + \overline{c} \cdot \mu) \cdot \lambda \equiv C \cdot \lambda \equiv \Upsilon , \quad (3.5)$$

where $c = c \cdot \partial + \overline{c} \cdot \overline{\partial} = \Upsilon \cdot \partial Z + \overline{\Upsilon} \cdot \partial Z$ is the (smooth) Faddeev-Popov ghost vector field expressed respectively in the holomorphic coordinates $z$ and $Z$, and $C$ reflects a change of basis in the Lie algebra of diffeomorphisms leading to the holomorphic factorization property of bidimensional conformal models [2, 3, 5, 6]. Here, in matrix notation, we have

$$\begin{pmatrix} C, \Upsilon \end{pmatrix} = (c, \overline{c}) \begin{pmatrix} I & \mu \ \\ \mu & I \end{pmatrix} , \quad \begin{pmatrix} \Upsilon, \Upsilon \end{pmatrix} = (C, \overline{C}) \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} , \quad (3.6)$$

the latter being the BRS counterpart of eq.(2.13).
On the one hand, the infinitesimal version of (3.1) can be expressed in terms of the nilpotent BRS operation, \( s^2 = 0 \), as

\[
\begin{align*}
    s \lambda &= c \lambda + (\partial c + \partial c \cdot \mu) \cdot \lambda, \\
    s \mu &= c \mu + (\overline{\partial} - \mu \cdot \partial)c + (\overline{\partial} - \mu \cdot \partial)c \cdot \mu, \\
    s c &= \frac{1}{2}[c, c],
\end{align*}
\]

(3.7)

where the graded bracket is that of vector fields. Note that the variation of \( \mu \) is non-linear in \( \mu \). Moreover, we know that the integrability condition (2.6) is \( s \)-invariant as shown in Appendix A.

On the other hand, with respect to the integrable complex structure defined by \( \mu \), the infinitesimal action of diffeomorphisms on the complex coordinate \( Z \) writes locally in \( U \), with \( s^2 = 0 \),

\[
\begin{align*}
    s Z &= c Z = \Upsilon, \\
    s \Upsilon &= 0, \\
    s d + d s &= 0 \implies [s, \partial Z] = -\partial c
\end{align*}
\]

(3.8)

Accordingly, one has the following remarkable identities, together with their complex conjugate,

\[
\begin{align*}
    [s - c, \partial Z] &= 0, \\
    s \partial + \partial s &= \partial \Upsilon \cdot \partial Z - \overline{\partial} \Upsilon \cdot \partial Z.
\end{align*}
\]

(3.9)

By computing directly \( s d Z = -d s Z \), one finds,

\[
\begin{align*}
    s \lambda &= \partial (C \cdot \lambda), \\
    s \mu &= -\overline{\partial}(C \cdot \partial) + [\mu, C \cdot \partial], \\
    s (C \cdot \partial) &= \frac{1}{2}[C \cdot \partial, C \cdot \partial] = (C \cdot \partial C) \cdot \partial,
\end{align*}
\]

(3.10)

where the component \( C \) of the vector field \( C \cdot \partial \), a smooth section of the holomorphic tangent bundle \( T^{1,0} \) [15], is defined by (3.6).

The reader’s attention is drawn to the fact that, these two sytems of infinitesimal variations for \( \mu \) and \( \lambda \), (3.7) and (3.10), are equivalent only upon the use of both the integrability conditions (2.6) and (2.11). The former represents the true infinitesimal action of diffeomorphisms, while the latter, which holds only in the case of an integrable complex structure, will govern the holomorphic factorization property. In the integrable case, the system (3.10) will also represent the infinitesimal action of diffeomorphisms, and the integrability condition (2.6) is seen to be stable under diffeomorphisms by

\[
\overline{\partial} \mu - \mu^2 = 0 \implies s (\overline{\partial} \mu - \mu^2) = [C \cdot \partial, \overline{\partial} \mu - \mu^2] = 0.
\]

(3.11)

4 Fields and Lagrangians

According to what has been done in the two dimensional case [16, 17, 5, 6, 13], one can take advantage of the complex coordinate system \( Z \) pertaining to an integrable complex structure \( \mu \), in order to exhibit some simple Lagrangians built on a \( n \)-dimensional complex manifold.
Before going further, let us emphasize that a classical action \( \Gamma_{0}^{Cl} \) has to be well defined on the complex manifold \( M \), i.e.

\[
\Gamma_{0}^{Cl} = \int_{M} dZ^{A_{n}} \wedge d\overline{Z}^{\overline{A}_{n}} L_{A_{n}}(Z, \overline{Z}),
\]

where in the integrand a multi-index notation has been used, see eq.(5.26) down below, with \( |A_{n}| = |\overline{A}_{n}| = n \). This means that the Lagrangian density is a skew-symmetric covariant tensorial density under holomorphic change of local coordinates \( Z \rightarrow Z'(Z) \),

\[
L_{A_{n}}(Z', \overline{Z'}) = \left| \frac{\partial (Z^{1}, \ldots, Z^{n})}{\partial (Z'^{1}, \ldots, Z'^{n})} \right|^{-2} L_{A_{n}}(Z, \overline{Z}), \quad |A_{n}| = |A'_{n}| = |\overline{A}_{n}| = |\overline{A}'_{n}| = n. \tag{4.2}
\]

For instance, one may ask whether analogue Lagrangians of the \( bc \)-systems as well as a generalisation of the bosonic string can be obtained over such a higher dimensional complex manifold and what is the role of the complex structure.

Our purpose is first to use the holomorphic coordinates \( Z \) and then to make explicit the coupling of matter fields with the integrable complex structure \( \mu \).

The simplest higher-dimensional analogue of the bidimensional \( bc \)-system [5, 6] is constructed by considering a holomorphic vector bundle \( E \) over \( M \) and the fields \( \Psi \) as a smooth section of \( T^{k,0*} \otimes E, \ 0 \leq k \leq n-1 \), and \( \Psi^{*} \) a smooth section of \( T^{n-k,n-1*} \otimes E^{*} \). Our choice for the \( bc \) fields is rather different from the one written in [1] and reduces to the usual \( bc \)-system for \( n = 1 \). One has following well-defined action

\[
S_{bc} = \int_{M} \Psi^{*} \wedge (\overline{\partial} + \overline{\mathcal{A}}) \Psi, \tag{4.3}
\]

where the \((0,1)\)-connection form \( \mathcal{A} \) parametrizes the holomorphic structure on \( E \) provided the integrability condition, \( \overline{\partial}A + \overline{\mathcal{A}}^{2} = 0 \), is satisfied [18].

The natural generalisation of the free bosonic string action reads

\[
S(\Phi) = \int_{M} \overline{\Phi} \wedge (\overline{\partial} \overline{\Phi}), \tag{4.4}
\]

where, in a multi-index notation, \( \Phi = \Phi_{I} dZ^{I} \) is a smooth complex-valued \((n-1,0)\)-form which can be chosen to be real only in the case \( n = 1 \). One can see that it is not possible to consider scalar fields in such a complex model unless there exists a hermitian metric \( g \) on \( M \).

In any case, the important point is that the above Lagrangian densities depend locally on the integrable complex structure \( \mu \). One can directly check that the two above actions are actually local in \( \mu \). Indeed, defining the following change of field variables \( (\lambda^{I}_{J}) \Phi_{I} = \phi_{J} \), with \( (\lambda^{I}_{J}) = \lambda^{n}_{n-1} \cdots \lambda^{n-1}_{1} \), according to the form degree, the action (4.4) reads

\[
S(\phi, \mu, \overline{\mu}) = \int_{M} \det(I - \mu \cdot \overline{\mu})(\nabla \phi + \iota(\tau)\phi) \wedge (\nabla \phi + \iota(\tau)\phi), \tag{4.5}
\]

where \( \nabla = dz \cdot \nabla \) (\( \nabla \) defined in (2.15)) and \( \iota(\tau)\phi = (n-1)\tau \wedge \phi_{r} \) is a \((n-1,1)\)-form given by the inner product on the \((n-1,0)\)-form \( \phi = \phi_{I} dZ^{I} \) with respect to the vector-valued \((1,1)\)-form \( \tau = d\tau \cdot (I - \mu \overline{\mu})^{-1} \wedge \partial_{\mu} \overline{\partial} \). In the course of the computation the condition (2.11) on the integrating factor has been used.
Similarly, for the bc-system the action takes the following local expression in $\mu$

$$S_{bc}(\psi, \psi^*, \mu, \overline{\mu}) = \int_M \det(I - \mu \cdot \overline{\mu}) \psi^* \wedge \left( \nabla + i(\tau) + d\sigma(I - \mu \cdot \overline{\mu})^{-1} \cdot \sigma \right) \psi,$$

(4.6)

with $i(\tau) \psi = k \tau^r \wedge \psi_r$ is a $(k,1)$-form with values (as well $\psi$) in a copy of the holomorphic vector bundle $E$ with respect to the complex analytic coordinates $z$, and where we have set for the components of the $(0,1)$-type connection $A_\sigma = \lambda^{-1} (I - \mu \overline{\mu})^{-1} \cdot \sigma$. In this framework, (integrable) complex structures on $E$ are parametrized by $\mu$ and $\overline{\mu}$ subject to the integrability conditions $\mathcal{L}_\mu = 0$ and $\mathcal{L}_{\overline{\mu}} + \overline{\sigma}^2 = 0$, respectively. However, the obvious holomorphic factorization property in $\mu$ for $n = 1$, $(\psi^* \wedge (\overline{\partial} - \mu + \overline{\sigma}) \psi$ [6]) is spoilt, and the action $S_{bc}$ strongly differs from the one proposed in [1] which is assumed to enjoy that property in any dimension.

The action of diffeomorphisms gives rise, for instance, to the following local (in $\mu$) transformation laws on the $(n-1,0)$-form $\phi$ and on the bc fields $\psi$ and $\psi^*$,

$$s \phi = c \phi + i(\Omega) \phi, \quad s \psi = c \psi + i(\Omega) \psi, \quad s \psi^* = c \psi^* + i(\Omega) \psi^* + i(\overline{\Omega}) \psi^*$$

(4.7)

where we have introduced the following vector-valued $(1,0)$-form carrying ghost grading one $\Omega = (\partial c + \overline{\partial} c, \mu) \cdot \overline{\partial}$ which governs the parallel transport along the ghost vector field $c$.

5 Study of diffeomorphism anomalies

From now on, since matter fields have not been specified yet, their will be collectively denoted by $\phi$, and together with the integrable complex structure $\mu$, one introduces the short-hand notation $f_0 \equiv \{\phi, \psi, \psi^*, \mu, \overline{\mu}\}$, and for the diffeomorphism ghost fields as well $f_1 \equiv \{c, \overline{\sigma}\}$, and $f \equiv f_0 \cup f_1$. In order to suppress the index summation and the integration over the manifold $M$ as well, we also introduce the dual contraction by the bracket $<,>$, defined as

$$< \tilde{f}, f > \equiv \int_M dz^n \wedge d\overline{z}^n < \tilde{f}(z, \overline{z}), f(z, \overline{z}) > .$$

(5.1)

The diffeomorphism symmetry (3.7) can be encoded in the following nilpotent BRS functional operator

$$\delta_0 = \sum_f < s f, \frac{\delta}{\delta f} >, \quad s f(z, \overline{z}) = c f(z, \overline{z}) + V(f, c, \overline{c})(z, \overline{z}),$$

(5.2)

where $V$ is a local differential polynomial. The diffeomorphism invariance of the classical theory is stated in a Ward identity, $\delta_0 \Gamma_0^{[k]} [f_0] = 0$, together with the integrability condition on the $\mu$’s, which may be seen as a local constraint

$$[\mathcal{L}_\tau, \mathcal{L}_\sigma] = (\mathcal{L}_\tau \mu^k_\sigma - \mathcal{L}_\sigma \mu^k_\tau) \partial_k \equiv F^k_{\tau\sigma}(\mu) \partial_k, \quad \text{with} \quad F^k_{\tau\sigma}(\mu) = 0,$$

(5.3)

with of course the complex conjugate expression. According to the diffeomorphism invariance of the integrability condition, one has

$$\delta_0 F^k_{\tau\sigma}(\mu) = 0 = \delta_0 \overline{F}^k_{\tau\sigma}(\overline{\mu}).$$

(5.4)
Following the spirit of [3], in the vacuum sector ($\phi \equiv 0$), we can rephrase the Ward identity in terms of local Ward operators with respect to the ghost (parameter) fields, $c$'s, as

$$W_k(z, \bar{z})\Gamma_0^{c_i}[\mu, \bar{\mu}] \equiv \left( \partial_k \mu^r \frac{\delta}{\delta \mu^r} - \partial_r \frac{\delta}{\delta \mu^r} + \partial_r \left( \mu^r \frac{\delta}{\delta \mu^r} \right) + \right.$$

$$+ \bar{\mu}^l \left( \partial_k \bar{\mu}^l \frac{\delta}{\delta \mu^r} + \partial_l \left( \bar{\mu}^l \frac{\delta}{\delta \mu^r} \right) \right) \left( z, \bar{z} \right) \Gamma_0^{c_i}[\mu, \bar{\mu}] = 0,$$

(5.5)

together with the complex conjugate expression $\bar{W}_k \Gamma_0^{c_i}[\mu, \bar{\mu}] = 0$. One can directly check that the combination

$$\left( W_k - \bar{W}_k \right)(z, \bar{z}) \Gamma_0^{c_i}[\mu, \bar{\mu}] = \left( F^j_{kl}(\bar{\mu}) \frac{\delta}{\delta \mu^l} - \bar{\mu}^l F^j_{kl}(\mu) \frac{\delta}{\delta \mu^l} + \right.$$

$$+ \left( I - \bar{\mu} \cdot \mu \right)_k^l \left( \partial_r \mu^r \frac{\delta}{\delta \mu^r} - \partial_l \frac{\delta}{\delta \mu^r} + \mu^r \partial_r \frac{\delta}{\delta \mu^r} + \partial_l \mu^r \frac{\delta}{\delta \mu^r} \right) \left( z, \bar{z} \right) \Gamma_0^{c_i}[\mu, \bar{\mu}] = 0,$$

(5.6)

does no longer depend on the $\bar{\mu}$'s if the integrability conditions $F(\mu(z, \bar{z})) = F(\bar{\mu}(z, \bar{z})) = 0$ are taken into account. With these constraints one obtains a new local Ward operator

$$\mathcal{W}_k(z, \bar{z})\Gamma_0^{c_i}[\mu, \bar{\mu}] \equiv \left( \left[ \left( I - \bar{\mu} \cdot \mu \right)_k^l \right] (W_k - \bar{W}_k) \right)(z, \bar{z}) \Gamma_0^{c_i}[\mu, \bar{\mu}]

$$

$$= \left( \partial_r \mu^r \frac{\delta}{\delta \mu^r} - \partial_l \frac{\delta}{\delta \mu^r} + \mu^r \partial_r \frac{\delta}{\delta \mu^r} + \partial_l \mu^r \frac{\delta}{\delta \mu^r} \right) \left( z, \bar{z} \right) \Gamma_0^{c_i}[\mu, \bar{\mu}],$$

(5.7)

which reflects the linear change of parameter $C^k = c^k + \mu^k \bar{\tau}^\sigma$ corresponding to the the variation (3.10) of $\mu$. Recall that the latter is equivalent to the Lie derivative of $\mu$ only if the integrability condition is considered.

On the physical side, the meaning of the geometrical quantity $\Theta^c(z, \bar{z}) \equiv \frac{\delta \Gamma_0^{c_i}}{\delta \mu^r}(z, \bar{z})$ is still obscure. It does not correspond to the energy-momentum tensor as in the bidimensional case since the equivalence between conformal and integrable complex structures does not hold anymore in higher dimensions. Moreover, the $\mu$'s which serve as classical sources for the components of $\Theta$ are not all independent due to the integrability condition. The question whether the latter induces some constraints on the theory is not yet under control. Nevertheless, one may write down the classical Ward identity (5.7) in the vacuum sector with respect to the ghost $C$, (the tree level)

$$\left( \mathcal{L}_T(z, \bar{z}) \frac{\delta}{\delta \mu^r}(z, \bar{z}) - \partial_r \mu^r(z, \bar{z}) \frac{\delta}{\delta \mu^r}(z, \bar{z}) - \partial_l \mu^r(z, \bar{z}) \frac{\delta}{\delta \mu^r}(z, \bar{z}) \right) \Gamma_0^{c_i}[\mu, \bar{\mu}] = 0.$$

(5.8)

Geometrically, going to the $(Z, \bar{Z})$ complex coordinates, $\Theta$ behaves like a $(T^{1,0})^* \otimes (T^{0,1})$-valued density according to the following transformation law

$$\vartheta(Z, \bar{Z}) \equiv \frac{1}{J} \chi^{-1} \cdot \Theta(z, \bar{z}) \cdot (I - \mu \cdot \bar{\mu}) \cdot \bar{\lambda},$$

(5.9)

where $J$ is the Jacobian defined in (2.9). In terms of the $(Z, \bar{Z})$ complex coordinates, the classical Ward identities (5.8) translate into the vanishing of the following $n$ divergences, (see
Appendix C for some details),
\[ \partial_{\xi^r} \partial_{\tilde{\kappa}}^r = 0 , \quad k = 1, \ldots, n. \] (5.10)

Remark that for \( n = 1 \), (5.9) reduces to the transformation law of a quadratic differential [17].

We now introduce external fields \( \beta \equiv \beta_{-1} \cup \beta_{-2} \equiv \{ \gamma, \eta^l_k, \eta^k_j \} \cup \{ \zeta, \tilde{\zeta}_j \} \), with an assigned (negative) ghost grading respectively coupled to the BRS variation of the fields \( f \equiv \{ \phi, \mu^k_l, \tilde{\mu}_l^k, c^i, \tilde{c}^i \} \), in order to have a \( \Phi\Pi \) neutral action, written in a short-hand notation as
\[ \Gamma_{\text{Cl}}^\beta[f] = \sum_f < \beta, s f > . \] (5.11)

One has \( \delta_0 \Gamma_{\text{Cl}}^\beta[f] = 0 \) thanks to \( s^2 = 0 \). Within the enlarged classical action, \( \Gamma_{\text{Cl}}^\beta[f] = \Gamma_{\text{Cl}}^0[f_0] + \Gamma_{\text{Cl}}^1[\beta, f] \), the BRS functional operator writes
\[ \delta_0 = \sum_f < \delta \Gamma_{\text{Cl}}^\beta / \delta \beta , \delta f > \] (5.12)

and the diffeomorphism invariance of \( \Gamma_{\text{Cl}}^\beta \) simplifies into the so-called Slavnov identity [19]
\[ \delta_0 \Gamma_{\text{Cl}}^\beta[f] = 0. \] (5.13)

In order to preserve the classical diffeomorphism invariance on the functional \( \Gamma_{\text{Cl}}^\beta \), the BRS operation can be extended to the classical sources \( \beta \), either trivially by saying that these external fields are \( s \)-invariant, or by defining
\[ s_{\beta_{-1}}(z, \bar{z}) = \frac{\delta \Gamma_{\text{Cl}}^\beta[f]}{\delta f_0(z, \bar{z})} = \frac{\delta \Gamma_{\text{Cl}}^0[f_0]}{\delta f_0(z, \bar{z})} + c \beta_{-1}(z, \bar{z}) + \tilde{V}(\beta_{-1}, f_1)(z, \bar{z}), \]
\[ s_{\beta_{-2}}(z, \bar{z}) = \frac{\delta \Gamma_{\text{Cl}}^\beta[f]}{\delta f_1(z, \bar{z})} = c \beta_{-2}(z, \bar{z}) + \tilde{V}(\beta_{-2}, f_1)(z, \bar{z}) + X(f_0, \beta_{-1})(z, \bar{z}) , \] (5.14)

where \( \tilde{V} \) is the formal dual of the differential operator \( V \) given in (5.2), and \( X \) is the local differential polynomial
\[ X_i(f_0, \beta_{-1})(z, \bar{z}) = \frac{\delta}{\delta c^i(z, \bar{z})} \sum_{f_0} < \beta_{-1}, s f_0 > , \] (5.15)

together with the complex conjugate expression. With these variations we can define the following nilpotent functional differential operator
\[ \delta = \delta_0 + \sum_\beta < s_\beta, \delta / \delta \beta > , \] (5.16)

which corresponds to the linearized Slavnov operator acting on the space of local functionals in both the fields \( f \) and the sources \( \beta \). Finding a quantum extension \( \Gamma[\beta, f] \) of the classical functional \( \Gamma_{\text{Cl}}^\beta \) such that \( \delta \Gamma = 0 \) amounts to studying the \( \delta_0 \)-cohomology in the space of local functionals with respect to the ghost grading (the \( \Phi\Pi \) charge). In the lagrangian field theory language, a quantum extension will exist if the cohomology is trivial at the ghost grading one,
i.e. there is no anomaly. It is now customary [20, 12] to translate this cohomological problem into the space of local polynomials in the fields and sources and their derivatives spanned by the infinite set of local independent variables of the type

\[ \partial_1 \partial_2 \chi(z, \bar{z}), \quad \chi \equiv \{ \phi, \mu, \bar{\mu}, \bar{\eta}, \eta, \gamma, \bar{\gamma}, \zeta, \bar{\zeta}, \}, \quad |I| \geq 0, \quad |J| \geq 0, \]

\[ \partial_I \equiv (\partial_1)^{i_1} \cdots (\partial_n)^{i_n}, \quad I = (i_1, \ldots, i_n), \quad |I| = i_1 + \cdots + i_n, \]

\[ \partial_J \equiv (\partial_1)^{j_1} \cdots (\partial_n)^{j_n}, \quad J = (j_1, \ldots, j_n), \quad |J| = j_1 + \cdots + j_n, \]

\[ \delta I^{M+N} \frac{I!}{M!N!} \equiv \prod_{k=1}^{n} \delta_{i_k}^{m_k+n_k} \frac{i_k!}{m_k!n_k!} \]

\[ I + 1_I = (i_1, \ldots, i_l + 1, \ldots, i_n), \quad J + 1_J = (j_1, \ldots, j_l + 1, \ldots, j_n), \]

where we have conveniently used a multi-index notation and $|I|$ denotes the length of the multi-index $I$. The space of local polynomials in the fields and their derivatives can be endowed with the structure of a Fock space. The interchange between the two systems of local complex coordinates $(z, \bar{z})$ and $(Z, \bar{Z})$, can be taken into account. According to a previous work [13], by considering the coordinates $Z$ and $\bar{Z}$ as independent variables but non local in $\mu$, we enlarge the set $\chi = f \cup \beta$ of fields. The local cohomology will be that of the following differential operator

\[ \delta = <sZ(z, \bar{z}), \frac{\partial}{\partial Z(z, \bar{z})}> + <s\bar{Z}(z, \bar{z}), \frac{\partial}{\partial \bar{Z}(z, \bar{z})}> + \sum_{|I|,|J|\geq 0} \left( \sum_{|\chi|} <\partial_I \partial_J s\chi(z, \bar{z}), \frac{\partial}{\partial \partial_I \partial_J \chi(z, \bar{z})}> \right. \]

\[ + \left. <\partial_I \partial_J s\lambda(z, \bar{z}), \frac{\partial}{\partial \partial_I \partial_J \lambda(z, \bar{z})}> + <\partial_I \partial_J s\bar{\lambda}(z, \bar{z}), \frac{\partial}{\partial \partial_I \partial_J \bar{\lambda}(z, \bar{z})}> \right), \]

which turns out to be nilpotent $\delta^2 = 0$ upon using both the integrability conditions (2.11) and (2.6). Note that the non local part in the $\mu$’s has been isolated.

For a given $\Phi$-II charge $p$, solutions will be obtained modulo total derivatives, and we get the following descent equations

\[
\begin{align*}
\delta \Delta_{2n}^p(z, \bar{z}) + d\Delta_{2n-1}^{p+1}(z, \bar{z}) &= 0 \\
\delta \Delta_{2n-1}^p(z, \bar{z}) + d\Delta_{2n-2}^{p+2}(z, \bar{z}) &= 0 \\
& \vdots \\
\delta \Delta_{2n-r}^p(z, \bar{z}) + d\Delta_{2n-r-1}^{p+r+1}(z, \bar{z}) &= 0 \\
& \vdots \\
\delta \Delta_1^{p+2n-1}(z, \bar{z}) + d\Delta_0^{p+2n}(z, \bar{z}) &= 0 \\
\delta \Delta_0^{p+2n}(z, \bar{z}) &= 0
\end{align*}
\]

where the lower index will label the Poincaré form degree and the upper index denotes the ghost grading. The last of these equation has general solution

\[ \Delta_0^{p+2n}(z, \bar{z}) = \Delta_0^{p+2n,0}(z, \bar{z}) + \delta \Delta_0^{p+2n-1}(z, \bar{z}) \]
where $\Delta_0^{p+2n,1}(z,\bar{z})$ is an element of the $\delta$-cohomology space in the space of local functions. It is readily seen [12, 13] that

$$d = dz^k \left\{ \delta, \frac{\partial}{\partial \sigma^k(z,\bar{z})} \right\} + d\bar{z}^k \left\{ \delta, \frac{\partial}{\partial \sigma^k(z,\bar{z})} \right\}, \quad (5.21)$$

which allows one to define the derivative operator on the local polynomials as a formal derivative

$$\partial_i = \left\{ \delta, \frac{\partial}{\partial \sigma^i(z,\bar{z})} \right\} = \lambda_i(z,\bar{z}), \frac{\partial}{\partial Z(z,\bar{z})} > + \lambda_i(z,\bar{z}), \frac{\partial}{\partial \bar{Z}(z,\bar{z})} > \quad (5.22)$$

and one gets $\partial_i$ by complex conjugation.

Going up through the system of descent equations (5.19) we subtitute Eq(5.20) and obtain

$$\delta \left[ \Delta_1^{p+2n-1}(z,\bar{z}) + \frac{\partial \Delta_0^{p+2n,1}(z,\bar{z})}{\partial \sigma(z,\bar{z})} dz^i + \frac{\partial \Delta_0^{p+2n,1}(z,\bar{z})}{\partial \sigma(z,\bar{z})} d\bar{z}^i + d\hat{\Delta}_0^{p+2n-1}(z,\bar{z}) \right] = 0. \quad (5.23)$$

This implies that

$$\Delta_1^{p+2n-1}(z,\bar{z}) = \frac{\partial \Delta_0^{p+2n,1}(z,\bar{z})}{\partial \sigma(z,\bar{z})} dz^i + \frac{\partial \Delta_0^{p+2n,1}(z,\bar{z})}{\partial \sigma(z,\bar{z})} d\bar{z}^i + d\hat{\Delta}_0^{p+2n-1}(z,\bar{z}) = \Delta_1^{p+2n-1,0}(z,\bar{z}) + \delta \hat{\Delta}_1^{p+2n-2}(z,\bar{z}) \quad (5.24)$$

and proceeding step by step, we reach the top equation of (5.19) which gives the general solution of the $\delta$-cohomology in the sector of ghost grading $p$ [12, 13]

$$\Delta_2^n(z,\bar{z}) = \sum_{r,s=1}^n \frac{\partial^r}{\partial \sigma(z,\bar{z})} \frac{\partial^s}{\partial \sigma(z,\bar{z})} \Delta_2^{p+r+s,0}(z,\bar{z}) dz^{A_r} \wedge d\bar{z}^{A_s} + \Delta_2^n(z,\bar{z}) + d\hat{\Delta}_2^{p-1}(z,\bar{z}) \quad (5.25)$$

where for the sake of compactness an increasing ordered multi-index notation has been used,

$$A_r = (a_1,\ldots,a_r), \quad a_1 < \cdots < a_r, \quad dz^{A_r} = dz^{a_1} \wedge \cdots \wedge dz^{a_r}, \quad \frac{\partial^r}{\partial \sigma(z,\bar{z})} \frac{\partial^s}{\partial \sigma(z,\bar{z})} = \frac{\partial^r}{\partial \sigma(z,\bar{z})} \frac{\partial^s}{\partial \sigma(z,\bar{z})}, \quad (5.26)$$

with of course the complex conjugate expressions. Equation (5.25) links the elements of the diffmod $d$ cohomology to those of the local functional cohomology, once the $\Delta_2^{p+r+s,0}$, $r,s = 0,\ldots,n$ are known as $\delta$-cocycles.

We are interested in finding possible anomalies in the quantum field theoretic sense, hence with $p = 1$ in eq.(5.25). According to our previous analysis in [13], it turns out that the only relevant tensorial index $2n - r - s$ is of scalar type, namely for $r = s = n$. The problem reduces into solving the cocycle equation

$$\delta \Delta_0^{2n+1,0}(z,\bar{z}) = 0. \quad (5.27)$$
Following the line of Refs [12, 13, 14] we decompose the general $\Delta_0^{p,0}$, for $p \geq 2n$, with respect to its underived ghosts content, namely
\[ \Delta_0^{p,0}(z, \bar{z}) = \sum_{r,s=0}^{n} c^{Ar}_{rs}(z, \bar{z}) \bar{\Omega}(z, \bar{z}) \mathcal{D}^{p-r-s}_{A_r \bar{A}_s}(z, \bar{z}), \] (5.28)
where according to the summation on all possible ordered multi-indices (5.26), the $\mathcal{D}$'s are independent skew-symmetric tensors containing only derivatives of the ghost fields. This canonical decomposition identifies independent sectors according to the ghost grading. Inserting the decomposition (5.28) into (5.27), and introducing the nilpotent operator
\[ \hat{\delta} = \delta - (c'(z, \bar{z}) \partial_t + c'(z, \bar{z}) \partial_{\bar{t}}), \quad \hat{\delta}^2 = 0, \] (5.29)
the cocycle condition can be canonically decomposed with respect to the underived ghost fields onto each ghost sector, and find the following chain of $2^{2n}$ equations
\[
\begin{cases}
\hat{\delta} \mathcal{D}^p(z, \bar{z}) = 0 \\
-\hat{\delta} \mathcal{D}_{f}^{-1}(z, \bar{z}) + \partial_t \mathcal{D}_{f}^{-1}(z, \bar{z}) + \partial_{\bar{t}} \mathcal{D}_{f}^{-1}(z, \bar{z}) + \partial_t \mathcal{D}^p(z, \bar{z}) = 0 \\
-\hat{\delta} \mathcal{D}_{r}^{-1}(z, \bar{z}) + \partial_t \mathcal{D}_{r}^{-1}(z, \bar{z}) + \partial_{\bar{t}} \mathcal{D}_{r}^{-1}(z, \bar{z}) + \partial_t \mathcal{D}^p(z, \bar{z}) = 0 \\
\vdots \\
(-)^{k+l} \hat{\delta} \mathcal{D}_{A_k \bar{A}_l}^{-p-k-l}(z, \bar{z}) + c^{Ar}_{Ak} \left( \sum_{l=1}^{k} (-)^{l-1+i} \partial_t c^{Al}_{Ar+1} \mathcal{D}_{A_{r+1} \bar{A}_l}^{-p-k-l}(z, \bar{z}) \\
+ \sum_{l=1}^{l+1} (-)^{k+l+i} \partial_t c^{Al}_{Ar} \mathcal{D}_{A_{r+1} \bar{A}_l}^{-p-k-l}(z, \bar{z}) + (-)^{k-1} \partial_t \mathcal{D}_{A_r \bar{A}_s}^{-p-k-l}(z, \bar{z}) \right) \\
+ c^{Al}_{A_l} \left( \sum_{l=1}^{k+1} (-)^{l-1} \partial_t c^{A_{l+1}}_{A_{l+1}} \mathcal{D}_{A_{l+1} \bar{A}_s}^{-p-k-l}(z, \bar{z}) + \sum_{l=1}^{l+1} (-)^{k+l} \partial_t \mathcal{D}_{A_{l+1} \bar{A}_s}^{-p-k-l}(z, \bar{z}) \right) \\
\vdots \\
\hat{\delta} - \partial_t c'(z, \bar{z}) - \partial_{\bar{t}} c'(z, \bar{z}) \mathcal{D}_{A_n \bar{A}_n}^{-2n}(z, \bar{z}) + (-1)^{n-1} c^{Al}_{A_n} \partial_t \mathcal{D}_{A_{n-1} \bar{A}_n}^{-2n+1}(z, \bar{z}) \\
- c^{Al}_{A_n} \partial_t \mathcal{D}_{A_{n-1} \bar{A}_n}^{-2n+1}(z, \bar{z}) = 0,
\end{cases}
\] (5.30)
where $c^{Ar}_{Ak}$ is the generalized Kronecker skew-symmetric tensor with respect to the multi-indices, and $A_{r+1}$ means $a_1 \prec \cdots \prec a_t \prec \cdots \prec a_r$ for a given $A_r$. In the above formulas summation on repeated multi-indices must be performed and $\delta^{Al}_{A_l}$ equals 1 if the ordered multi-indices are identical and 0 otherwise.

Since the $\delta$-cohomology is translated in terms of the $\hat{\delta}$-cohomology, the first step is to solve the top equation of the system (5.30). Also, note that the $\hat{\delta}$ operator will act on the space of local polynomials in the following extended set of fields $\chi \equiv f \cup \beta \cup \{\lambda, \bar{\lambda}\}$. Solving the $\hat{\delta}$-cohomology is the technical part of the paper and in Appendix D the following theorem will be established.

**Theorem.** Define in matrix notation the $n^2$ local expressions of ghost grading one,
\[ \hat{\Omega}(z, \bar{z}) = (\partial c + \partial \bar{c} \cdot \mu)(z, \bar{z}) = (\hat{\delta} \lambda \cdot \lambda^{-1})(z, \bar{z}), \] (5.31)
and note the $\hat{\delta}$-coboundary
\[
\text{tr} \tilde{\Omega}(z, \bar{z}) = \hat{\delta} \ln \det \lambda(z, \bar{z}).
\] (5.32)

Then, in the scalar sector of the space of analytic (local) functions in the fields and their derivatives, the ghost sectors with grading $p$ of the $\hat{\delta}$-cohomology are generated by the following non-trivial cocycles
\[
\text{for even } p : \quad \text{tr} \tilde{\Omega}^{2r+1}(z, \bar{z}) \text{tr} \tilde{\Omega}^{2s+1}(z, \bar{z}), \quad \text{for odd } p : \quad \text{tr} \tilde{\Omega}^{2k+1}(z, \bar{z}),
\] (5.33)

while the cocycle $\text{tr} \tilde{\Omega}$ can be reabsorbed by completing the set of generators with $\ln \det \lambda$ seen as an independent variable. The same statements hold true for the complex conjugate expressions.

A non trivial local anomaly modulo $d$ will then be given by (cf. eq.(5.25))
\[
\Delta^{1}_{2n}(z, \bar{z}) = \frac{\partial^{n}}{\partial c^{A_{n}}(z, \bar{z})} \frac{\partial^{n}}{\partial c^{A_{n}}(z, \bar{z})} \Delta^{2n+1, \ast}_{0}(z, \bar{z}) dz^{A_{n}} \wedge d\bar{z}^{A_{n}},
\] (5.34)

with the local $\delta$-cocycle $\Delta^{2n+1, \ast}_{0}$ given by the decomposition (5.28) and also subject to the reality condition $\Delta^{1}_{2n}(z, \bar{z}) = \Delta^{1}_{2n}(\bar{z}, z)$.

According to the definition of both $\hat{\delta}$ and (5.31), one considers in matrix notation the $n^{2}$ variables of ghost grading one,
\[
\Omega(z, \bar{z}) = (\delta \lambda \cdot \lambda^{-1})(z, \bar{z}) = \tilde{\Omega}(z, \bar{z}) + (c \lambda \cdot \lambda^{-1})(z, \bar{z}),
\]
\[
\text{tr} \Omega(z, \bar{z}) = \delta \ln \det \lambda(z, \bar{z}).
\]

Unfortunately, $\Omega$ is not local in the $\mu$ fields due to the presence of the $\lambda$ terms. One has $\delta \Omega(z, \bar{z}) = \Omega^{2}(z, \bar{z})$. Moreover, setting $\Lambda_{i} \equiv \partial_{i} \lambda \cdot \lambda^{-1}$, it is easily shown that
\[
\Omega(z, \bar{z}) = (C^{r} \Lambda_{r})(z, \bar{z}) + \partial C(z, \bar{z}),
\]
\[
\delta \ln \det \lambda(z, \bar{z}) = (C^{r} \partial_{r} \ln \det \lambda)(z, \bar{z}) + \text{tr}(\partial C)(z, \bar{z}),
\] (5.35)

after the use of eq.(2.11) and the definition (3.6) of the factorized ghosts $C$. Note that the above variations contain a local part in the $\mu$ fields.

In turns out, after a rather lengthy analysis based on the above theorem, that the possible $\delta$-cocycles $\Delta_{0}^{2n+1, \ast}$ are not local in the $\mu$ external fields due to the presence of the non-local $\lambda$ terms. The former are listed hereafter
\[
\text{tr} \Omega^{n} \text{tr} \tilde{\Omega}^{n} \text{tr} \Omega = \delta(\text{tr} \Omega^{n} \text{tr} \tilde{\Omega}^{n} \ln \det \lambda), \quad \text{for odd } n,
\]
\[
\text{tr} \Omega^{2r+1} \text{tr} \tilde{\Omega}^{2s+1} \text{tr} \Omega = \delta(\text{tr} \Omega^{2r+1} \text{tr} \tilde{\Omega}^{2s+1} \ln \det \lambda), \quad \text{for even } n, \quad n = r + s + 1,
\] (5.36)
\[
\text{tr} \Omega^{2n+1} + \text{tr} \tilde{\Omega}^{2n+1} \neq \delta(\cdots), \quad n \geq 3.
\]

The first two cocycles are trivial while the last non-trivial $\delta$-cocycle can only produce a local cocycle of ghost grading $2n$ and not $2n + 1$ as required. This shows that the local $\delta$-cohomology modulo $d$ is not trivial in that sector.
Therefore, if there were an anomaly, the latter is a non-trivial cocycle of the $\delta$-cohomology modulo $\mathbf{d}$. So it is not surprising that the last possibility to be considered is that the local $\delta$-cocycle $\Delta_0^{2n+1}$ matching with the principle of locality will appear as the variation of a non-local expression in the $\mu$ fields according to,

$$
\Delta_0^{2n+1} = \delta \hat{\Delta}_0^{2n},
$$

(5.37)

where $\hat{\Delta}_0^{2n}$ does not belong to the non-local $\delta$-cohomology modulo $\mathbf{d}$ and will provide non-local counterterms in the $\mu$ fields which are defined up to a total derivative. Since the non-local generators are the $\lambda$ terms and the variations (5.35) contain a local part, $\hat{\Delta}_0^{2n}$ can be formally expanded in terms of $\lambda$ and its (independent) derivatives. By using once more eq.(2.11), it assumes in the scalar sector the following finite expansion

$$
\hat{\Delta}_0^{2n} = \sum_{|I|=0}^{2n} \left( T^I \partial_I \ln \det \lambda + \text{tr} \left( T^{I+1} \partial_I \Lambda_1 \right) + \text{c.c.} \right),
$$

(5.38)

where in multi-index notation (5.17) both the scalar valued, $T^I = T^{i_1 \cdots i_n}$ and matrix valued, $T^{I+1}$, symmetric tensors are local expressions in the $\mu$ external fields with $\dim T^I = \dim T^{I+1} = 2n - |I|$, and $Q_{\Phi \Pi}(T^I) = Q_{\Phi \Pi}(T^{I+1}) = 2n$. Recalling that in a Lagrangian formulation counterterms are defined up to a total derivative, so it is for those generated by $\hat{\Delta}_0^{2n}$. Thus, discarding the explicit form of the divergences one can write

$$
\hat{\Delta}_0^{2n} = \left( \sum_{|I|=0}^{2n} (-1)^{|I|} \partial_I T^I \right) \ln \det \lambda + \text{tr} \left( \sum_{|I|=0}^{2n} (-1)^{|I|} \partial_I T^{I+1} \right) \Lambda_i + \partial_i \Gamma^i + \text{c.c.}
$$

(5.39)

with $\dim R = \dim \mathcal{R}^i + 1 = 2n = Q_{\Phi \Pi}(R) = Q_{\Phi \Pi}(\mathcal{R}^i)$. Substitution of $\hat{\Delta}_0^{2n}$ in eq.(5.34), together with $\frac{\partial}{\partial \mathcal{R}^0(\mathbf{r}, \mathbf{z})} \partial_0 = 0$, provides the anomaly up to a total derivative, one can thus discard the divergences. The $\delta$-variation of $\Delta_0^{2n}$ writes

$$
\delta \hat{\Delta}_0^{2n} = R \text{tr} \partial C + \text{tr} \left( \mathcal{R}^i \partial_i \partial C \right) + \delta R \ln \det \lambda + R C^r \partial_r \ln \det \lambda
$$

$$
+ \text{tr} \left( \delta \mathcal{R}^i \Lambda_i + \mathcal{R}^i \partial_i \Lambda_1 + \partial_i C^r \Lambda_1 + [\partial C, \Lambda_1] \right) + \text{c.c.},
$$

(5.40)

and by requiring the vanishing of the non-local part one gets

$$
(a) \begin{cases} 
RC^r = 0, & \forall r = 1, \ldots, n, \\
\mathcal{R}^i C^r = 0,
\end{cases} 
\quad (b) \begin{cases} 
\delta R = 0, \\
\delta \mathcal{R}^i = \mathcal{R}^k s \partial_s C^i - \mathcal{R}^k s \partial_s C^i - \mathcal{R}^k s \partial_s C^i.
\end{cases}
$$

(5.41)

Remark that $\mathcal{R}^k s$ transform in a rigid manner. Eqs.(5.41)(a) are solved by

$$
(a) \begin{cases} 
R = \Sigma A_n C^A_n, \\
\mathcal{R}^i = \Sigma A_n C^A_n,
\end{cases} 
\quad (b) \begin{cases} 
\partial \Sigma A_n = \frac{\partial \Sigma A_n}{\partial C^r} = 0, & \forall r = 1, \ldots, n,
\end{cases}
$$

(5.42)
with $C^{A_n} \equiv C^1 C^2 \ldots C^n$, $\delta C^{A_n} = -\text{tr}(\partial C) C^{A_n}$, and $Q_{\Phi}(\Xi_{A_n}) = Q_{\Phi}(\Sigma_{A_n}^i) = n$. The local cocycle (5.37) then reads
\[
\Delta_0^{2n+1,5} = C^{A_n} \left( \Xi_{A_n} \partial_r C^r + \Sigma_{l,A_n}^{k,i} \partial_l \partial_k C^l \right) + \partial_r \Gamma_{loc}^r + \text{c.c.} \tag{5.43}
\]
A direct substitution of (5.42)(a) into Eqs. (5.41)(b) gives,
\[
(\delta \Xi_{A_n} - \partial_r C^r \Xi_{A_n}) C^{A_n} = 0 \implies \delta \Xi_{A_n} = \text{tr}(\partial C) \Xi_{A_n} + C^r \xi_{r,A_n}, \tag{5.44}
\]
with $Q_{\Phi}(\xi_{r,A_n}) = n$, and furthermore,
\[
\delta \Sigma_{l,A_n}^{k,i} = \text{tr}(\partial C) \Sigma_{l,A_n}^{k,i} + \partial_l C^s \Sigma_{s,A_n}^{k,i} - \partial_s C^i \Sigma_{l,A_n}^{k,s} - \partial_s C^k \Sigma_{l,A_n}^{s,i} + C^r \sigma_{r,l,A_n}^{k,i}, \tag{5.45}
\]
with $Q_{\Phi}(\sigma_{r,l,A_n}^{k,i}) = n$. Obviously, due to (5.42)(b), the $C$ dependence in both Eqs. (5.44) and (5.45) comes from the undervived ghost content of the BRS operator and thus
\[
\frac{\partial \xi_{A_n}}{\partial C^r} = \frac{\partial \sigma_{l,A_n}^{k,i}}{\partial C^r} = 0, \quad \forall r = 1, \ldots, n. \tag{5.46}
\]
The terms $\Xi_{A_n}$, $\xi_{r,A_n}$, $\Sigma_{l,A_n}^{k,i}$ and $\sigma_{r,l,A_n}^{k,i}$ are local in the fields $\{\phi, \mu, \pi, c, \bar{c}\}$ and their explicit form has to be separately discussed according to the presence of the $\phi$ matter fields. The same considerations hold for the complex conjugate expressions.

### 5.1 Vacuum Anomalies

In the vacuum sector, it is readily seen that $\partial_r = \left\{ \frac{\partial}{\partial C^r}, \delta \right\}$. This easily implies together with Eqs. (5.42)(b) and (5.46) that $\xi_{r,A_n} = \partial_r \Xi_{A_n}$, and $\sigma_{r,l,A_n}^{k,i} = \partial_r \Sigma_{l,A_n}^{k,i}$. Eqs. (5.44) and (5.45) rewrite respectively as
\[
\delta \Xi_{A_n} = \partial_r (C^r \Xi_{A_n}), \quad \delta \Sigma_{l,A_n}^{k,i} = \partial_r (C^r \Sigma_{l,A_n}^{k,i}) + \partial_l C^s \Sigma_{s,A_n}^{k,i} - \partial_s C^i \Sigma_{l,A_n}^{k,s} - \partial_s C^k \Sigma_{l,A_n}^{s,i}, \tag{5.47}
\]
showing that $\Xi_{A_n}$ and $\Sigma_{l,A_n}^{k,i}$ are respectively a scalar and a tensorial densities. The inhomogeneous term in the $\delta$-variation of $\mu$ eliminates the possible $\mu$ dependence of both $\Xi_{A_n}$ and $\Sigma_{l,A_n}^{k,i}$; one has to figure out the latter in terms of local expressions in the derivated ghosts $C$’s only. According to both the power counting index $2n$ and the ghost grading $n$, it turns out that the most general expressions are labelled by the set of permutations $\pi$ of the integers $\{1, \ldots, n\}$. Thus there are $n!$ scalar densities of the following type
\[
\Xi_{A_n,\pi} = \epsilon^{a_1 \cdots a_n} \partial_{a_1} \partial_{a_1} C^{d_1} \cdots \partial_{a_n} \partial_{a_n} C^{d_n} \delta_{\pi(1)}^{i_1} \cdots \delta_{\pi(n)}^{i_n}, \tag{5.48}
\]
the $\pi$’s cover all the possible contractions. Saturation with the product $C^{A_n}$ of the $C$’s provides the solutions (5.42). The latter are numerized according to the decompositions of the integer $n$ into sums of integers independently to the order. Let $p(n)$ denote such a decomposition. The independent solutions of (5.42) can be seen as skewsymmetrizations of products of already skewsymmetric tensors of maximal rank,
\[
R(p(n)) = \left( -1 \right)^{\frac{n(n-1)}{2}} \frac{k_1! \cdots k_n!}{k_1! \cdots k_n!} \text{tr} \left( V^{k_1} \right) \cdots \text{tr} \left( V^{k_n} \right), \tag{5.49}
\]
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for a given partition \( p(n) \) of the integer \( n \) with \( k_1 + \cdots + k_n = n, k_1 \leq \cdots \leq k_n \) and where in matrix notation the \( n^2 \) ghost graded 2 quantities \( V_i^l \equiv C^r \partial_r \partial_l C^l \) rigidly transforms as

\[
\delta V = [\partial C, V], \quad V \equiv C \cdot \partial (\partial C),
\]

(5.50)
a transformation law very similar to that of the adjoint of \( gl(n, C) \) with matrix parameter \( \partial C \). The first contributions \( R_{p(n)}(\partial C) \) to the cocycle (5.43) are developed with respect to the monomial \( \partial^C \) while the second are developed with respect to the monomials \( \partial_i \partial_k C^l \). It is easily seen that \( \sum_{k,i} s,A \nabla^2_{k,i} C^s \) is a scalar density of power counting index \( n+1 \) and ghost grading \( n+1 \), so that \( \sum_{k,i} s,A \nabla^2_{k,i} C^s = \Xi_A \nabla (\partial C) \). One has, as diffeomorphism cocycles labelled by all the partitions of \( n \),

\[
R_{p(n)}(\partial C) = R_{k,i,p(n)}^l \partial_k \partial_l C^l.
\]

(5.51)

Thus, in the vacuum sector, the \( \delta \)-cocycle modulo \( d \) (5.43) reduces to the following linear combination

\[
\Delta_{2n+1,2}^0 = \sum_{p(n)} A_{p(n)} R_{p(n)}(\partial C) \nabla \partial C
\]

(5.52)

where the \( A_{p(n)} \)'s are complex numbers depending on the field content of the model. The cocycles (5.51) are higher dimensional analogues of the well-known Gel’fand-Fuchs cocycle [21]. For complex dimension \( n = 1 \), one recovers the usual Gel’fand-Fuchs cocycle \( C \partial C \partial^2 C \) [13].

Inserting the cocycle (5.52) into the computational formulae (5.34), one finds the local expression of (U-V part) the vacuum anomaly modulo \( d \) (5.43) reduces to the following linear combination

\[
\Delta_{2n}^1 = (-1)^{n(n-1)/2} \sum_{p(n)} A_{p(n)} \frac{1}{n!} \left[ \nabla (\partial C) \nabla \left( (\partial V)^{k_1} \right) \cdots \nabla \left( (\partial V)^{k_n} \right) \right]
\]

\[
+ \nabla (\partial V) \sum_{l=1}^n k_l \left[ \nabla \left( (\partial V)^{k_1} \right) \cdots \nabla \left( (\partial V)^{k_{l-1}} \partial C \right) \cdots \nabla \left( (\partial V)^{k_n} \right) \right],
\]

(5.53)

where the sum is performed over all the partitions \( p(n) \) of the integer \( n \), \( k_1 + \cdots + k_n = n \), with \( k_1 \leq \cdots \leq k_n \), and where we have introduced the \( n^2 \) vector-valued \((1,1)\)-forms

\[
V \equiv \nabla (d \bar{z} \cdot \mu) = \left( d \bar{z} \cdot \frac{\partial}{\partial \bar{c}} \right) \left( d \bar{z} \cdot \frac{\partial}{\partial \bar{c}} \right) V,
\]

(5.54)

fulfilling the following Bianchi-like identities

\[
\nabla V = 0, \quad \nabla V + [V, \mu] = 0,
\]

(5.55)

thanks to the integrability condition (2.6). In the \( n = 1 \) case, the well-known holomorphically split anomaly \( \partial C \partial^2 \mu \) [2, 3] is recovered.

Note that there is a very particular linear combination of the anomaly (5.53) given by the component of degree \( 2n + 2 \) of the Todd class according to formula (5.19) of [1]. However, formula (1.11) or (8.5) of the quoted reference does not match as a diffeomorphism anomaly in the spirit of our construction in the sense that it is not a diffeomorphism \( \delta \)-cocycle whereas each term in the summand (5.53) is.
In concluding, since the same argument holds for the complex conjugate part, the anomalous Ward id’s (5.8) are holomorphically split at the quantum level, similarly to the bidimensional case, it is a strong indication that the vacuum functional turns out to be as well holomorphically factorized in the Beltrami differential parametrizing an integrable complex structure. The anomalous Ward id’s correspond to the non-vanishing of the divergence (5.10) at the quantum level, a similar phenomenon to the bidimensional case, [17].

A non-local functional $\Gamma[^{\mu}]$ analogous to the Wess-Zumino-Polyakov action remains to be found out as functional in the integrating factor $\lambda$.

### 5.2 Matter field anomalies

Having extracted in the previous subsection all the C ghost dependence of both $\Xi_{A_n}$ and $\Sigma_{k,A_n}$ it remains to study their matter field dependence under the constraints (5.44) and (5.45) respectively. Due to the transformation laws of matter fields under diffeomorphisms the ghost grading is this time carried by the C ghost fields. According to the power counting, one finds for $\Xi$,

$$\Xi_{A_n} = C^1 \cdots C^n K_{A_n,\overline{\lambda}_n} \equiv \overline{C^n} K_{A_n,\overline{\lambda}_n},$$

where $K_{A_n,\overline{\lambda}_n}$ is a differential polynomial in both the $\mu$ and matter fields with $Q_{\Phi \Pi} K_{A_n,\overline{\lambda}_n} = 0$ and $\dim(K_{A_n,\overline{\lambda}_n}) = 2n$. Due to the reality condition the cocycle (5.43) writes in the matter field sector as

$$\Delta^{2n+1,\lambda} = C^A \overline{C^n} K_{A_n,\overline{\lambda}_n} \det(\partial C + \overline{\partial C}) = c^A \overline{C^n} \det(I - \mu \cdot \overline{\mu}) K_{A_n,\overline{\lambda}_n} \det(\partial c + \mu \cdot \partial \overline{c} + \overline{\mu} \cdot \partial c).$$

(5.57)

where it is more useful to rewrite the variations with respect to the true ghos fields. The cocycle condition (5.41)(b) is equivalent to

$$\delta \left( c^A \overline{C^n} \det(I - \mu \cdot \overline{\mu}) K_{A_n,\overline{\lambda}_n} \right) = 0$$

(5.58)

and yields

$$\left( \delta - \det(\partial c + \overline{\partial c}) \right) \left( \det(I - \mu \cdot \overline{\mu}) K_{A_n,\overline{\lambda}_n} \right) = c^r \kappa_{r,A_n,\overline{\lambda}_n} + \overline{c^r} \overline{\kappa}_{r,A_n,\overline{\lambda}_n}.$$ (5.59)

A combination of the derivative operator (5.22) with the above equation yields

$$\kappa_{r,A_n,\overline{\lambda}_n} = \partial_r \left( \det(I - \mu \cdot \overline{\mu}) K_{A_n,\overline{\lambda}_n} \right), \quad \overline{\kappa}_{r,A_n,\overline{\lambda}_n} = \overline{\partial}_r \left( \det(I - \mu \cdot \overline{\mu}) K_{A_n,\overline{\lambda}_n} \right),$$ (5.60)

which implies

$$\delta \left( \det(I - \mu \cdot \overline{\mu}) K_{A_n,\overline{\lambda}_n} \right) = \partial_r \left( c^r \det(I - \mu \cdot \overline{\mu}) K_{A_n,\overline{\lambda}_n} \right) + \overline{\partial}_r \left( \overline{c^r} \det(I - \mu \cdot \overline{\mu}) K_{A_n,\overline{\lambda}_n} \right).$$ (5.61)

The r.h.s. shows that $\det(I - \mu \cdot \overline{\mu}) K_{A_n,\overline{\lambda}_n}$ belongs to the uncharged scalar density sector of the diff mod $d$ cohomology. Given the following variation

$$\delta \ln \det(I - \mu \cdot \overline{\mu}) = (c \cdot \partial + \overline{c} \cdot \overline{\partial}) \ln \det(I - \mu \cdot \overline{\mu}) - \text{tr}(\mu \cdot \partial \overline{c} + \overline{\mu} \cdot \overline{\partial} c)$$ (5.62)
together with eq. (5.61) allows to express (5.57) as
\[
\Delta_{2n+1} = \frac{c_{A_n} \alpha_n}{\det(I - \mu \cdot \bar{\mu})} K_{A_n, \bar{\alpha}_n} \frac{\partial c + \partial \bar{\alpha}}{\partial c} - \delta \left( \frac{c_{A_n} \alpha_n}{\det(I - \mu \cdot \bar{\mu})} K_{A_n, \bar{\alpha}_n} \ln \det(I - \mu \cdot \bar{\mu}) \right) .
\] (5.63)

Finally, we consider the contribution coming from \( \Sigma \). Rewriting the term
\[
C^A_{\alpha n} \Sigma^{k,i}_{k,A_n} \partial_i \partial_k C^i \equiv \text{tr}(V S),
\] (5.64)
inside the cocycle (5.43), the \( \delta \)-cocycle condition yields \( \delta S = [\partial C, S] \), showing that the \( 2n - 1 \) ghost graded quantity \( S \) depends only on the \( C \) ghost fields and cannot depend on matter fields. So no contribution arises from the \( \Sigma \) term in the matter sector.

Summing up, the insertion of the cocycle (5.63) in the constructive equation (5.34) gives rise to the well-known trace anomaly which breaks the holomorphic splitting of the partition function when matter fields are involved.

6 Concluding remarks

Even if the physical motivation in studying higher complex dimensional manifolds is not well stated, the previous considerations can be regarded as an exercise. The link between Beltrami differentials as sources of relevant physical tensors in higher dimension is not yet known. The integrability condition on the \( \mu \)’s is taken into account and insure the nilpotency of the BRS operator for the diffeomorphisms. In fact, the integrability condition and the nilpotency are actually equivalent. In other words, demanding an integrable complex structure gives the nilpotency. Moreover, the integrability condition is preserved under diffeomorphisms. The computation of the generalized Gel’fand-Fuchs unintegrated \( \delta \)-cocycles (5.51) (modulo \( d \)) is valid upon the use of the integrability condition on the Beltrami differential because of the hidden \( \mu \) dependence in the \( C \) ghost fields and their diffeomorphism variation \( \delta C \). However, if one wants to directly check that the anomaly (5.53) (which explicitly depends on \( \mu \)) is a \( \delta \)-cocycle then the use of the integrability condition is explicitly required.

Once more, we emphasize the role of the holomorphic sector and the locality principle which is at the origin of a holomorphically split diffeomorphism anomalies in the vacuum sector. The holomorphic property is obtained at the price of restricting to integrable complex structures. Contrary to the anomaly given in [1] for the so-called “chiral diffeomorphisms”, the anomaly computed in the present work is more general since we have considered the whole invariance under reparametrizations. However, only the local part of the anomaly has been identified, i.e. the “universal” or ultra-violet contribution independent from the matter fields. The construction was inspired by the bidimensional case and relies on the use of an integrating factor which is non local in the Beltrami differential. While in the bidimensional situation, the conformal (Weyl) anomaly is equivalent to the well-known holomorphically factorized anomaly [4], in higher dimensions, we do not know to which diffeomorphic invariants the factorized anomalies are equivalent; this amounts to finding a globally defined version of the anomalies. Moreover, in the presence of matter fields, one recovers the usual trace anomaly.

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Finally, a further careful analysis based on the local index theorem of Bismut-Gillet and Soulé as pioneered in [5, 6] for the bidimensional case is needed. An attempt in that direction has been made in [1].

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Appendix A Diffeomorphism invariance of the integrability condition

In this appendix, the diffeomorphism invariance of the integrability condition on $\mu$ is proved. With $\mu^\varphi$ defined by eq.(3.2) of the main text, let us compute directly the integrability condition and similarly to the first order differential operator $L$ defined by eq.(2.5), one set $L^\varphi = \overline{\partial} - \mu^\varphi$. The $(0,2)$-form with $(1,0)$-vector values reads

$$
\overline{\partial} \mu^\varphi - (\mu^\varphi)^2 = L^\varphi \mu^\varphi = -L^\varphi(\varphi \cdot \varphi) \cdot \left( \partial \varphi + \partial \varphi \cdot (\mu \circ \varphi) \right)^{-1} \cdot \partial
$$

where the third equality is obtained by using $\partial = \partial \varphi \cdot \partial + \partial \varphi \cdot \partial \varphi$ and $\overline{\partial} = \overline{\partial} \varphi \cdot \partial + \overline{\partial} \varphi \cdot \partial \varphi$, while the fourth one comes from the identity $L^\varphi \varphi = -(L^\varphi \varphi \cdot (\mu \circ \varphi))$. Now, the integrability condition for $\mu$ writes

$$
\overline{\partial} \mu - \mu^2 = 0 \iff L \mu^b_2 = L \mu^b_2 ,
$$

and shows that $L \mu$ is a mixed tensor, symmetric in the lower indices. This symmetry implies the vanishing of the last expression (A.1),

$$
\overline{\partial} \mu - \mu^2 = 0 \implies \overline{\partial} \mu^\varphi - (\mu^\varphi)^2 = 0 .
$$

Appendix B On the equivalence between complex structures

Any diffeomorphism $\varphi$ when expressed in terms of its local representative will induce a smooth change of local coordinates. This change of variables can be written in matrix notation (with matrix entries), as well the inverse change of variables, as

$$
\begin{pmatrix}
\partial \varphi \\
\overline{\partial} \varphi
\end{pmatrix}
= \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
$$

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with the following relations

\[
\begin{align*}
A \cdot \partial \varphi + B \cdot \partial \overline{\varphi} &= I_w, \\
A \cdot \partial \overline{\varphi} + B \cdot \partial \varphi &= 0, \\
C \cdot \partial \varphi + D \cdot \partial \overline{\varphi} &= 0, \\
C \cdot \partial \overline{\varphi} + D \cdot \partial \varphi &= I_w, \\
\partial \varphi \cdot A + \partial \overline{\varphi} \cdot C &= I_z, \\
\partial \varphi \cdot B + \partial \overline{\varphi} \cdot D &= 0, \\
\partial \overline{\varphi} \cdot A + \partial \varphi \cdot C &= 0, \\
\partial \overline{\varphi} \cdot B + \partial \varphi \cdot D &= I_z.
\end{align*}
\] (B.2)

Starting with the equation in the target chart with respect to \( \varphi \),

\[
\left( \partial_w - (\mu \circ \varphi) \cdot \partial_w \right) Z = 0,
\] (B.3)

and by using the two last identities of the right column of the above equations (B.2), one gets in the source chart of \( \varphi \),

\[
\left( \partial_z - (\partial \varphi + \partial \overline{\varphi} \cdot (\mu \circ \varphi)) \cdot (B \cdot \partial_z + A \cdot \partial Z) \right) (Z \circ \varphi) = 0,
\] (B.4)

and by eq. (3.1) of the main text combined with the two first identities of the right column of (B.2), one has

\[
\left( \partial_z - (\mu^\varphi) \cdot \partial Z \right) (Z \circ \varphi) = 0.
\] (B.5)

The terms involving \( \partial \overline{\varphi} \) turns out to be proportional to eq.(B.3) due to the laws for the change of variables, and thus the last parenthesis vanishes. It remains

\[
\left( \partial_z - (\mu^\varphi) \cdot \partial Z \right) (Z \circ \varphi) = 0,
\] (B.6)

which shows that \( (Z \circ \varphi) \) also provide \( n \) linearly independent solutions of the Beltrami equations on the source of the diffeomorphism \( \varphi \), see eq.(3.4) in the text which is here recalled

\[
\left( \partial_z - (\mu^\varphi) \cdot \partial Z \right) Z^\varphi = 0.
\] (B.7)

Thus the mapping \( (Z^\varphi, Z^{\overline{\varphi}}) \mapsto (Z, Z) \) is indeed bi-holomorphic, see Theorem 5.3 in [15]. In other words, this result states the equivalence between the complex structures parametrized by \( \mu \) and \( \mu^\varphi \).

**Appendix C  True holomorphic divergence**

In the following the computation of the divergence (5.10) is performed. In doing so, several results are required and are listed hereafter.

Firsts, the computation of the inverse matrix

\[
\begin{pmatrix}
I & \overline{\mu} \\
\mu & I
\end{pmatrix}^{-1} = \begin{pmatrix}
(I - \overline{\mu} \cdot \mu)^{-1} & -(I - \overline{\mu} \cdot \mu)^{-1} \cdot \overline{\mu} \\
-(I - \mu \cdot \overline{\mu})^{-1} \cdot \mu & (I - \mu \cdot \overline{\mu})^{-1}
\end{pmatrix},
\] (C.1)

yields in particular the following two identities

\[
(I - \overline{\mu} \cdot \mu)^{-1} - \overline{\mu} \cdot (I - \mu \cdot \overline{\mu})^{-1} \cdot \mu \equiv I, \\
(I - \mu \cdot \overline{\mu})^{-1} \cdot \overline{\mu} - \mu \cdot (I - \mu \cdot \overline{\mu})^{-1} \equiv 0.
\] (C.2)
Second, one will use the identities
\[
\partial Z^k \left( \frac{1}{f} \lambda^k \right) + \partial Z^l \left( \frac{1}{f} \mu^l \lambda^k \right) \equiv 0 , \quad \partial Z^k \left( \frac{1}{f} \mu^k \lambda^l \right) + \partial Z^l \left( \frac{1}{f} \lambda^l \right) \equiv 0 ,
\]
coming from the Jacobi multipliers.

We are now in the position for computing the divergences (5.10) of the main text. Taking into account (2.11) and (2.15) one can show,
\[
\left( I - \mu \cdot \mu \right) \Theta^l = \frac{1}{\det(\lambda)} \left( \lambda \right) = \frac{1}{\det(\lambda)} \left( \lambda^{-1} \right) \Theta^l = \frac{1}{\det(\lambda)} \left( \lambda^{-1} \right) \Theta^l = \frac{1}{\det(\lambda)} \left( \lambda^{-1} \right) \Theta^l
\]
which relates to the Ward id’s (5.8) by virtue of the non-singularity of \( \lambda \). It is straightforward to show that
\[
\partial Z^r \left( \frac{1}{f} \det(\lambda) \right) = 0 ,
\]
thanks to the identities (C.3) and (C.2).

**Appendix D The \( \hat{\delta} \) Cohomology**

The \( \hat{\delta} \) operator, given by formulæ (5.29) in the text, is defined on the space of local functions considered, according to the power counting, as differential polynomials in both the matter and the \( \Phi-I \) charged fields, and as analytic in the components of the Beltrami differential which are of zero dimension. The following set of fields
\[
\chi \equiv f_0 \cup f_1 \cup \beta \cup \{ \lambda, \lambda \}, \quad f_0 \equiv \{ \phi, \mu, \bar{\mu} \}, \quad f_1 \equiv \{ c, \bar{c} \}, \quad \beta \equiv \{ \gamma, \bar{\gamma}, \zeta, \bar{\zeta} \},
\]
will serve as generators for our space of local functions. Thanks to the definition of \( \hat{\delta} \), these local functions do not contain any underived ghosts \( c \)'s.

Let us recall that the fields and their derivatives will be considered as independent coordinates, and in practice, will play the role of creation operators, while the annihilators will be the formal derivatives with respect to these coordinates, both acting on the Fock space structure of the space of local functions. This Fock space is graded according to the ghost grading. For any operator, its adjoint will be given by the formal replacement of the derivative with respect to some coordinate by the formal multiplication with respect to the same coordinate and vice versa.

According to this rule, let us now introduce the following self-adjoint operator,
\[
\nu = \sum_{|I|, |J| \geq 0} \frac{(|I| + |J|)}{(|I| + |J| + 1)} \left( \sum_{f_1} \frac{\partial}{\partial \partial f_1} f_1(z, \bar{z}) \right) \quad \text{D.1}
\]
whose eigenvalues will provide the order of the derivatives of the ghost fields. The operator \( \nu \) will decompose the space of local functions into a direct sum of subspaces according to its eigenvalues while the operator \( \hat{\delta} \) will be filtered according to
\[
[\nu, \hat{\delta}] = \sum_{|I|, |J| \geq 0} \frac{(|I| + |J|)}{(|I| + |J| + 1)} \left( \sum_{|I|, |J| \geq 0} \hat{\delta} (|I| + |J|) \right), \quad \hat{\delta} = \sum_{|I|, |J| \geq 0} \hat{\delta} (|I| + |J|) .
\]

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The general spectral method [20, 12] insures that the \( \hat{\delta} \)-cohomology is isomorphic to the solutions \( \tilde{\Delta} \) of the system

\[
\begin{aligned}
\hat{\delta}(|I| + |\mathcal{J}|)\tilde{\Delta}(z, \overline{z}) &= 0 , \\
\hat{\delta}^{\dagger}(|I| + |\mathcal{J}|)\tilde{\Delta}(z, \overline{z}) &= 0 , \\
|I| + |\mathcal{J}| &\geq 0.
\end{aligned}
\] (D.3)

In other words, \( \tilde{\Delta} \) is in the kernel of the Laplacians \( \{\hat{\delta}(|I| + |\mathcal{J}|), \hat{\delta}^{\dagger}(|I| + |\mathcal{J}|)\} \).

In the following, we present in full details the solution of the cohomology by means of the spectral sequences. The zero eigenvalue of \( \nu \) will select the part of \( \hat{\delta} \) in which no ghost field is involved, namely

\[
\hat{\delta}(0) = \sum_{|I|, |\mathcal{J}| \geq 0} \left( \sum_{\beta \leq 1} < \partial_{I} \partial_{\mathcal{J}} \delta \Gamma_{0}^{(i)}, \frac{\partial}{\partial \partial_{I} \partial_{\mathcal{J}} \beta_{-1}}(z, \overline{z}) > \\
+ \sum_{\beta \geq 2} < \partial_{I} \partial_{\mathcal{J}} X(f_{0}, \beta_{1})(z, \overline{z}), \frac{\partial}{\partial \partial_{I} \partial_{\mathcal{J}} \beta_{-2}}(z, \overline{z}) > \right).
\] (D.4)

Note that the latter is a nilpotent operator and is an annihilator with respect to the external sources coupled to the BRS-variations. One will prove that the \( \hat{\delta} \)-cohomology does not actually depend on the external sources coupled to the BRS-variations by performing a double-filtration.

One first filtrates according to the counting operator

\[
\mathcal{N}(\beta) = \sum_{|I|, |\mathcal{J}| \geq 0} < \partial_{I} \partial_{\mathcal{J}} \beta(z, \overline{z}), \frac{\partial}{\partial \partial_{I} \partial_{\mathcal{J}} \beta}(z, \overline{z}) > ,
\] (D.5)

with respect to only one of the external sources \( \beta \equiv \{\gamma, \eta_{k}, \eta_{\hat{\alpha}}, \zeta_{l}, \zeta_{\overline{\alpha}}\} \). Next, the second step consists in a filtration with respect to the full counting operator

\[
\mathcal{N} = \sum_{\chi} \sum_{|I|, |\mathcal{J}| \geq 0} < \partial_{I} \partial_{\mathcal{J}} \chi(z, \overline{z}), \frac{\partial}{\partial \partial_{I} \partial_{\mathcal{J}} \chi}(z, \overline{z}) > .
\] (D.6)

By virtue of the isomorphism (D.3), the Laplacian corresponding to the given operator in the null subsector of the double filtration \( [\mathcal{N}, [\mathcal{N}(\beta), \hat{\delta}(0)]] \), shows that the \( \hat{\delta} \)-cohomology does not depend on any negatively ghost graded external sources \( \beta \), thus restricts the set of generators to \( \chi \equiv f \cup \{\lambda, \overline{\lambda}\}, f \equiv f_{0} \cup f_{1} \).

The second operator \( \hat{\delta}(1) \) in the filtration (D.2) will necessarily contain the ghost part, \( \hat{\delta}_{\text{gh}} \) of \( \hat{\delta} \) since \([\nu, \hat{\delta}_{\text{gh}}] = \hat{\delta}_{\text{gh}} \). The latter will play a central role in the computation of the cohomology. For both this reason and the sake of compactness in the notation it is written as

\[
\hat{\delta}_{\text{gh}} = \sum_{|I| \geq 0} \delta_{I}^{M+N} \frac{(I + 1)!!}{(M + 1)!!} (\partial_{I} \partial_{\mathcal{J}} \chi(0, \overline{z}), \overline{z}) \frac{\partial}{\partial \partial_{I} \partial_{\mathcal{J}} \chi(0, \overline{z})},
\] (D.7)

\[
\alpha, \beta, \rho = 1, \ldots, n, \ldots, 2n, \quad \alpha \equiv \begin{cases} i, & \text{for } \alpha = 1, \ldots, n, \\
\bar{i}, & \text{for } \alpha = n + 1, \ldots, 2n; \end{cases}
\]

\[
I = (i_{1}, \ldots, i_{2n}), \quad i_{\alpha} = \begin{cases} i_{k}, & \text{for } \alpha = k = 1, \ldots, n; \\
\bar{i}_{l}, & \text{for } \alpha = n + 1, \ldots, 2n, \quad l = \alpha - n \end{cases}
\]

\[
e^{\alpha} = \begin{cases} e^{i}, & \text{for } \alpha = 1, \ldots, n; \\
\overline{e^{i}}, & \text{for } \alpha = n + 1, \ldots, 2n. \end{cases}
\]
where, according to the ghost grading, one can make the following decomposition for the operator $U_β^α = (U_0)^β_α + (U_1)^β_α$, where $(U_0)^β_α$ is given by the Lie derivatives of the fields described as components of geometrical objects and $(U_1)^β_α$ comes from $\hat{\delta}_{Gh}$. For instance, in the case of matter fields $U_j^i \equiv S_j^i$ one has

\[
(S_0)_j^i = \sum_{|M|, |N| \geq 0} (m_i + 1) \sum_{\chi \equiv f_0 \cup (\alpha, \lambda)} < \partial_{M+1, j} \partial_N \chi(z, \bar{z}), \frac{\partial}{\partial \partial_{M+1}, \partial_N \chi(z, \bar{z})} > \\
+ \sum_{A_{n-1}} \delta^A_{n-1} \partial_M \partial_N \phi_{A_{n-1}}(z, \bar{z}), \frac{\partial}{\partial \partial_M \partial_N \phi_{A_{n-1}}(z, \bar{z})} > \\
+ \partial_M \partial_N \lambda_j^i(z, \bar{z}) \frac{\partial}{\partial \partial_M \partial_N \lambda_j^i(z, \bar{z})} + \partial_M \partial_N \mu_j^i(z, \bar{z}) \frac{\partial}{\partial \partial_M \partial_N \mu_j^i(z, \bar{z})}
\]

(D.9)

\[
(S_1)_j^i = \sum_{|M| \geq 0} (m_i + 1) \partial_{M+1, j} c^α(z, \bar{z}) \frac{\partial}{\partial \partial_{M+1}, c^α(z, \bar{z})} \\
- \sum_{|M| \geq 1} \partial_M c^i(z, \bar{z}) \frac{\partial}{\partial \partial_M c^i(z, \bar{z})}
\]

(D.10)

Note in addition that the trace $S^i_j = \text{tr}S$ is nothing else but the “little z” indices counting operator

\[
\text{tr}S = N_\partial + N_\lambda + N_\pi + N_\phi - N_\mu - N_c \equiv N_z(\downarrow) - N_z(\uparrow)
\]

(D.11)

One has $S^i_j = S^\sim_j \equiv \bar{S}_j^i$ and $\bar{S} \equiv N_\pi(\downarrow) - N_\pi(\uparrow)$ with $S^\sim = S^\sim$. In full generality, one has for the ghost contribution

\[
(U_1)^β_α = \sum_{|I| \geq 0} (i_α + 1) \partial_{I+1, β} c^β(z, \bar{z}) \frac{\partial}{\partial \partial_{I+1, α} c^β(z, \bar{z})} - \partial_{I+1, α} c^α \frac{\partial}{\partial \partial_{I+1, α} c^β(z, \bar{z})}
\]

(D.12)

and the remaining piece $R(1)$ concerns only the ghost derivatives of order greater or equal to three,

\[
R(1) = \sum_{|M| \geq 0} \partial_{M+1, α+1, β} c^β(z, \bar{z}) R_{µ}^{M+1, α+1, β}(z, \bar{z})
\]

(D.13)

\[
R_{µ}^{M+1, α+1, β}(z, \bar{z}) = \sum_{|N| \geq 0} \frac{(M+1, α+1, β+1)}{(M+1, α)!(N+1, β)!} \partial_N c^α(\bar{z}, z) \frac{\partial}{\partial \partial_M N+1, α+1, β+1, c^β(z, \bar{z})}
\]

Furthermore, the nilpotency, $\hat{\delta}(1)^2 = 0$, allows one to apply once more the spectral sequence analysis to the operator $\hat{\delta}(1)$. The filtration will be performed with respect to the following counting operator

\[
\nu' = 1 + \partial_α c^β(z, \bar{z}) \frac{\partial}{\partial \partial_α c^β(z, \bar{z})}
\]
Remarkably, the cohomology reduces to the following finite filtration

\[
\left[ \nu', \hat{\delta} (1) \right] = \sum_{a=1}^{2} a \hat{\delta}'(a),
\]
where

\[
\begin{align*}
\hat{\delta}'(1) &= R(1), \\
\hat{\delta}'(2) &= \partial_{\alpha} c^{\beta}(z, \bar{z}) U^{a}_{\beta},
\end{align*}
\]

and \( \hat{\delta} (1) = \hat{\delta}'(1) + \hat{\delta}'(2) \). According to (D.3), one has to solve the system

\[
\begin{align*}
\hat{\delta}'(a) \tilde{\Delta}(z, \bar{z}) &= 0 \\
\hat{\delta}''(a) \tilde{\Delta}(z, \bar{z}) &= 0,
\end{align*}
\]
for \( a = 1, 2 \),

which turns out to be equivalent to

\[
< \tilde{\Delta}(z, \bar{z}) | \left\{ \hat{\delta}'(a), \hat{\delta}'(a) \right\} | \tilde{\Delta}(z, \bar{z}) > = ||\hat{\delta} (a) \tilde{\Delta}(z, \bar{z})||^{2} + ||\hat{\delta}'(a) \tilde{\Delta}(z, \bar{z})||^{2} = 0,
\]

where the scalar product is the one induced by the definition of the adjoint operator in the space of local functions. For \( a = 1 \), the lowest order of formal derivatives with respect to the ghost derivatives in the Laplacian involved in eq.(D.17) can be seen of order three. So, the cocycle \( \tilde{\Delta} \) depends on the first and second order derivatives of the ghost fields. The second filtration \( a = 2 \), selects the scalar sector, and eliminates the second order ghost derivatives. It is then useful to perform the following change of variables in the first order derivatives of the ghost fields. Define, in matrix notation, the \( n^{2} \) variables of ghost grading one,

\[
\tilde{\Omega}(z, \bar{z}) = (\partial c + \partial c^{\gamma} \cdot \mu)(z, \bar{z}) = (\hat{\delta} \lambda \cdot \lambda^{-1})(z, \bar{z}),
\]

with \( \text{tr} \tilde{\Omega} = 0 \), and note the \( \hat{\delta} \)-coboundary

\[
\text{tr} \tilde{\Omega}(z, \bar{z}) = \hat{\delta} \ln \det \lambda(z, \bar{z}).
\]

One has \( \hat{\delta} \tilde{\Omega}(z, \bar{z}) = \hat{\delta}'(2) \tilde{\Omega}(z, \bar{z}) = \tilde{\Omega}^{2}(z, \bar{z}) \), and

\[
\hat{\delta} \tilde{\Omega}^{2k+1}(z, \bar{z}) = \tilde{\Omega}^{2k+2}(z, \bar{z}),
\]

showing that the \( \tilde{\Omega}^{2k+2} \) are \( \hat{\delta} \)-coboundaries. Taking the trace in order to project onto the scalar sector, one readily checks that

\[
\hat{\delta} \text{tr} \tilde{\Omega}^{2k+1}(z, \bar{z}) = \text{tr} \tilde{\Omega}^{2k+2}(z, \bar{z}) = -\text{tr} \tilde{\Omega}^{2k+2}(z, \bar{z}) = 0.
\]

This yields the following theorem which is the main result of this appendix.

**Theorem.** In the scalar sector of the space of analytic functions in the fields and their derivatives, the ghost sectors with grading \( p \) of the \( \hat{\delta} \)-cohomology are generated by the following nontrivial cocycles

\[
\text{for even } p : \quad \text{tr} \tilde{\Omega}^{2r+1}(z, \bar{z}) \text{tr} \tilde{\Omega}^{2s+1}(z, \bar{z}),
\]

\[
\text{for odd } p : \quad \text{tr} \tilde{\Omega}^{2k+1}(z, \bar{z}),
\]

while the cocycle \( \text{tr} \tilde{\Omega} \) can be reabsorbed by completing the set of generators with \( \ln \det \lambda \) seen as an independent variable. The same statements hold true for the complex conjugate expressions.
References


