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A GENERAL PROOF OF THE POMERANCHUK THEOREM AND RELATED THEOREMS

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A B S T R A C T

It is shown that under general analyticity properties which have been recently demonstrated from the L.S.Z. formalism for all forward scattering amplitudes of stable particles by Bros, Epstein and Glaser, one can prove the Pomeranchuk theorem on total cross-sections if one makes the following additional assumptions :

- i) Inside the analyticity domain, the forward scattering amplitude is bounded by $C(\epsilon) \exp \epsilon |s|$ for arbitrarily small ϵ , where $|s|$ is the square of the c.m. energy in one channel.
- ii) For physical energies, $s \rightarrow \pm\infty$, $|F(s,0)|/|s| \log |s| \rightarrow 0$.
- iii) The difference between particle-particle and particle-anti-particle cross-sections has a limit.

If any of these three assumptions is released, counter-examples can be found. For the real parts of the amplitudes, similar results can be found. Extensions to the moduli of the amplitudes are also considered. It is shown that if $|F(s,0)|$ is bounded by s^N and if $\lim |F(s,0)/F(-s,0)|$ exists, it is unity. A similar result holds for fixed transfer if one makes the additional assumption that $\text{Im} F(s,t)$ has a finite number of oscillations for $s > s_0$ and $s < -s_0$.

INTRODUCTION

There exist now many proofs of the Pomeranchuk theorem ^{1),2)} according to which the difference of the particle-particle and particle-antiparticle cross-sections $\sigma - \bar{\sigma}$ has a limit zero at infinite energies. It is satisfactory to see that in these proofs the degree of rigour and the number of assumptions are, respectively, an increasing and decreasing function of time. The latest and most satisfactory proof, by Meimann ²⁾, states that if the forward scattering amplitude is analytic in a cut plane (or quasi-cut plane) and bounded by $|s|$ (where s is the square of the c.m. energy in one channel) then the set of limiting values of $\sigma - \bar{\sigma}$ contains zero, so that if $\sigma - \bar{\sigma}$ has a limit, this limit is zero. No smoothness assumptions are needed.

The proof we present here was obtained by the author while preparing a review talk on the subject. It is, we believe, simpler than that of Meimann, and contains the slight improvement that the condition $|F| < |s|$ is replaced by $F/|s|\log|s| \rightarrow 0$. The advantage of this improvement is that then one reaches the limit where the Pomeranchuk theorem breaks down. Namely, if $F/s \log s$ has finite limits for $s \rightarrow +\infty$ and $s \rightarrow -\infty$, explicit counter-examples can be constructed, in which not only the difference of the cross-sections is non-zero, but also the ratio of these cross-sections differs from unity.

The method we present is trivially extended to the study of the real parts. Again if $|F(s,0)|/|s|\log|s| \rightarrow 0$, and if $(\operatorname{Re} F(s) + \operatorname{Re} F(-s))/s$ has a limit, this limit is zero.

It is slightly more delicate to extend these considerations to the ratios of the moduli $|F(s,0)/F(-s,0)|$ if one does not want to assume anything on the real and imaginary parts separately. Here the positivity properties due to unitarity play a major role. However, the final result holds with the still less restrictive assumption that $|F(s,0)|$ is bounded by $|s|^N$. Then if the limit of $|F(s,0)/F(-s,0)|$ exists

(including zero and infinity), this limit is unity. This latter result can be extended to fixed transfer $t < 0$ if one adds the assumption that $\text{Im} F(s, t)$ does not oscillate infinitely many times for large energies. This is true also for the fixed "crossed transfer" amplitude.

In all these considerations the analyticity domain we shall use follows rigorously from axiomatic field theory according to Bros, Epstein and Glaser³⁾. In particular, we have the desired analyticity domain even in cases where dispersion relations have not been proved, for instance pp scattering in the forward direction, or πp scattering for large negative transfers, or nearly backward πp scattering, which has been recently considered by Białawski and Czyzewski⁴⁾, and is connected with the reaction $p\bar{p} \rightarrow \pi\pi$.

Section I contains a description of the analyticity domain obtained by Bros, Epstein and Glaser and its connection with the physical amplitude. Section II contains the proof of the Pomeranchuk theorem for cross-sections and its extension to the case of the real parts. In Section III we study the behaviour of the moduli of the amplitudes.

I. THE ANALYTICITY DOMAIN

We consider the reactions :

$$(I) \quad A + B \rightarrow A' + B'$$

$$(II) \quad A + \bar{B} \rightarrow A' + \bar{B}'$$

$$(III) \quad A + \bar{A} \rightarrow B + \bar{B}$$

where A and B are stable particles.

We shall consider first only reactions (I) and (II).

The scattering amplitude for reaction (I), $F(s, t)$ where $s = (\sqrt{M_A^2 + k^2} + \sqrt{M_B^2 + k^2})^2$ is the square of the centre-of-mass energy and $t = -2k^2(1 - \cos\theta)$, $t \leq 0$, is the square of the momentum transfer from A to A', is according to Bros, Epstein and Glaser ³⁾ the boundary value $\lim_{\epsilon \rightarrow 0} F(s + i\epsilon, t)$ of an analytic function of s , analytic in the upper half s plane minus a finite region. The physical region for fixed negative or zero t , i.e.,

$$\left(\sqrt{M_A^2 + \frac{|t|}{4}} + \sqrt{M_B^2 + \frac{|t|}{4}} \right)^2 < s < +\infty$$

lies entirely on the boundary of the domain. The analytic continuation of $F(s, t)$ to negative s is such that

$$\lim_{\epsilon \rightarrow 0} F(s + i\epsilon, t)$$

coincides with the complex conjugate of the scattering amplitude for reaction (II) with a centre-of-mass energy u given by :

$$u = 2M_A^2 + 2M_B^2 - t - s \quad (1)$$

and a transfer t from A to A'. Again the region

$$\left(\sqrt{M_A^2 + \frac{|t|}{4}} + \sqrt{M_B^2 + \frac{|t|}{4}} \right)^2 < u < +\infty$$

which corresponds to :

$$-\infty < s < 2M_A^2 + 2M_B^2 - t - \left(\sqrt{M_A^2 + \frac{|t|}{4}} + \sqrt{M_B^2 + \frac{|t|}{4}} \right)^2$$

lies entirely on the boundary of the analyticity domain. The domain is represented in Fig. 1. One can, of course, also define an analytic function in a symmetric region $F^*(s^*, t)$. In favourable cases, where

dispersion relations have been proved for instance, these two functions coincide on a finite segment of the real axis (Fig.2). This occurs also in some cases where dispersion relations have not been established, for instance for forward pp scattering in the neighbourhood of the pp threshold ⁵⁾ (Fig.3). However, anyway, Bros, Glaser and Epstein have shown that there is a complex path in the s, t complex space along which one can perform an analytic continuation from the upper half plane to the lower half plane.

For fixed $u < 0$ a similar analyticity domain can be obtained, but then one connects the physical region for reaction (I) and the physical region for reaction (III). It is important to notice that if $M_A \neq M_B$, u should be taken strictly negative, to ensure that physical points of both reactions are on the boundary of the analyticity domain in s .

Finally, let us remind an important property of the forward scattering amplitude ($t=0$): for $s > (M_A + M_B)^2$ [physical region (I)] $\text{Im} F(s+i\epsilon, 0)$ is positive or zero, as a consequence of unitarity (in fact, with stronger analyticity assumptions such as the Mandelstam representation, one can show that it never vanishes). For $s < (M_A - M_B)^2$ [physical region (II)], $\text{Im} F(s+i\epsilon, 0)$ is negative or zero.

II. PROOF OF THE POMERANCHUK THEOREM

Since we are dealing here with the forward scattering amplitude, it is more convenient to use as a variable the laboratory energy E of particle B, particle A being at rest. E is in fact proportional to s for $s \rightarrow \infty$:

$$s = M_A^2 + M_B^2 + 2 M_A E \quad (2)$$

The advantage of the variable E is that $s(E) = u(-E)$, i.e., equal magnitudes of $|E|$ correspond to equal centre-of-mass energies in channel (I) and channel (II). Then, if we introduce the laboratory momentum $q = \sqrt{E^2 - M_B^2}$, we have

$$\begin{aligned} \operatorname{Im} F(E + i\varepsilon) &= \frac{q}{4\pi} \sigma(E) & E > M_B \\ \operatorname{Im} F(E + i\varepsilon) &= -\frac{q}{4\pi} \bar{\sigma}(|E|) & E < -M_B \end{aligned} \quad (3)$$

where σ and $\bar{\sigma}$ are the total cross-sections associated with reactions (I) and (II).

We now make another change of variables and introduce the variable $z = E^2$.

The function $F(z)$ is now analytic in the region indicated on Fig. 4, which is a cut plane with a single cut from $z = M_B^2$ to $z = \infty$ minus a finite region of the z plane which certainly lies entirely inside a circle C centered at $z = 0$ with a sufficiently big radius. Then consider the function

$$\phi(z) = 4\pi \frac{F(z) - F^*(z^*)}{\sqrt{z - M_B^2}} \quad (4)$$

where $\sqrt{z} = E$ for $E > 0$. $\phi(z)$ is analytic in the same domain and is purely real for z real < 0 .

The discontinuity of $\phi(z)$ across the cut is given according to (3) by :

$$\operatorname{Im} \phi(z + i\varepsilon) = \sigma(z) - \bar{\sigma}(z) \quad (5)$$

6.

Now come the assumptions :

$$i) \quad F(s) < C(\varepsilon) \exp \varepsilon |s| \quad (6)$$

$$ii) \quad \lim_{s \rightarrow \pm\infty} \frac{|F(s)|}{|s| \log |s|} = 0 \quad (7)$$

These two assumptions, according to the Phragmén-Lindelöf theorem, imply that in all directions

$$\lim_{|z| \rightarrow \infty} \frac{|\phi(z)|}{\log |z|} = 0 \quad (8)$$

Therefore it is allowed to write for $\phi(z)$ a once subtracted dispersion relation :

$$\phi(z) = \phi(z_0) + \frac{z-z_0}{2i\pi} \int_{\Gamma} \frac{\phi(z') dz'}{(z'-z)(z'-z_0)} + \frac{z-z_0}{\pi} \int_{z_1}^{\infty} \frac{\text{Im} \phi(z')}{(z'-z)(z'-z_0)} dz' \quad (9)$$

where Γ surrounds the finite singularity region and z_0 is taken sufficiently negative, inside the analyticity domain.

Then comes assumption :

$$iii) \quad \lim_{z \rightarrow \infty} \sigma(z) - \overline{\sigma}(z) \quad (10)$$

exists.

Suppose now that this limit differs from zero. Suppose, for instance, that it is $C > 0$. Then by definition of the limit, given ε one can find z_2 such that for $z > z_2$, $\text{Im} \phi(z) > C - \varepsilon$. We shall take $\varepsilon < C$. Then we have

$$\begin{aligned}
\phi(z) - \phi(z_0) &+ \frac{z-z_0}{2i\pi} \int_{\gamma} \frac{\phi(z') dz'}{(z'-z)(z'-z_0)} + \frac{z-z_0}{\pi} \int_{z_1}^{z_2} \frac{\operatorname{Im} \phi(z') dz'}{(z'-z)(z'-z_0)} \\
&= \frac{z-z_0}{\pi} \int_{z_2}^{\infty} \frac{\operatorname{Im} \phi(z') dz'}{(z'-z)(z'-z_0)} \quad (11)
\end{aligned}$$

Now let us study the behaviour of both sides of equation (11) for $z \rightarrow -\infty$. The right-hand side is larger in modulus than

$$\frac{|z-z_0|}{\pi} \int_{z_2}^{\infty} \frac{(C-\varepsilon) dz'}{(z'-z)(z'-z_0)} = (C-\varepsilon) \log \left| \frac{z_2-z}{z_2-z_0} \right|$$

In the left-hand side, two integrals over a finite integration region appear. In these integrals one can take the limit $z \rightarrow \infty$ under the integral; therefore the left-hand side behaves like $\phi(z) + \text{const.}$, i.e., goes to infinity less fast than $\log z$ according to assumptions i) and ii). We have therefore a contradiction unless $C = 0$.

The same argument can be carried through if C is negative. Therefore under assumptions i), ii) and iii), $\lim \sigma(z) - \bar{\sigma}(z) = 0$. Notice that the case $C = \infty$ is also excluded by this type of argument.

It is perhaps worth while to present a more general derivation, which is maybe more adapted to the experimental situation. Assume that for $E > E_A$ one knows that

$$C - \Delta < \sigma - \bar{\sigma} < C + \Delta$$

where Δ represents the experimental uncertainty. Then the Pomeranchuk theorem states that the closed interval $C - \Delta$ $C + \Delta$ contains zero. If it did not, the same argument could be carried through and would lead to a contradiction.

We want to show now that if one abandons conditions i), ii) or iii), counter-examples can be found. To construct a counter-example to i), satisfying ii) and iii), it is enough to remark that one can write $F(E)$ as $i q \phi(E)$ and take $\phi(E)$ to be a real entire function. Now, with $\phi(E)$, one can approach uniformly any given continuous function from $E = -\infty$ to $E = +\infty$ ⁶⁾.

Concerning condition ii), one can be contented with the example :

$$F(E) = A E \log(M_B - E) + B E \log(M_B + E) \quad (12)$$

which is such that $|\sigma - \bar{\sigma}(\infty)| = |A - B| 4\pi^2$, and $\sigma/\bar{\sigma} = A/B$.

If one does not like to have logarithmic branch points at $E = M_B$ and $E = -M_B$, one can always push them away in the "second" sheet, replacing $M_B - E$ by $(1 + \sqrt{M_B - E})^2$ and $E + M_B$ by $(1 + \sqrt{E + M_B})^2$. As for condition iii), we can take

$$F(E) = \sqrt{M_B^2 - E^2} \left(1 + \alpha \exp - \sqrt{M_B - E} \right) \quad (13)$$

which has a limit for $E \rightarrow -\infty$ but no limit for $E \rightarrow +\infty$. By taking α small enough, one can still manage to fulfil the requirements on the sign of $\text{Im } F(E)$.

An analogous result can be obtained for the real part of the scattering amplitude : one maintains conditions i) and ii) and one replaces iii) by :

$$\text{iii')} \quad \lim_{E \rightarrow \infty} \frac{\text{Re } F(E) + \text{Re } F(-E)}{E} \quad (14)$$

exists.

Then :
$$\lim_{E \rightarrow \infty} \frac{R_e F(E) + R_e F(-E)}{E} = 0 \quad (15)$$

All one has to do is to replace $\phi(z)$ by

$$\psi(z) = \frac{F(z) + F^*(z^*)}{i\sqrt{z}}$$

in the argument.

III. THE LIMIT OF THE RATIO OF TWO CROSSED AMPLITUDES

We want to investigate the behaviour of

$$\left| \frac{F(s, t)}{F(-s, t)} \right| \quad \text{and also} \quad \left| \frac{F(s, \Sigma M^2 - s, u)}{F(-s, \Sigma M^2 + s, u)} \right|$$

respectively for fixed $t \leq 0$ and fixed $u < 0$. It is enough to consider the first case (incidentally, it is as easy to study $|F(s, t)/F(-s, t)|$).

What we shall prove is the following. If $|F(s, t)|$ is bounded by $|s|^N$ for $s \rightarrow \pm\infty$ and by $O(\varepsilon) \exp \varepsilon |s|$ for all ε for complex s , if $\text{Im} F(s, t)$ has a constant sign for $s > s_0$ and $s < -s_0$, and if $|F(s, t)/F(-s, t)|$ has a limit for $s \rightarrow \infty$, then this limit is unity. The theorem excludes, in particular, the case when $\lim |F(s, t)/F(-s, t)| = 0$ or infinity.

In the special case of the forward scattering amplitude, the assumption about the sign of $\text{Im} F(s, t)$ is hardly necessary, because by unitarity $\text{Im} F(s + i\varepsilon, 0) \geq 0$ for $s > (M_A + M_B)^2$ and ≤ 0 for $s < (M_A - M_B)^2$.

We already mentioned that if it vanishes, this is in contradiction with stronger analyticity assumptions such as Mandelstam representation. It must be realized that our assumptions are much weaker than the now familiar assumption ⁷⁾ that separately $F(s,t)/\phi(s)$ and $F(-s,t)/\phi'(s)$ have a limit for $s \rightarrow \infty$ where $\phi(s)$ and $\phi'(s)$ are the so-called "admissible functions".

First we want to show that in the intersection of the half plane $\text{Im } s > 0$ and of the circle $|s| \leq s_1$ the phase of $F(s,t)$ varies in a finite interval and the number of zeros of $F(s,t)$ is finite.

From $s = s_1$ to $s = \infty$ the phase of $F(s,t)$ stays between 0 and π (modulo 2π), from $s = -\infty$ to $s = -s_1$ it stays between 0 and $-\pi$. In the complex neighbourhood of $s = s_1$ and $s = -s_1$, $\text{Im } F$ does not vanish if $\lim_{\epsilon \rightarrow 0} \text{Im } F(s+i\epsilon, t)$ exists and is, as we assumed, non-zero. Then, since the half circle $|s| = s_1$, $\text{Im } s > 0$ lies inside the analyticity domain (except for the extremities which have been taken care of), then along this half circle $\text{Im } F$ has a finite number of changes of sign. Therefore, all along the border of the domain $|s| \geq s_1$, $\text{Im } s \geq 0$, the phase of $|F(s,t)|$ is bounded. One can then map this region on a half plane. It is clear that the mapping variable $s'(s)$ will be such that $\lim_{|s| \rightarrow \infty} \frac{s'(s)}{s} = 1$. Therefore we obtain a function which is analytic in a half plane bounded by $|s'|^N$, and has a continuous imaginary part with a finite number of changes of sign along $\text{Im } s' > 0$. According to Jin and Martin ⁸⁾, such a function has a finite number of zeros in $\text{Im } s' \geq 0$ and according to Jin and MacDowell ⁹⁾, it admits a phase representation. The phase of F is not only bounded on $\text{Im } s' = 0$ but also in the whole region $\text{Im } s' \geq 0$.

We shall not need to work with s' and shall use directly s . The first remark is that we can now factor out the zeros of $F(s,t)$. So

$$\frac{F(s,t)}{F^*(-s^*,t)} = \frac{G(s,t)}{G^*(-s^*,t)} \frac{\prod_1^N (s-s_i)}{\prod_1^N (-s-s_i^*)} \quad (16)$$

and so if
$$\lim_{s \rightarrow \infty} \left| \frac{F(s+i\varepsilon, t)}{F(-s+i\varepsilon, t)} \right|$$

exists, including zero and infinity, we have

$$\lim_{s \rightarrow \infty} \left| \frac{F(s+i\varepsilon, t)}{F(-s+i\varepsilon, t)} \right| = \lim_{s \rightarrow \infty} \left| \frac{G(s+i\varepsilon, t)}{G(-s+i\varepsilon, t)} \right| \quad (17)$$

We shall, from now on, work with the function G which has no zeros in the region $|s| \gg s_1, \operatorname{Im} s \gg 0$. The function G has also, obviously, a bounded phase in this region. In addition, in complex directions, its modulus cannot grow faster than s^N or decrease faster than s^{-P} ⁸⁾, where P is some integer which can be known.

Now we have that $\log G(s, t)$ is analytic in $|s| \gg s_1, \operatorname{Im} s > 0$ and from the polynomial boundedness condition $\log |G(s, t)| < C \log |s|$ for real and complex s , and

$$|\log |G(s, t)|| < C \log |s| \quad (18)$$

for $\varepsilon < \operatorname{Arg} s < \pi - \varepsilon$. So the integral

$$\int_{-\infty}^{+\infty} \frac{\log^+ |G(s, t)| ds}{1+s^2} \quad (19)$$

converges, where $\log^+ |G|$ stands for $\log |G|$ for $|G| > 1$ and zero otherwise. Now, according to Boas ¹⁰⁾, this implies that also the integral

$$\int_{-\infty}^{+\infty} \frac{|\log |G(s, t)|| ds}{1+s^2} \quad (20)$$

converges.

Let us now define, in analogy with Section II,

$$\chi(s) = i \log \frac{G(s, t)}{G^*(-s^*, t)} \quad (21)$$

Notice that $\chi(s)$ is purely real along $\text{Re } s = 0$. It is therefore convenient to introduce the variable $z = s^2$. Then, in complete analogy with Eq. (9), we can write

$$\chi(z) = \chi(z_0) + \frac{z - z_0}{2\pi i} \int_{\Gamma} \frac{\chi(z') dz'}{(z' - z)(z' - z_0)} + \frac{z - z_0}{\pi} \int_{z_1}^{\infty} \frac{\text{Im } \chi(z') dz'}{(z' - z)(z' - z_0)} \quad (22)$$

where
$$\text{Im } \chi(z) = \log \left| \frac{G(s, t)}{G(-s, t)} \right|$$

This representation is valid because, according to (18), $\text{Im} |\chi(z)|$ does not grow faster than $\log z$ in complex directions, $\text{Re } \chi(z)$ is bounded, and finally the convergence of integrals (19) and (20) guarantees the convergence of the last integral in the representation.

Then we make the remark that since for $z < 0$, $\chi(z)$ is purely real, it is bounded by a constant. On the other hand, it is easy to see, by the same argument as in Section II, that if $\lim_{z' \rightarrow +\infty} \text{Im } \chi(z')$ exists and differs from zero, or if $\text{Im } \chi(z') \rightarrow +\infty$ or $-\infty$, the right-hand side of (22) grows at least like $\log |z|$ when $z \rightarrow -\infty$. Therefore if

$$\lim_{s \rightarrow +\infty} \log \left| \frac{G(s, t)}{G(-s, t)} \right|$$

exists (including infinity), it has to be zero. Hence if

$$\lim_{s \rightarrow \infty} \left| \frac{F(s, t)}{F(-s, t)} \right|$$

exists (including 0 and ∞), it has to be unity.

As we said, for $t = 0$ the assumption on $\text{Im} F(s, 0)$ is almost automatically satisfied.

It is clear that the same argument can be applied to the fixed u amplitudes. Our result strengthens in this way the result obtained recently by Białas and Czyzewski⁴⁾, which can be stated as follows for the case of pion-nucleon scattering: if the imaginary part of the fixed u pion-nucleon scattering amplitude has a definite sign for sufficiently large positive and negative energies, and if the ratio of the fixed u (i.e., almost backward) pion-nucleon cross-section and fixed u nucleon-antinucleon annihilation into two pions (i.e., almost forward or almost backward, according to the charge of the pion) has a limit or tends to infinity, this limit - apart from a statistical weight factor - is unity.

IV. CONCLUDING REMARKS

We have tried to present here a proof of the Pomeranchuk theorem and related theorems which uses as few assumptions as possible and we have shown that in the case of the Pomeranchuk theorem our assumptions are minimal. We have tried to avoid as much as possible smoothness assumptions because in the present state of physics, even though this smoothness property of high energies seems to be present, it has not been so far justified in any way. We have tried to avoid assumptions on the existence of limits, except when they were necessary. For instance, the Pomeranchuk theorem is true if $\sigma - \bar{\sigma}$ has a limit, but σ may have no limit, $\bar{\sigma}$ may have no limit and the real part of the amplitude can

have any behaviour one likes. Also the difference $\sigma - \bar{\sigma}$ can have a completely arbitrary derivative with respect to energy. For instance, if

$$\sigma - \bar{\sigma} = C + \frac{\sin(\exp s)}{\log s}$$

one can still prove that $C = 0$. We feel that this latter point is important because if it is possible to get reasonable limits on $\sigma(s) - \bar{\sigma}(s)$ from experiment, it is, on the other hand, extremely difficult to put limits on its derivative. Experiment will never exclude an oscillatory structure between the limits of error.

We have also carefully avoided to assume dispersion relations which may well be true but which, so far, have not yet been deduced from axiomatic field theory. In doing this, we want to indicate that the verification of asymptotic theorems does not constitute in any way a proof that full dispersion relations hold.

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R E F E R E N C E S

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FIGURE CAPTIONS

Figure 1 : The general analyticity domain, according to Bros, Epstein and Glaser, for fixed $t \leq 0$ or fixed $u < 0$ in the s or E variable. The shaded region may contain singularities or natural boundaries.

Figure 2 : The analyticity domain if dispersion relations hold.

Figure 3 : The analyticity domain for forward proton-proton scattering.

Figure 4 : The analyticity domain after mapping $\text{Im } s > 0$ with the variable $z = s^2$, and the integration contour.

FIG. 1

FIG. 2

FIG. 3

FIG. 4