CM-P00057107

ON THE SOLUTIONS OF PARTIAL WAVE DISPERSION RELATIONS

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A B S T R A C T

Two results are obtained on partial wave dispersion relations, the method used, in both cases, avoids the N/D separation and takes into account inelastic effects.

First, a criterion for the existence of bound states is obtained as a sum rule over the left-hand cut contribution. It is explicitly shown that inelasticity favours the formation of bound states.

Secondly, the one particle exchange model for the unphysical cut contribution is shown to be inconsistent with unitarity and threshold behaviour if the partial wave under consideration has an angular momentum larger than the spin of the exchanged particle. However, we remark that "approximate" solutions may satisfy both unitarity and threshold behaviour.

I. INTRODUCTION

We want to present two theorems on the solutions of partial wave dispersion relations. The first one is constructive. It is a simple criterion on the left-hand cut discontinuity for the presence or the absence of bound states, and as will be illustrated by a potential scattering example, this criterion seems to work rather well. The second one is negative. It shows that if the forces are due to a one-particle exchange the partial wave dispersion relations have no solution if the angular momentum ℓ is larger than the spin of the exchanged particle J . This just means that when the angular momentum is large the left-hand cut contribution cannot just be due to the exchange of one particle. This is not a new fact. "Empirically" it was noticed long ago (I acknowledge a conversation on this subject with Professor H.P. Noyes in 1961) that additional singularities on the left-hand cut were required to simulate the centrifugal barrier. Theoretically it was shown by Balachandran ¹⁾ that the contradiction appears for $J = 0$ $\ell \gg 2$. Here we show that the contradiction occurs already for $\ell = 1, J = 0$.

The common feature of the two theorems is that one avoids the N/D method and that one does not need to neglect inelasticity.

II. REMOVAL OF SUBTRACTIONS FROM THE DISPERSION RELATION

We take the simple case of equal mass (M) particles. The scattering amplitude in a given angular momentum state is analytic in the k^2 plane with a left-hand cut from $-x$ to $-a$ and a physical cut from $k^2 = 0$ to $k^2 = +\infty$. The normalization is such that $T(k^2)$, the scattering amplitude, satisfies the unitarity condition

$$\begin{aligned} \text{Im } T(k^2) &= \frac{k}{\sqrt{M^2 + k^2}} |T(k^2)|^2 && \text{in the elastic region} \\ \text{Im } T(k^2) &\geq \frac{k}{\sqrt{M^2 + k^2}} |T(k^2)|^2 && \text{in the elastic region} \end{aligned} \quad (1)$$

from which follows

$$|T(k^2)| < \frac{\sqrt{M^2 + k^2}}{k} \quad (2)$$

We define the left-hand cut contribution $T_0(k^2)$ such that $T(k^2) - T_0(k^2)$ has no left-hand cut. This does not yet fix uniquely T_0 because one could add a polynomial to T_0 . Now we make the assumption that there exists a T_0 such that

$$\lim_{k^2 \rightarrow \infty} |T_0(k^2)| = 0$$

Then if this is the case, this $T_0(k^2)$ is unique and can be represented as

$$T_0(k^2) = \frac{1}{\pi} \int_{-\infty}^{-a} \frac{\Delta T(k'^2) dk'^2}{k'^2 - k^2} \quad (3)$$

Now we can write the dispersion relation if no bound states are present as

$$T(k^2) = T_0(k^2) + \frac{k^2}{\pi} \int_0^{\infty} \frac{\text{Im} T(k'^2) dk'^2}{k'^2(k'^2 - k^2)} + P_n(k^2) \quad (4)$$

where $P_n(k^2)$ is a polynomial in k^2 . The integral over $\text{Im} T$ certainly converges at infinity by condition (2). It converges at the origin because if normal threshold behaviour is assumed

$$\lim_{k \rightarrow 0} \frac{\text{Im} T}{k}$$

is finite.

Let us show first that $P_n(k^2) \equiv \text{const.}$ $T - T_0$ is analytic in the cut plane with only one cut from $k^2 = 0$ to $k^2 = +\infty$. Along this cut T_0 goes to zero as $k^2 \rightarrow \infty$ and T is bounded by $\sqrt{M^2 + k^2}/k$. Therefore $T - T_0$ is bounded by 1 at infinity on both sides of the cut. Hence by the Phragmén-Lindelöf theorem $T - T_0$ is also bounded by 1 when $k^2 \rightarrow -\infty$. Now since

$$\int_0^{\infty} \frac{\text{Im} T(k'^2) dk'^2}{k'^2(k'^2 - k^2)}$$

converges and since $\text{Im} T$ is positive

$$\lim_{k^2 \rightarrow -\infty} \int_0^{\infty} \frac{\text{Im} T(k'^2) dk'^2}{k'^2(k'^2 - k^2)} = 0$$

Hence for $k^2 \rightarrow -\infty$ we have

$$P_n(k^2) = O(1) + o(k^2)$$

4.

Hence $P_n(k^2)$ is at most a constant and we can replace (4) by

$$T(k^2) = T_0(k^2) + \frac{k^2}{\pi} \int_0^\infty \frac{\text{Im} T(k'^2) dk'^2}{(k'^2 - k^2) k'^2} + C \quad (5)$$

Next consider $T - T_0 - C$. This function has only the right-hand cut as singularity, and for sufficiently large k^2 , along the cut, is bounded by a constant. Hence it is also bounded by a constant as $k^2 \rightarrow -\infty$. Now, for $k^2 = -\kappa^2$ $\kappa^2 > 0$ we have

$$\frac{\kappa^2}{\pi} \int_0^\infty \frac{\text{Im} T(k'^2) dk'^2}{k'^2 (k'^2 + \kappa^2)} > \frac{1}{2\pi} \int_0^\kappa \frac{\text{Im} T(k'^2) dk'^2}{k'^2}$$

So the right-hand side integral should be bounded and this implies, since $\text{Im} T$ is positive, that

$$\int_0^\infty \frac{\text{Im} T(k'^2) dk'^2}{k'^2}$$

converges. Therefore

i) we can write the right-hand cut contribution in non-subtracted form

$$T = T_0 + C + \frac{1}{\pi} \int_0^\infty \frac{\text{Im} T(k'^2) dk'^2}{k'^2 - k^2}$$

ii)

$$\lim_{k'^2 \rightarrow +\infty} \text{Im} T(k'^2) = 0$$

This, by the unitarity condition (1) implies

$$\lim_{k'^2 \rightarrow +\infty} |T(k'^2)| = 0$$

Now if we do not want to be too pedantic let us assume that $T(k'^2)$ is a "smooth" function, more precisely that

$$\lim_{k'^2 \rightarrow \infty} |T(k'^2)| = \lim_{k'^2 \rightarrow \infty} T(k'^2)$$

Then applying again the Phragmén-Lindelöf theorem we easily see that we are forced to take $C = 0$. Indeed we prove first that $T - T_0 \rightarrow 0$ for $k^2 \rightarrow -\infty$, secondly that

$$\lim_{k^2 \rightarrow -\infty} \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im} T(k'^2) dk'^2}{k'^2 - k^2} = 0$$

and hence we get $C = 0$.

In fact the smoothness condition is not necessary. In the Appendix we prove that if $T_0(k^2)$ decreases faster than $(\log k)^{-\frac{1}{2}-\epsilon}$ for $k^2 \rightarrow +\infty$ (a condition which is fulfilled if the left-hand cut is produced by the exchange of a spin 0 particle) then $C = 0$, even if T is not smooth.

The final form is

$$T = T_0 + \frac{1}{\pi} \int \frac{\text{Im} T(k'^2) dk'^2}{k'^2 - k^2} \quad (6)$$

III. A CRITERION FOR BOUND STATES

What we want to do is to find a necessary condition for the absence of bound states by making use of the unitarity condition (1) in its most general form, i.e., including inelasticity. It will be convenient, to see the role of inelasticity, to introduce an inelasticity parameter $\lambda(k^2)$, defined by

$$\operatorname{Im} T(k^2) = \frac{k \lambda(k^2)}{\sqrt{M^2 + k^2}} |T(k^2)|^2 \quad (7)$$

with $\lambda(k^2) \geq 1$. Then, for convergence reasons which will appear more clearly later, we introduce a function $\phi(k^2)$ and write a dispersion integral for

$$(T - T_0) \phi(k^2)$$

The function $\phi(k^2)$ should have the following properties

- i) it is holomorphic in the cut plane, with the cut $0 - \infty$
 - ii) it is real and positive for k^2 real < 0
 - iii) it goes to zero for $|k^2| \rightarrow \infty$
 - iv) $\operatorname{Im} \phi(k^2 + i\varepsilon) > 0$ for $k^2 > 0$.
- (8)

Further conditions will appear later on. That such functions exist is proved by taking the example

$$\phi(k^2) = \frac{1}{C - ik}$$

where k is taken to be positive for $k^2 = \text{positive number} + i\varepsilon$. Then we have

$$(T - T_0) \phi(k^2) = \frac{1}{\pi} \int \frac{\text{Im}[T(k'^2) \phi(k'^2)] dk'^2}{k'^2 - k^2} - \frac{1}{\pi} \int \frac{[\text{Im} \phi(k'^2)] T_0(k'^2) dk'^2}{k'^2 - k^2} \quad (9)$$

and hence, grouping differently the various terms and noticing that $\text{Im } A < |A|$, we get for negative k^2

$$\left| (T - T_0) \phi(k^2) + \frac{1}{\pi} \int_0^{\infty} \frac{[\text{Im} \phi(k'^2)] T_0(k'^2) dk'^2}{k'^2 - k^2} \right| < \frac{1}{\pi} \int_0^{\infty} \frac{|T(k'^2)| |\phi(k'^2)| dk'^2}{k'^2 - k^2}$$

Now we apply the Schwartz inequality so as to make appear in one of the integrals the right-hand side of (7). We get, (for $k^2 < 0$)

$$\left| (T - T_0) \phi(k^2) + \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im} \phi(k'^2) T_0(k'^2) dk'^2}{k'^2 - k^2} \right|^2 \quad (10)$$

$$< \frac{1}{\pi} \int \frac{k' \lambda(k')}{\sqrt{M^2 + k'^2}} \frac{|T(k')|^2 dk'^2}{k'^2 - k^2} \times \frac{1}{\pi} \int \frac{\sqrt{M^2 + k'^2}}{k' \lambda(k')} \frac{|\phi(k')|^2 dk'^2}{k'^2 - k^2}$$

8.

The first integral appearing in the right-hand side of (10) is nothing but $T - T_0$, by application of the unitarity condition (7) and representation (6). Hence, calling

$$\begin{aligned} I(k^2) &= \frac{1}{\pi} \int \frac{\text{Im} \phi(k'^2) T_0(k'^2) dk'^2}{k'^2 - k^2} \\ J(k^2) &= \frac{1}{\pi} \int_0^\infty \frac{\sqrt{M^2 k'^2}}{k' \lambda(k')} \frac{|\phi(k'^2)| dk'^2}{k'^2 - k^2} \end{aligned} \quad (11)$$

we get a quadratic inequality for $T - T_0$

$$(T - T_0)^2 \phi^2 + (T - T_0)(2\phi I - J) + I^2 < 0$$

The discriminant of this quadratic form must be positive and, since J is positive, we get

$$4\phi I < J \quad (12)$$

At this stage we see that the further constraint to impose on ϕ is that the integrals I and J have a meaning. There are, of course, infinitely many conditions of type (12) depending on what choice is made for ϕ and for k^2 , condition (12) should be interpreted as a necessary condition for the absence of bound states. Indeed it is easy to see, taking for simplicity the S state case, that if we replace (6) by

$$T = T_0 + \frac{g^2}{-x^2 - k^2} + \frac{1}{\pi} \int \frac{\text{Im} T(k'^2) dk'^2}{k'^2 - k^2} \quad (13)$$

$g^2 > 0$, which corresponds to the presence of a bound state, condition (12) is replaced by

$$4\phi \left[I - \frac{g^2}{\pi} \int \frac{\text{Im} \phi(k'^2) dk'^2}{(k^2 + k'^2)(k'^2 - k^2)} \right] < J \quad (14)$$

so that even if (12) is violated one can manage to satisfy (14) by taking g^2 big enough. So conversely we shall say that if for some ϕ and some k^2 we have

$$4\phi I > J. \quad (15)$$

This is a sufficient condition to have a bound state and, further, from (14) we then have a lower limit on the residue of the pole associated with this bound state.

There is one important aspect of (14) and (15) which is that we can predict the existence of a bound state irrespective of the inelasticity. Indeed, if we have

$$4\phi(k^2) \frac{1}{\pi} \int \frac{\text{Im} \phi(k'^2) T_0(k'^2) dk'^2}{k'^2 - k^2} > \frac{1}{\pi} \int \frac{\sqrt{M^2 + k'^2}}{k'} \frac{|\phi(k'^2)|^2 dk'^2}{k'^2 - k^2}$$

then it follows that condition (15) is necessarily fulfilled because the inelasticity parameter is larger than one. More precisely we see that as $\lambda(k^2)$ increases J decreases and therefore we prove rigorously a fact which has been anticipated in many instances, namely that inelasticity helps the formation of bound states.

We would also like to stress that so far the only assumptions made are $T_0(k^2) \rightarrow 0$ for $|k^2| \rightarrow \infty$ and the smoothness of $|T(k^2)|$ for $k^2 \rightarrow +\infty$. There are no approximations in these results.

At this point it would be desirable to optimize with respect to k^2 and $\phi(k^2)$. But for this purpose something must be known about $T_0(k^2)$. We shall first consider a simple case which we shall call the "extreme short range approximations". Then $T_0(k'^2)$ in the S wave case will be practically constant on the effective range of integration in I and the necessary condition for the absence of bound states becomes

$$4|\phi(k^2)|^2 T_0(0) < \frac{1}{\pi} \int \frac{\sqrt{M^2 + k'^2} |\phi(k'^2)|^2 dk'^2}{k' (k'^2 - k^2)} \quad (16)$$

Then it can be shown that the optimum ϕ is precisely

$$\phi(k^2) = \frac{1}{(M - ik)^{1/2}}$$

Indeed we have

$$\begin{aligned} \frac{(\phi(k'^2))^2 (M - ik)}{-ik} &= \frac{1}{\pi} \int \frac{\text{Re} [(M - ik') \phi^2(k'^2)] dk'^2}{k' (k'^2 - k^2)} \\ &\leq \frac{1}{\pi} \int \frac{\sqrt{M^2 + k'^2} |\phi(k'^2)|^2 dk'^2}{k' (k'^2 - k^2)} \end{aligned}$$

and we want to have a ratio between the right-hand side and the left-hand side as small as possible. This will be the case when the equality sign holds, i.e., $\text{Im} [(M - ik) \phi^2(k^2)] = 0$ so the function $(M - ik) \phi^2(k^2)$ has not cut in the k^2 plane. ϕ goes to zero at infinity and hence the product is bounded by $|k|$ at large distances. Hence it is a constant. So (16) becomes

$$4 T_0(0) < \frac{M + |k|}{|k|}$$

which is optimum for $|k| \rightarrow \infty$, i.e.,

$$T_0(0) < \frac{1}{4} \quad (17)$$

Condition (17) is simple. However, it does not cover realistic cases, so it would be useful to try the more realistic case of a force due to the exchange of a spin 0 particle with finite mass μ . Here we have not been able to get the best trial function ϕ . However, we shall justify the choice

$$\phi(k^2) = \frac{1}{\mu/2 - ik} \quad (18)$$

$$k^2 = -\mu^2/4$$

by performing a potential scattering "experiment".

We consider scattering by a Yukawa potential

$$V(r) = -\lambda \frac{e^{-\mu r}}{r}$$

We normalize the scattering amplitude as

$$T(k^2) = \frac{e^{i\delta} \sin \delta}{k}$$

where δ is the phase shift. Then, in this situation, we can carry again our argument, the only difference being that the relativistic factor $\sqrt{M^2 + k^2}$ should be omitted in all equations where it appears. So the analogue of (12) is

$$4 \phi(k^2) \frac{1}{\pi} \int \frac{\operatorname{Im} \phi(k'^2) T_0(k'^2) dk'^2}{k'^2 - k^2} < \int \frac{|\phi(k'^2)|^2 dk'^2}{k'(k'^2 - k^2)}$$

where

$$T_0 = \frac{2M \log\left(1 + \frac{4k^2}{\mu^2}\right)}{4k^2}$$

Inserting the special choice (18) which is designed to make appear only dimensionless integrals, we get

$$\frac{2M}{\mu} \frac{1}{\pi} \int_0^\infty \frac{dx \log(1+x^2)}{(1+x^2)^2} < \frac{1}{\pi} \int \frac{dx}{(1+x^2)^2}$$

The left-hand integral can be found in Bierends de Haan tables ³⁾. This gives

$$\frac{2M}{\mu} < \frac{1}{2 \log 2 - 1} \simeq 2.5 \quad (19)$$

Now the exact condition for the absence of bound states can be found in Blatt and Weisskopf ⁴⁾. It is

$$\frac{2M}{\mu} < 1.68 \quad (20)$$

This, considering the various steps in which information has been lost (neglect of further cuts, majorization of $\operatorname{Im} [\phi(k^2) T(k^2)]$ by $|\phi| |T|$, use of the Schwartz inequality), seems to us a rather impressive agreement. We feel that it is not worth while to look for a better trial function (at least in the S wave case).

Returning to the relativistic problem we take the one particle exchange contribution to be

$$\Lambda \frac{\log \left(1 + \frac{4k^2}{\mu^2} \right)}{4k^2} \quad (21)$$

and we want to find a limit on Λ to guarantee the absence of bound states. We keep the choice (18) for k^2 and ϕ and we get

$$\frac{\Lambda}{\mu} \frac{\pi}{2} \left(\log 2 - \frac{1}{2} \right) < \int_0^\infty dx \frac{[M^2 + \frac{\mu^2}{4} x^2]^{1/2}}{(1+x^2)^2}$$

the right-hand side integral can be calculated exactly. However, we shall content ourselves with an estimate which has a simple form. We get

$$\Lambda < \frac{\mu \sqrt{M^2 + M^2/6}}{2 \log 2 - 1} \quad (22)$$

As an illustration of the present method let us set a limit on the pion-nucleon coupling constant. Assume that in single S neutron-proton scattering, the over-all contribution of the distant left-hand cut singularities is positive (attractive), i.e., $T_0(k^2) > T_0$ (one-pion exchange) for $k^2 > 0$. Then we can apply inequality (12), replacing $T_0(k^2)$ by T_0 (one-particle exchange), since the singlet np system has no bound state. T_0 (one-pion exchange) is given by

$$\frac{1}{4} g^2 \frac{\mu^2}{4k^2} \log \left(1 + \frac{4k^2}{\mu^2} \right) \quad (23)$$

where μ is the pion mass, and hence we get

$$\frac{1}{4} g^2 \mu^2 < \frac{\mu \sqrt{M^2 + \mu^2/6}}{2 \log 2 - 1}$$

i.e.,

$$g^2 < 67$$

(24)

This is a very poor result which just indicates what is well known, namely that other forces are needed to account for nucleon-nucleon scattering.

Dr. Abers has pointed to me a possible application of this technique which is the following : exchange of vector particles, such as the ρ meson, requires a cut-off which, in a self-consistent calculation of the ρ parameters (mass and width), remains an arbitrary parameter when N/D equations are solved "exactly". One can now, using (12), get a limit on this cut-off by imposing that the $\pi\pi$ system has no bound state in the S state.

IV. INCONSISTENCY OF STRICT ONE-PARTICLE EXCHANGE
FOR LARGE ANGULAR MOMENTUM STATES

We shall restrict ourselves to the case where the left-hand cut is produced by the exchange of a spin 0 particle and demonstrate that then the dispersion relation has no solution for angular momenta larger than zero which satisfies at the same time unitarity (including inelastic processes) and correct threshold behaviour.

The basic remark is that $T_0^{\ell}(k^2)$, projection over the angular momentum ℓ of the one-particle exchange, has automatically the correct threshold behaviour and vanishes at infinity. More precisely

$$T_0^{\ell}(k^2) = \Lambda \frac{Q_{\ell}(1 + \frac{k^2}{2k^2})}{2k^2} \quad (25)$$

Since this T_0^{ℓ} vanishes at infinity it is just that definition of T_0^{ℓ} which we need in the argument of Section II. Therefore, it is possible, using unitarity, to eliminate, as was done in Section II, all subtraction terms and get finally Eq. (6) which we repeat here

$$T^{\ell}(k^2) - T_0^{\ell}(k^2) = \frac{1}{\pi} \int \frac{\text{Im} T(k'^2) dk'^2}{k'^2 - k^2} \quad (26)$$

Now we have $T^{\ell}(k^2) \sim k^{2\ell}$ and $T_0^{\ell}(k^2) \sim k^{2\ell}$ around $k^2 = 0$. So for $k^2 \rightarrow 0$, $T^{\ell} - T_0^{\ell} \rightarrow 0$ if the angular momentum is larger than zero. However, due to the positive character of $\text{Im} T$ we have, for $-\kappa^2 < k^2 < 0$

$$\frac{1}{\pi} \int \frac{\text{Im} T(k'^2) dk'^2}{k'^2 - k^2} > \frac{1}{\pi} \int \frac{\text{Im} T(k'^2) dk'^2}{k'^2 + \kappa^2} > 0 \quad (27)$$

and hence, the right-hand side of (26) cannot vanish at $k^2 = 0$, which is the contradiction. It is important to notice that this contradiction persists if one includes inelastic effects on the right-hand cut.

Now, let us make the remark that this theorem says that there are no exact solutions of (26). However, "approximate" solutions may have both the correct threshold behaviour and satisfy the unitarity condition. For instance, following Zachariasen and Zemach ⁵⁾ one may be contented with the first iteration of the N/D equations which merely consists in replacing N by $T_0(k^2)$. Then one gets, in the elastic approximation

$$T^l(k^2) = \frac{T_0^l(k^2)}{1 - \frac{k^2 + \chi^2}{\pi} \int_0^\infty \frac{k'}{\sqrt{M^2 + k'^2}} \frac{T_0^l(k'^2) dk'^2}{(k'^2 - k^2)(k'^2 + \chi^2)}} \quad (28)$$

where the left-hand cuts of T^l and T_0^l coincide only for $k^2 = -\chi^2$. So the original problem is not solved exactly. However, unitarity is satisfied by construction, since

$$\frac{\text{Im } T}{|T|^2} = \frac{k}{\sqrt{M^2 + k^2}}$$

and since the denominator is in general non-vanishing for $k = 0$ we also have the correct threshold behaviour. So one may consider that approximate solutions of the type (28) are in fact better than forced solutions of (26) which would necessarily exhibit ghosts on the left-hand cut. At least (28) satisfies general requirements in which we believe rather firmly such as the unitarity property. As for the threshold behaviour it can be established for sufficiently large ℓ from Mandelstam representation and may in fact be shown to hold down to $\ell = 2$ ⁶⁾. For $\ell = 0$ and 1 experiment seems to be sufficiently demonstrative.

Now one may ask whether the presence of a bound state upsets the argument. The important remark to make is that the residue of a bound state has alternate signs with ℓ . It contributes to T by

$$- \frac{(-1)^\ell g^2}{k^2 + \kappa_0^2} \quad g^2 > 0$$

Since we have made no distinction between the nature of the various singularities of the left-hand cut we can include it in T_0 formally to carry through the derivation of Section II. Then we get, returning to the old T_0

$$T - T_0 = -(-1)^\ell \frac{g^2}{\kappa_0^2 + k^2} + \frac{1}{\pi} \int \frac{\text{Im } T(k'^2) dk'^2}{k'^2 - k^2}$$

So, clearly all odd ℓ are excluded because the right-hand side is positive near $k^2 = 0$. To exclude even $\ell \geq 2$ we perform one subtraction

$$\begin{aligned} T - T_0 = & \left[-\frac{g^2}{\kappa_0^2} + \frac{1}{\pi} \int \frac{\text{Im } T(k'^2) dk'^2}{k'^2} \right] \\ & + k^2 \left[\frac{g^2}{(k^2 + \kappa^2) \kappa_0^2} + \frac{1}{\pi} \int \frac{\text{Im } T(k'^2) dk'^2}{k'^2 (k'^2 - k^2)} \right] \end{aligned}$$

The first bracket is zero because $T - T_0 = 0$ at $k^2 = 0$. Hence

$$T - T_0 = k^2 \left[\frac{g^2}{(k^2 + \kappa^2) \kappa_0^2} + \int \frac{\text{Im } T(k'^2) dk'^2}{k'^2 (k'^2 - k^2)} \right]$$

Thus we get

$$\lim_{k^2 \rightarrow 0} \frac{T - T_0}{k^2} > 0$$

which contradicts the threshold behaviour $T - T_0 \sim k^{2\ell}$ for $\ell \gg 2$. Therefore we have a confirmation of our assertion that the only possible solutions of (26) have ghosts ; i.e., poles with residues of the wrong sign.

Now one can check that the "approximate" solution (28) may have no ghosts and exhibits only "honest" bound states. Indeed, take the case $\Lambda > 0$. The denominator can be written

$$(k^2 + \kappa^2) \left[\frac{1}{k^2 + \kappa^2} - \int_0^\infty \frac{k'}{\sqrt{M^2 + k'^2}} \frac{T_0^\ell(k'^2) dk'^2}{(k'^2 - k^2)(k'^2 + \kappa^2)} \right]$$

T_0^ℓ is positive for $k^2 > 0$. Hence the bracket is negative for $k^2 < -\kappa^2$. For $-\kappa^2 < k^2 < 0$ the bracket is a decreasing function of k^2 ; and $(k^2 + \kappa^2) > 0$. If $-\kappa^2$ is chosen sufficiently close to the beginning of the cut ($k^2 = -\frac{\mu^2}{4}$) the bracket can vanish at most once in the interval $k^2 = -\frac{\mu^2}{4}$ to $k^2 = 0$. If it does vanish at $k^2 = -\kappa^2$ then the residue has the sign of $-T_0^\ell(k^2)$ which is $-(-1)^\ell$, i.e., the correct sign for a bound state. Expression (28) is also useful in another respect, because it illustrates a general theorem previously stated by Jin and the author⁷⁾ which says the following (we present it in the special case where there are no bound states).

If the discontinuity across the left-hand cut ΔT^ℓ has the same sign at the beginning of the cut as an attractive one particle exchange cut then, if threshold behaviour and unitarity are imposed, the number of oscillations of ΔT^ℓ around zero is larger or equal to the number of oscillations of the one particle exchange cut, i.e., ℓ . Example (28), taken in the case where D does not vanish, shows that this theorem cannot be strengthened because ΔT has a number of oscillations exactly equal to the number of oscillations of the one particle exchange cut.

Finally what can we say about the exchange of a particle with spin larger than zero? Formally, there is no difficulty to extend the proof we gave to this case. One can work with T/k^{2J} where J is the spin of the exchanged particle and then prove in exactly the same way that one gets a contradiction for $\ell > J$. However, this is not a very meaningful result because we know that we encounter other kinds of difficulties, divergence difficulties, which force us anyway to make a cut off on the left-hand cut. Then it is no longer possible to define T_0^ℓ , the left-hand cut contribution as vanishing as $k^2 \rightarrow \infty$ and having at the same time the correct threshold behaviour. More explicitly we can define T_0^ℓ either as

$$T_0^\ell(k^2) = \frac{1}{\pi} \int_{-A}^{-1/4} \frac{\Delta T_0^\ell(k'^2) dk'^2}{k'^2 - k^2} \quad (29)$$

or as

$$\tilde{T}_0^\ell(k^2) = \frac{k^{2\ell}}{\pi} \int_{-A}^{-1/4} \frac{\Delta T_0^\ell(k'^2) dk'^2}{k'^{2\ell} (k'^2 - k^2)} \quad (30)$$

These two definitions no longer coincide. Since (29) vanishes at infinity we shall prefer to define T_0 by (29), so that the argument of Section II remains still valid. Then, for $\ell > 0$ we have

$$T^\ell(0) = 0 = \frac{1}{\pi} \int_{-A}^{-1/4} \frac{\Delta T_0^\ell(k'^2) dk'^2}{k'^2} + \frac{1}{\pi} \int \frac{\text{Im } T(k'^2) dk'^2}{k'^2}$$

Everything now depends on the sign of the first integral. If

$$\frac{1}{\pi} \int_{-A}^{-1/4} \frac{\Delta T_0^\ell(k'^2) dk'^2}{k'^2} < 0$$

there is no contradiction. If

$$\frac{1}{\pi} \int_{-A}^{-1/4} \frac{\Delta T_0^{\ell}(k'^2) dk'^2}{k'^2} > 0$$

again, in spite of the cut-off, the equations have no solution ; higher order restrictions can be obtained for sufficiently large ℓ involving higher momenta of ΔT_0^{ℓ} . These have been investigated by Balachandran and Von Hippel ¹⁾, so we shall not develop this any further. What we just want to say is that the cut-off position $-A$ cannot be completely independent of ℓ . Indeed, it is easy to see (on the ρ exchange for instance) that as ℓ increases the number of oscillations of ΔT_0^{ℓ} increases and, at the same time, the first zero of ΔT_0^{ℓ} (starting from $k^2 = -\infty$) moves to the left. So for sufficiently large ℓ one or more oscillations will be eliminated by the cut-off and then the theorem previously quoted ⁷⁾ tells us that there are no solutions.

ACKNOWLEDGEMENTS

I thank Drs. E.S. Abers, V.L. Teplitz and B. Diu for stimulating discussions on this subject.

A P P E N D I X

We want to prove, without any smoothness assumption on T , that in the dispersion relation

$$T = T_0 + C + \frac{1}{\pi} \int_0^\infty \frac{\text{Im} T(k'^2) dk'^2}{k'^2 - k^2} \quad (\text{A.1})$$

C is equal to zero. The additional condition we shall need, besides

$$T_0(k^2) \rightarrow 0 \quad \text{for } |k^2| \rightarrow \infty \quad (\text{A.2})$$

will be that

$$\int_0^\infty \frac{|T_0(k'^2)|^2 dk'^2}{k'^2} \quad (\text{A.3})$$

converges. First, notice that by unitarity

$$\text{Im} T \geq \frac{k}{\sqrt{M^2 + k^2}} |T|^2$$

and hence since we already know that

$$\int_{k_0^2}^\infty \frac{\text{Im} T(k'^2) dk'^2}{k'^2}$$

converges, it follows that

$$\int_{k_0^2}^{\infty} \frac{|T(k'^2)|^2 dk'^2}{k'^2} \quad (\text{A.4})$$

converges. Now we have

$$|T - T_0|^2 < 2[|T|^2 + |T_0|^2] \quad (\text{A.5})$$

Thus, combining (A.3) and (A.4) we get that

$$\int_{k_0^2}^{\infty} \frac{|T - T_0|^2 dk'^2}{k'^2} \quad (\text{A.6})$$

converges. On the other hand, we know that for $|k^2| \rightarrow \infty$ $|T - T_0| < \text{const.}$
We are, therefore, led to prove :

Theorem :

Let a function $f(z)$ be analytic in the whole z plane with a cut from $z = 0$ to $z = +\infty$, bounded by a constant for $|z| > R$, finite along the cut and such that

$$\int_{x_0}^{\infty} \frac{|f(x)|^2 dx}{x}$$

converges. Then for $x \rightarrow -\infty$ $|f(x)| \rightarrow 0$.

Proof :

Consider the function

$$\frac{(f(x))^2}{\sqrt{-x}}$$

This function goes to zero at infinity like $1/\sqrt{|x|}$. It therefore satisfies an unsubtracted dispersion relation

$$(f(x))^2 = \frac{\sqrt{-x}}{\pi} \int_0^{\infty} \frac{\text{Re}[f(x')]^2 dx'}{\sqrt{x'}(x' - x)}$$

consider the case $x < 0$. We have

$$|f(x)|^2 < \frac{1}{\sqrt{x}} \frac{1}{\pi} \int_0^A \frac{|f(x')|^2 dx'}{\sqrt{x'}} + \frac{1}{\pi} \int_A^{\infty} \frac{|f(x')|^2 dx'}{x'}$$

where A will be chosen larger than x_0 . Now since (A.6) converges one can choose A sufficiently large so as to make the second contribution arbitrarily small, in particular less than ε given in advance. The first contribution is a finite integral. By choosing now $|x|$ large enough, it can be made arbitrarily small. Hence $|f(x)| \rightarrow 0$ as $x \rightarrow -\infty$.

Referring to our problem, we take $x = k^2$, $f(x) = T(k^2) - T_0(k^2)$. All conditions are fulfilled and we get $T - T_0 \rightarrow 0$ for $k^2 \rightarrow -\infty$. Now we also have

$$\frac{1}{\pi} \int_0^{\infty} \frac{\text{Im} T(k'^2) dk'^2}{k'^2 - k^2} \rightarrow 0$$

for $k^2 \rightarrow -\infty$ because

$$\frac{1}{\pi} \int_0^{\infty} \frac{|\text{Im} T(k'^2)| dk'^2}{k'^2}$$

converges. Therefore, the constant C , appearing in (A.1), is zero.

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