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CM-P00057068

A PHYSICAL SUM RULE ON THE REAL PART OF THE
FORWARD SCATTERING AMPLITUDE

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10056/TH.503
21 December 1964

Recently, Kinoshita and Khuri ¹⁾ have proposed a sum rule on the real part of the forward scattering amplitude, which could constitute a test of dispersion relations. In this note we just want to remark that one can avoid the difficulty of evaluating the real part of the scattering amplitude in part of the unphysical region and write a sum rule containing only physical quantities.

Following Ref. ¹⁾ we consider the case of pion-nucleon scattering. E is the total energy of the pion-nucleon scattering amplitude and $f_{\pm}(E)$ refer respectively to $\pi^{\pm}p$ scattering. We consider the symmetrized amplitude

$$f(E) = \frac{1}{2} [f_{+}(E) + f_{-}(E)] - \text{pole terms} \quad (1)$$

$f(E)$, as in Ref. ¹⁾, is supposed to increase less fast than $E^{2-\epsilon}$ at infinity. This is a very weak assumption according to Ref. ¹⁾. Let us just say that a constant high energy cross-section corresponds to $f(E) \sim E$. The symmetrized amplitude $f(E)$ is such that $f(E) = f(-E)$, with two cuts from $E = -\infty$ to $E = -\mu$ and $E = \mu$ to $E = +\infty$. It is convenient then to use the variable $z = E^2$ which maps the upper half E plane on the whole z plane with a cut from $z = \mu^2$ to $z = \infty$. From now on we shall use this variable without changing the name of the function. Since $f(\mu^2)$, the scattering amplitude at threshold is finite, as experimentally established, it is permissible to perform a subtraction at this point (only one subtraction in z is necessary) :

$$f(z) = f(\mu^2) + \frac{z - \mu^2}{\pi} \int_{\mu^2}^{\infty} \frac{dz' \operatorname{Im} f(z')}{(z' - \mu^2)(z' - z)} \quad (2)$$

Let us now calculate the quantity

$$g(z_0) = \int_{\mu^2}^{z_0} \frac{f(z) - f(\mu^2)}{(z - \mu^2)^{3/2}} dz \quad (3)$$

2.

we get

$$g(z_0) = \frac{1}{\pi} \int_{\mu^2}^{\infty} \frac{dz'}{z' - \mu^2} \operatorname{Im} f(z') \int_{\mu^2}^{z_0} \frac{dz}{\sqrt{z - \mu^2} (z' - z)} \quad (4)$$

The inversion of the orders of integration can easily be justified for complex z_0 , integrating along a line $\operatorname{Arg}(z - \mu^2) = \operatorname{Arg}(z_0 - \mu^2)$, and then one can go to the limit of real z_0 . In this way one gets for real z_0

$$\int_{\mu^2}^{z_0} \frac{(\operatorname{Re} f(z) - f(\mu^2)) dz}{(z - \mu^2)^{3/2}} = \frac{1}{\pi} \int_{\mu^2}^{\infty} \frac{dz' \operatorname{Im} f(z')}{(z' - \mu^2)^{3/2}} \log \left| \frac{\sqrt{z' - \mu^2} + \sqrt{z_0 - \mu^2}}{\sqrt{z' - \mu^2} - \sqrt{z_0 - \mu^2}} \right| \quad (5)$$

We notice that the logarithm in the integral is always larger than 0. The fact that it becomes infinite for $z' = z_0$ is not a difficulty because for finite $\operatorname{Im} f(z')$ this singularity can be integrated.

Since, by unitarity $\operatorname{Im} f(z') > 0$, we get the inequality

$$\int_{\mu^2}^{z_0} \frac{\operatorname{Re} f(z) - f(\mu^2)}{(z - \mu^2)^{3/2}} dz > 0 \quad (6)$$

which is the analogue of sum rule (4) in Ref. 1).

More generally, one can write

$$\int_{\mu^2}^{z_0} \frac{\operatorname{Re} f(z) - f(\mu^2)}{(z - \mu^2)^{3/2}} dz > \frac{1}{\pi} \int_{\mu^2}^{z_1} \frac{dz' \operatorname{Im} f(z')}{(z' - \mu^2)^{3/2}} \log \left| \frac{\sqrt{z' - \mu^2} + \sqrt{z_0 - \mu^2}}{\sqrt{z' - \mu^2} - \sqrt{z_0 - \mu^2}} \right| \quad (7)$$

where one can take z_1 arbitrarily, in particular such that under the integral only experimentally known cross-sections are inserted.

This sum rule involves only physical quantities. Its disadvantage is that it requires rather precise information on the low energy behaviour of the scattering amplitude. Essentially, what one needs to know in the low energy region is

$$\lim_{z \rightarrow \mu^2} \frac{\operatorname{Re} f(z) - f(\mu^2)}{z - \mu^2}$$

Fortunately, the low energy pion-nucleon scattering has been rather well studied experimentally. Equations (6) or (7) show of course that if the dispersion formalism is valid, the real part of the scattering amplitude cannot stay negative, of the order of E , i.e., $\sqrt{z}$ in the limit of extremely high energies.

Now can one construct sum rules which put less emphasis on high energies? The answer is yes and we shall present a systematic procedure to do so. We can calculate

$$\int_{\mu^2}^{z_0} \frac{[\operatorname{Re} f(z) - f(\mu^2)] \phi(z) dz}{(z - \mu^2)^{3/2}} = \frac{1}{\pi} \int_{\mu^2}^{\infty} \frac{\operatorname{Im} f(z')}{(z' - \mu^2)} I(z', z_0) dz' \quad (8)$$

4.

where

$$I(z', z_0) = P \int_{\mu^2}^{z_0} \frac{\phi(z) dz}{(z - \mu^2)^{1/2} (z' - z)} \quad (9)$$

It is easy to see, by partial integration, that if

$$\begin{aligned} \phi(z) &> 0 \\ \phi'(z) &\leq 0 \end{aligned} \quad (10)$$

$I(z', z_0)$ is positive for all $z_0 > \mu^2$. One can now compare I with $I_0(z', z_0)$ which corresponds to $\phi(z) = \phi_0(z) \equiv 1$. Normalizing $\phi(z)$ to unity at $z = \mu^2$, we easily see that one has :

$$\begin{aligned} I(\mu^2, z_0) / I_0(\mu^2, z_0) &> 1 \\ \lim_{z' \rightarrow z_0} I(z', z_0) / I_0(z', z_0) &< 1 \\ \lim_{z' \rightarrow \infty} I(z', z_0) / I_0(z', z_0) &< 1 \end{aligned}$$

so that lower energies are favoured both in the left-hand side and the right-hand side of (8). For any ϕ satisfying conditions (10), we get the sum rule

$$\int_{\mu^2}^{z_0} \frac{[\operatorname{Re} f(z) - f(\mu^2)] \phi(z) dz}{(z - \mu^2)^{3/2}} > \frac{1}{\pi} \int_{\mu^2}^{z_1} \frac{\operatorname{Im} f(z') I(z', z_0) dz'}{(z' - \mu^2)} \quad (11)$$

where $z_1 > \mu^2$ can be chosen at best convenience. A simple choice for $\phi(z)$ could be

$$\phi(z) = \left(\frac{C + \mu^2}{C + z} \right)^n$$

for which (9) can be calculated explicitly.

Generalizing further, one can abandon conditions (10) on $\phi(z)$ and retain only the positiveness condition $\phi(z) > 0$. Then $I(z', z_0)$ is no longer everywhere positive but will nevertheless be positive for $z' > z_0$. Hence, if one takes the precaution to choose $z_1 > z_0$, inequality (11) will still be satisfied. In this way one gets, for instance,

$$\int_{\mu^2}^{z_0} [\operatorname{Re} f(z_0) - f(\mu^2)] dz > \frac{1}{\pi} \int_{\mu^2}^{z_1} \operatorname{Im} f(z') \left[\log \left| \frac{z' - \mu^2}{z' - z_0} \right| - \frac{z_0 - \mu^2}{z' - \mu^2} \right] dz' \quad (12)$$

with $z_1 > z_0$. This sum rule avoids the difficulties associated with the threshold behaviour, but requires a knowledge of the cross-sections at sufficiently high energies.

I thank Professors Khuri and Kinoshita for communicating their work prior to publication.

REFERENCE

- 1) N. Khuri and T. Kinoshita, preprint Rockefeller Institute (1964).