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MARGINALLY SINGULAR FORCES AND THEIR ROLE
IN PRODUCING PSEUDOVECTOR RESONANCES

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ABSTRACT

We study the problem of two equal mass scalar bosons scattering in a P-state under the influence of "marginally singular" forces. Such forces arise from the exchange of a vector boson. We show that the solutions of the N/D equations have a one-parameter ambiguity which we relate to the existence in the theory of quasi-elementary particles. We speculate that the observed vector bosons might belong to this category. We also discuss the possibility of a special form of left-hand cut which would have the property that all solutions of the N/D equations would contain a resonance. We point out that if this were to occur in the bootstrap context, it could provide a new mechanism for the determination of the coupling constant by crossing symmetry.

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I. INTRODUCTION

There has been interest recently in the problem of scattering by "marginally singular" forces ^{1),2)}. Here we consider them in relation to the problem of producing resonances, in particular to the bootstrap problem for the pseudovector resonances ³⁾.

We work through the language of dispersion theory so that we define a problem with "marginally singular" forces as one for which the discontinuity function for the partial wave amplitude on the left cut tends to a constant at infinity. This happens for the Born term corresponding to the exchange of a vector boson. It is therefore highly relevant to the problem of bootstrapping vector resonances. Here, we shall not be concerned with the bootstrap problem as such. If we were, we would be confronted with the enforcement of crossing symmetry, the inclusion of contributions from all partial waves in the crossed channel and difficulties of the non-convergence of the partial wave series. We will confine our investigation to problems with given left-hand cuts with the above-mentioned asymptotic property.

The corresponding situation in potential scattering is one for which $V(r) \sim \lambda / r^2$ ($r \rightarrow 0$). This produces a phase shift which tends to a constant value at infinity not equal to an integral multiple of π . Similarly, we find in our relativistic problem that we have solutions which are not classifiable according to Levinson's theorem ⁴⁾ and that the distinction between dynamical solutions and those containing a CDD pole is different from that for the non-singular situation ^{5),6)}. We show the existence of a new sort of arbitrariness in the solutions of the dispersion relations depending on one arbitrary constant, not two as for the ordinary CDD poles ⁷⁾. We further show that if there exist left-hand discontinuity functions satisfying a special condition, all the corresponding solutions are of a resonant form. Such solutions would be of a different character from dynamical solutions ordinarily considered. It could possibly be easier to achieve narrow resonances in this framework than in the conventional non-singular situation.

This could be a very hopeful circumstances for the "bootstrap" programme. For bootstrapping has suffered from two disabilities: firstly, it has proved difficult to get the PV resonances to come out with sufficiently narrow widths; secondly, the work has been open to theoretical criticism⁸⁾. This has been directed at the claim to predict unique values of mass and coupling constant by the mere application of crossing symmetry. Certainly, if the problem is expressible in terms of Lagrangian field theory with solutions continuous in the mass and coupling constant and identically crossing symmetric, the bootstrap programme must necessarily be chimerical. Calculations will simply consist in a search for those values of the parameters which make ones chosen form of approximation most closely obey the relations which it ought to satisfy identically. The present investigations open a way out of this dilemma.

The remaing sections are laid out as follows. In Section II, we formulate the problem mathematically in terms of the N/D method. The essential task is shown in the usual way to be the solution of a non-singular integral equation. In Section III, we give a preliminary discussion of possible types of solution. In Section IV, we derive the formal solution of the integral equation and exhibit the one-parameter ambiguity referred to above. We also discuss the form of the solutions when the left-cut discontinuity satisfies the special condition such that all solutions contain a resonance. The final section is devoted to a brief discussion.

II. FORMULATION OF THE PROBLEM IN TERMS OF THE N/D METHOD

We consider the partial wave amplitude for the elastic scattering of two spinless bosons of equal mass. We treat explicitly the case of a P state. We use the amplitude $f(\nu)$ with the kinematic zero factored out so that $f(\nu) = \sqrt{\{(\nu+1)/\nu^3\}} e^{i\delta} \sin\delta$, for physical values of ν (ν = c.m. momentum squared). The assumption of equal masses for the scattered particles is not an important restriction since, as we shall see, the discussion will hinge on high energy effects. We consider the case where there is no bound state and assume that we can set up a dispersion relation for $f(\nu)$ of the form :

$$f(v) = \frac{1}{\pi} \int_0^{\infty} dv' \frac{\text{Im} f_R(v')}{v' - v} + \frac{1}{\pi} \int_{-\infty}^{v_L} dv' \frac{\text{Im} f_L(v')}{v' - v} \quad (2.1)$$

The function $\text{Im} f_R(v)$ is constrained by unitarity

$$\text{Im} f_R(v) = \sqrt{\frac{v+1}{v^3}} \sin^2 \delta = \sqrt{\frac{v^3}{v+1}} |f|^2 \quad (2.2)$$

We work throughout in units for which the mass of the bosons, μ , equals unity. The right-hand end point of the left cut, v_L , is less than zero. If the lightest mass that can be exchanged is 2μ , $v_L = -1$.

In writing (2.1), we assume in addition to analyticity, a certain boundedness condition for $f(v)$ at infinity :

$$|f(v)| = O(v^{-\varepsilon}) \quad (\varepsilon > 0) \quad (2.3)$$

For the discontinuity functions $\text{Im} f_R(v)$ and $\text{Im} f_L(v)$ we impose stronger conditions than (2.3). Firstly, from unitarity

$$\text{Im} f_R(v) = O\left(\frac{1}{v}\right) \quad (v \rightarrow \infty) \quad (2.4)$$

The function $\text{Im} f_L(v)$ which we regard as given and providing the force, we suppose to be in the "marginally singular" situation

$$\text{Im} f_L(v) \sim \frac{A}{v} \quad (v \rightarrow \infty) \quad (2.5)$$

where A is a constant. This is the form that results from the exchange of a vector boson.

Let us now explore the consequences of (2.5). Firstly, we define the Born approximation $B(\nu)$ by

$$B(\nu) = \frac{1}{\pi} \int_{-\infty}^{\nu} \frac{\text{Im } f_L(\nu')}{\nu' - \nu} d\nu' \quad (2.6)$$

For our case, governed by the relation (2.5), this has the asymptotic behaviour

$$B(\nu) \sim \frac{A}{\pi} \log \nu/\nu \quad (2.7)$$

Evidently, we are in a regime where $f(\nu)$ does not tend to $B(\nu)$ for large ν , for $B(\nu)$ at some point exceeds the unitarity bound. We might seem to be in an awkward situation but an obvious way round this is to have

$$\text{Im } f_R(\nu) \sim \frac{A}{\nu} \quad (\nu \rightarrow \infty) \quad (2.8)$$

We then have discontinuities which match at infinity and avoid the diverging feature of $B(\nu)$ persisting into the solution.

What are the implications of (2.8)? From (2.2), we deduce that

$$\sin^2 \delta \sim A \quad (\nu \rightarrow \infty) \quad (2.9)$$

This implies for one thing that A must obey the relation

$$0 \leq A \leq 1 \quad (2.10)$$

There is thus a limit on the strength of the coupling of the vector boson. (cf., Ref. 3) and Ref. 1).)

More generally, if several vector bosons contribute in the crossed channel some with negative and some with positive crossing coefficients, the resultant must be positive and less than our limit.

Equation (2.9) has infinitely many solutions for δ . If we define δ_A by

$$\delta_A = \arcsin \sqrt{A} \quad (0 \leq \delta_A \leq \pi/2) \quad (2.11)$$

we have

$$\begin{array}{ll} \delta(\infty) = \delta_A & \text{Case I} \\ \pi - \delta_A & \text{Case II} \\ -\delta_A & \text{Case III} \end{array} \quad (2.12)$$

etc.

The question is which of the listed alternatives obtains, or if there is an ambiguity which solution is to be preferred.

We now proceed to develop the N/D formalism ⁹⁾ for our problem. We thereby at once eliminate the conventional CDD ambiguity ^{10),5),6)}. We therefore set

$$f(v) = N(v) / D(v) \quad (2.13)$$

where N and D are constructed to have respectively only the left and right cuts and also to satisfy the appropriate boundary conditions. Our equations for N and D are

$$N(v) = \frac{1}{\pi} \int_{-\infty}^{v_L} dv' \frac{\text{Im} A_L(v') D(v')}{v' - v} \quad (2.14)$$

and hence $D(v)$ is determined by $N(v)$ through (2.15).

$$D(v) = 1 - \frac{v}{\pi} \int_0^\infty dv' \sqrt{\frac{v'}{v'+1}} \frac{N(v')}{v' - v} \quad (2.15)$$

We follow the method of Wong¹¹⁾ and eliminate $D(v)$ between (2.14) and (2.15) to obtain a linear integral equation for N ,

$$N(v) = B(v) - \frac{i}{\pi} \int_0^\infty dv' \sqrt{\frac{v'}{v'+1}} N(v') \left[\frac{vB(v) - v'B(v')}{v' - v} \right] \quad (2.16)$$

with $B(v)$ defined by (2.16). This non-singular integral equation is to be our main tool of investigation and its solution is the most difficult task in generating $f(v)$. Once $N(v)$ is obtained, $D(v)$ follows immediately from (2.15).

III. PRELIMINARY DISCUSSION

Before proceeding in Section IV to a detailed mathematical attack on Eq. (2.16), we develop some preliminary ideas.

Suppose that there exists a solution of the N and D equations with a given phase shift $\delta(v)$. Then we can construct $D(v)$ by the method of Omnès¹²⁾. We have assumed that there are no bound states. There is therefore no arbitrariness and we obtain

$$D(v) = \exp \left[-\frac{v}{\pi} \int_0^\infty dv' \frac{\delta(v')}{v'(v' - v)} \right] \quad (3.1)$$

From this we can deduce the asymptotic behaviour

$$D(v) \sim \text{const. } v^{\delta_{\infty}/\pi} \quad (3.2)$$

For our problem, we expect $\sin^2 \delta(\infty) \neq 0$ so that

$$\text{Re} f(v), \text{Im} f(v) = O\left(\frac{1}{v}\right) \quad (v \rightarrow \infty) \quad (3.3)$$

Hence

$$\begin{aligned} N(v) &= f(v) D(v) \\ &\sim \text{const. } v^{-1 + \delta_{\infty}/\pi} \end{aligned} \quad (3.4)$$

Thus our alternative kinds of solutions listed in (2.12) will manifest themselves as solutions of Eq. (2.16) with the respective asymptotic behaviours

$$\begin{aligned} N(v) &\sim \text{const. } v^{-1 + \delta_A/\pi} & (I) \\ &\sim \text{const. } v^{-\delta_A/\pi} & (II) \\ &\sim \text{const. } v^{-1 - \delta_A/\pi} & (III) \\ &\text{etc.} \end{aligned} \quad (3.5)$$

We now ask the question which of the alternatives do we expect to occur. We would expect in general that as $A \rightarrow 0$ and the singular force is switched off that we should tend to the non-singular situation with no CDD pole and $\delta(\infty) = 0$. This would imply for an attractive force that we should expect in general the Type I solution to obtain. For a partly repulsive force, we might obtain a Type III solution.

What about Type II solutions? Those have the property of necessarily containing a resonance and are thus of especial interest if they can be shown to occur.

Consider the following example. We construct a scattering amplitude of the form ¹³⁾

$$F(\nu) = \left[(\nu_R - \nu) / \Gamma - i \sqrt{\nu^3 / (\nu + 1)} \right]^{-1} \quad (3.6)$$

where ν_R and Γ are positive. This is the generalization to the relativistic case of the usual Breit-Wigner formula for a resonance. From the present point of view, it is interesting as an example of an amplitude with a left cut of the type under discussion and whose phase shift has the Type II behaviour at infinity $\delta'(\infty) = \pi - \arctan \Gamma$. If we consider the case where Γ is small, we have a left cut with the behaviour

$$\text{Im } f_L(\nu) \sim \Gamma^2 / \nu \quad (3.7)$$

associated with a resonance of width Γ . If we could show that the solutions of (2.16) were unique, then we could speak of arbitrarily weak forces producing arbitrarily narrow resonances. This would be a remarkable state of affairs! In fact, we shall show that equation (2.16) does not possess unique solutions but has a one-parameter arbitrariness associated with it. This means that the word "produce" has an ambiguous meaning in the present context. In contrast to the case with non-singular forces, there is an intermediate situation between a dynamical resonance and a fully elementary particle to which our example belongs. In Section V, we speculate on the relevance of this to physics.

IV. MATHEMATICAL DISCUSSION OF THE INTEGRAL EQUATION

(a) Introduction

Our task is to discuss Eq. (2.16) which we rewrite as

$$N(\nu) = B(\nu) + \int_0^{\infty} d\nu' K(\nu, \nu') N(\nu') \quad (4.1)$$

with

$$K(\nu, \nu') = -\frac{1}{\pi} \sqrt{\frac{\nu'}{\nu'+1}} \left[\frac{\nu B(\nu) - \nu' B(\nu')}{\nu' - \nu} \right] \quad (4.2)$$

This is to be solved for $N(\nu)$ with $B(\nu)$ given through Eq. (2.6).

Because of the asymptotic behaviour of $B(\nu)$ given in Eq. (2.7) the kernel fails to satisfy the Fredholm $\mathcal{L}^2$ condition. The failure is marginal and resides in the logarithmic divergence of an integral. We shall presently develop a frontal attack on Eq. (4.1) but first it is instructive to apply some non-rigorous mathematics to see how a power behaviour for N develops ¹⁵⁾.

(b) Heuristic Approach

We have from (2.7) $B(\nu) \sim \frac{A}{\pi\nu} \log \nu$. Hence

$$K(\nu, \nu') \sim \frac{A}{\pi^2} \left[\frac{\log \nu' - \log \nu}{\nu' - \nu} \right] \quad (4.3)$$

We now develop the iterative solution for N using the result

$$\int_1^\infty dv' (\log v')^n [\log v - \log v'] \left[\frac{1}{v'} - \frac{1}{v'-v} \right] \sim \frac{-(\log v)^{n+2}}{(n+1)(n+2)} \quad (4.4)$$

For each power of A , we keep only the highest power of $\log v$ so we produce an approximate solution valid for small A and large v . Iterating, we obtain

$$N(v) \sim \frac{A}{\pi} \frac{\log v}{v} + \frac{A^2}{\pi^3} \frac{1}{v} \frac{(\log v)^3}{2 \cdot 3} + \frac{A^3}{\pi^5} \frac{1}{v} \frac{(\log v)^5}{2 \cdot 3 \cdot 4 \cdot 5} + \dots \quad (4.5)$$

There thus results a situation analogous to a series of perturbation diagrams summing to Regge poles¹⁶⁾. For our case, the summation yields

$$\begin{aligned} N(v) &\sim \frac{\sqrt{A}}{v} \sinh \left[\frac{\sqrt{A}}{\pi} \log v \right] \\ &\sim \frac{\sqrt{A}}{2v} \left[v^{\sqrt{A}/\pi} - v^{-\sqrt{A}/\pi} \right] \end{aligned} \quad (4.6)$$

This agrees to first order in $\sqrt{A}$ with the form we previously obtained in (3.5) (Type I solution). It is possible to elaborate the argument a little further by the inclusion in $B(v)$ of the next leading term $C/\pi v$. The asymptotic form for $K(v, v')$ is unaltered and we now obtain

$$N(v) \sim \frac{\sqrt{A}}{v} \sinh \left[\frac{\sqrt{A}}{\pi} \log v \right] + \frac{C}{\pi v} \cosh \left[\frac{\sqrt{A}}{\pi} \log v \right] \quad (4.7)$$

We could then arrange for the Type III asymptotic behaviour, $N(v) \sim \text{const. } v^{-1-\delta A/\pi}$, by choosing $C = -\pi \sqrt{A}$. It is probably dangerous to take the argument any further along these lines. Note, however, that we can never obtain a Type II solution by iteration.

(c) Formal Solution of the Problem

Chew¹⁷⁾ has developed a method of solution for equations of the type under consideration. The essential idea is to split the kernel K into two pieces

$$K = AK_1 + K_2 \quad (4.8)$$

where AK_1 is in fact the asymptotic part of K set down in Eq. (4.3) and K_2 , the remainder, is a Fredholm kernel. An explicit construction is then given for the Green's function or resolvent kernel

$$O = (I - AK_1)^{-1} \quad (4.9)$$

by the Wiener-Hopf technique¹⁸⁾. The problem is then reduced to a Fredholm equation with the formal solution

$$N = O(I - K_2 O)^{-1} B \quad (4.10)$$

We now proceed to set this out in more detail. We start by making a series of charges of variable in Eq. (2.16). We first map the equation to a finite range through the transformation

$$v = (1 - z)/2z \quad (4.11)$$

We obtain

$$\begin{aligned} n(z) &= b(z) + \frac{1}{2\pi} \int_0^1 dz' \sqrt{\frac{1-z'}{1+z'}} \frac{[(1-z)b(z) - (1-z')b(z')]}{z' - z} \\ &\equiv b(z) + \int_0^1 dz' K(z, z') n(z') \end{aligned} \quad (4.12)$$

Note K has a different functional form from the kernel of (4.2).

Here

$$\left. \begin{aligned} n(z) &= \frac{1}{z} N\left(\frac{1-z}{2z}\right) \\ b(z) &= \frac{1}{z} B\left(\frac{1-z}{2z}\right) \end{aligned} \right\} \quad (4.13)$$

The equation has now been transformed onto a standard range of integration and would be of the Fredholm kind were it not for the divergence of the kernel at the origin. We now write the kernel of Eq. (4.12) as

$$K(z, z') = A K_1(z, z') + K_2(z, z') \quad (4.14)$$

where

$$K_1(z, z') = \frac{1}{\pi^2} \frac{[\log z' - \log z]}{z' - z} \quad (4.15)$$

We call the integral equation with K_2 omitted, the skeleton equation.

Our first task is to solve the skeleton equation, i.e., to find the resolvent of AK_1 . Following the methods of Ref. ¹⁷⁾, we make a further change of variable

$$\left. \begin{aligned} t &= \log z \\ F(t) &= e^{t/2} f(e^t) \quad \text{etc.} \end{aligned} \right\} \quad (4.16)$$

so that the integral operator K_1 has the new representation

$$\hat{K}_1[F(t)] = \frac{1}{\pi^2} \int_{-\infty}^0 dt' \frac{[t'-t]/2}{\sinh[(t'-t)/2]} F(t') \quad (4.17)$$

The skeleton equation which we are to solve takes the form

$$F(t) = F_0(t) + A \hat{K}_1[F(t)] \quad (4.18)$$

We suppose the inhomogeneous term $F_0(t)$ and the solution $F(t)$ to be extendable to values of $t > 0$. We make a Fourier analysis and write

$$\left. \begin{aligned} F(t) \Theta(-t) &\equiv F_-(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikt} F_-(k) \\ F(t) \Theta(t) &\equiv F_+(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikt} F_+(k) \\ F_0(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikt} F_0(k) \end{aligned} \right\} \quad (4.19)$$

(We use the same notation for a function and its Fourier transform. The meaning is always clear from the context.) Following Chew¹⁷⁾, we can write down the solution of the problem in terms of the Fourier transform (see also the Appendix). We obtain

$$F_-(k) = \Phi_-(k, \alpha) \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk' \frac{F_0(k')}{k' - k - i\varepsilon} \Phi_+(k', \alpha) \quad (4.20)$$

We thus have an explicit representation in terms of its two sided Fourier transform of the operator $\mathcal{O}$ referred to in Eq. (4.9). The quantities Φ_- and Φ_+ which tend to constants as $k \rightarrow \infty$ are defined thus :

$$\Phi_-(k, \alpha) = \frac{\Gamma(1 - ik - \alpha) \Gamma(-ik + \alpha)}{\Gamma(1/2 - ik)^2} \quad (4.21)$$

$$\Phi_+(k, \alpha) = \frac{\Gamma(\frac{1}{2} + ik)^2}{\Gamma(ik + \alpha)\Gamma(1 + ik - \alpha)} \quad (4.22)$$

$$(\quad = 1 / \Phi_-(-k, \alpha) \quad)$$

where α is related to A through

$$A = \cos^2 \pi \alpha \quad (\text{this means } \alpha = \frac{1}{2} - \delta_A / \pi) \quad (4.23)$$

We note in passing that the form of the solution (4.20) would change in character if A were outside the interval $(0, 1)$. The quantity α would then become complex and $F_-(k)$ would develop a branch cut in the k -plane. This confirms the conclusion of Eq. (2.10) (again cf. Ref. ³).

So much for the solution of the inhomogeneous skeleton equation. We now have to consider the corresponding homogeneous equation. We thus have to look for eigenfunctions of the operator $\hat{K}_1$ and we find the following function which we again define in terms of its Fourier transform

$$F_-^{(0)}(k, \alpha) = \frac{\Gamma(-ik + \alpha)\Gamma(-ik - \alpha)}{\Gamma(\frac{1}{2} - ik)^2} \quad (4.24)$$

Correspondingly, we define

$$F_+^{(0)}(k, \alpha) = \frac{\Gamma(\frac{1}{2} + ik)^2}{\Gamma(1 + ik + \alpha)\Gamma(1 + ik - \alpha)} \quad (4.25)$$

The quantities $F_-^{(0)}$ and $F_+^{(0)}$ have the asymptotic behaviour

$$F_-^{(0)}, F_+^{(0)} = O\left(\frac{1}{k}\right) \quad (4.26)$$

so that their Fourier inverses exist. For the homogeneous solution, the contour in the k -plane for the Fourier transformation back to the t -plane has to pass above the pole at $k = i\alpha$. For both the homogeneous and inhomogeneous solutions, we can deduce the asymptotic behaviour for large t from the position of the uppermost pole in the k -plane. Then, inverting the changes of variable (4.11) and (4.16), we can reinterpret our results in terms of the original variable v .

We thus conclude that the general solution for the skeleton equation is

$$N^{(0)}(v) = N_1^{(0)}(v) + \lambda N_2^{(0)}(v) \quad (4.27)$$

where λ is an arbitrary constant and $N_1^{(0)}$ and $N_2^{(0)}$ have the respective asymptotic behaviours

$$N_1^{(0)}(v) \sim \text{const. } v^{-1 + \delta_A/\pi} \quad (4.28)$$

$$N_2^{(0)}(v) \sim \text{const. } v^{-\delta_A/\pi} \quad (4.29)$$

(For more details, again see the Appendix).

Here then is the arbitrariness of the solution referred to in the Introduction. It is already manifest in the solution of the skeleton equation. As we shall see, it persists into the solution of the full equation to which we now turn.

We have now to include K_2 which we can regard as a perturbation to the skeleton equation. The details are set out in the Appendix [Eqs. (A.18) et seq.]. Here, in the main text, we employ operator notation. We have the inhomogeneous problem

$$N_1 = (I - OK_2)^{-1} OB \quad (4.30)$$

and the homogeneous problem

$$N_2 = (I - OK_2)^{-1} N_2^{(0)} \quad (4.31)$$

Since we are operating with a Fredholm kernel OK_2 , the resolvent kernel $(I - OK_2)^{-1}$ exists provided OK_2 does not have an eigenvector with eigenvalue one. Suppose this to be so. Then Eq. (2.16) has solutions of the same structure as does the skeleton equation.

$$N(v) = N_1(v) + \lambda N_2(v) \quad (4.32)$$

Here, $N_1(v)$ is the iterative solution of Eq. (2.16) and has the asymptotic form

$$N_1(v) \sim \text{const } v^{-1 + \delta_A/\pi} \quad (4.33)$$

The eigenfunction $N_2(v)$ has the asymptotic behaviour

$$N_2(v) \sim \text{const } v^{-\delta_A/\pi} + \text{const}' v^{-1 + \delta_A/\pi} \quad (4.34)$$

We shall presently, in Section V, discuss the physical meaning of (4.32). First, we discuss the exceptional case referred to above where OK_2 does have the eigenvalue one. We have not demonstrated that this can ever happen. However, let us suppose that it can and explore the consequences. As we shall see, they could be important.

The necessary and sufficient condition for a solution to (4.30) to exist when the kernel has eigenvalue one is for the inhomogeneous term to be orthogonal to all the eigenfunctions of the adjoint homogeneous equation ¹⁹⁾. We can arrange for this to occur by taking a suitable linear combination of (4.30) and (4.31). We denote the eigenfunction of OK_2 by $|N_0\rangle$ and that of its adjoint by $\langle N_0|$. (Since OK_2 is not a symmetric operator, its adjoint will have distinct eigenfunctions corresponding to a given eigenvalue. To make the formulae easy to read we use the bra and ket notation.) We therefore have

$$\langle N_0|OK_2 = \langle N_0| \quad (4.35)$$

$$OK_2|N_0\rangle = |N_0\rangle \quad (4.36)$$

We can now proceed to write down our solution. It is

$$|N_1^s\rangle = (I - OK_2)^{-1} [|OB\rangle - (\langle N_0|OB\rangle / \langle N_0|N_2^{(0)}\rangle) |N_2^{(0)}\rangle] \quad (4.37)$$

This is a particular solution. We can always add an arbitrary multiple $|N_0\rangle$ to give the general solution

$$|N\rangle = |N_1^s\rangle + \lambda |N_0\rangle \quad (4.38)$$

Again we have a one-parameter ambiguity but this time the two terms have changed roles in respect of their asymptotic behaviour. In this case, all solutions have the Type II behaviour and all contain a resonance.

If only we could show that such situations veritably occurred then we would have a possible new model for vector bootstraps. Such a model with its very special criterion for choice of coupling constant would certainly be safe against the criticism based on Lagrangian field theory alluded to in the Introduction. There would be difficulties in the arbitrariness remaining in the solution (4.38) (choice of the parameter λ). However, there is no point in taking the discussion further until the crucial question of the existence of these special kinds of left-hand cuts is established.

V. DISCUSSION

Having elaborated the mathematical details, we now proceed to discuss the physical interpretation and sketch how the present work should be generalized.

First, we discuss the one parameter ambiguity in the general case. We have solutions of the form of Eq. (4.32)

$$N(\nu) = N_1(\nu) + \lambda N_2(\nu) \quad (5.1)$$

and, correspondingly,

$$D(\nu) = D_1(\nu) + \lambda D_2(\nu) \quad (5.2)$$

where $D_1(\nu)$ is related to $N_1(\nu)$ through Eq. (2.15) and

$$D_2(\nu) = -\frac{\nu}{\pi} \int_0^\infty d\nu' \sqrt{\frac{\nu'}{\nu'+1}} \frac{N_2(\nu')}{\nu' - \nu} \quad (5.3)$$

We recall that $D_1(\nu) \sim \text{const.} (-\nu)^{\delta_{A/\pi}}$, $D_2(\nu) \sim \text{const.} (-\nu)^{1-\delta_{A/\pi}}$ and the phase shift δ is $\arg D$. What are the requirements on λ for generating physical solutions $f(\nu) = N(\nu)/D(\nu)$? Firstly, it is evident that λ must be real. Secondly, that for λ negative, a rapidly falling phase shift can result which may contradict causality²⁰⁾. For λ positive, a physical solution should result. Its phase shift must necessarily rise through $\pi/2$ for some positive ν so the solution will contain a resonance. We thus have a semi-elementary particle which can have arbitrary mass over some continuous range but whose width or coupling constant is determined once the mass is given. Following a fashion, we will term such particles quasons²¹⁾. One can also introduce ordinary CDD poles by making the usual modification to Eqs. (2.14) and (2.15) (see Refs. 5) and 6)). One could then extend Chew and Frautschi's definition⁵⁾ of the "potential" solution to the present case by requiring that there be no CDD poles and no quason. Conventional bootstrapping should presumably concern itself with the potential solution.

Perhaps, however, the vector mesons of nature are quasons²²⁾. To entertain this idea involves a partial retreat from the full bootstrap programme but a significant portion remains. Thus, to make self-consistency calculations for a vector meson (say the ρ) one would start with one parameter, in practice probably the mass, as given and then seek to determine the other parameter (the width) by imposing crossing symmetry on the solution. Such calculations will be attempted. One will then be able to see whether the quason hypothesis is capable of achieving more realistic widths than conventional bootstrapping. The pseudovector resonances do have rather narrow widths compared to manifestly dynamic entities such as the $N^*(1238)$ and dynamical explanations for them always involve substantial short-range forces²³⁾. There is certainly a case for considering new mechanisms. Of course, quasons can only live in the special environment provided by marginally singular forces. If they are to be invoked as a model for physical particles, it is essential to show in the full crossing symmetric theory that, however higher angular momentum exchanges may fade away on the far left cut, some vector exchange remains.

The present methods should be extended to include inelasticity. In a one channel formalism, this should not involve too much change since, experimentally, partial wave cross-sections appear to tend to constants at infinite energy. Indeed, the Pomeron, itself a vector particle, was invoked to explain this constancy. The problem should also be studied in a many channel formalism.

We have confined the discussion to the case of $\ell = 1$ in the direct channel. This was because we had the bootstrap question in mind but it is not an important restriction. What is important is the angular momentum of the exchanged systems. This could obviously be generalized within the present framework provided the vector exchanges remain dominant at $\sqrt{s} = -\infty$.

From the point of view of applications, one must of course remember that it is not strictly correct to identify the left cut of a Born approximation from particle exchange with the left cut of the corresponding scattering amplitude. This is very transparent in the pseudopotential approach²⁴⁾. For $\ell > 1$, to make the identification is not even internally consistent within a partial wave calculation since the correct threshold behaviour is not then obtained²⁵⁾.

To sum up, we have shown that for scattering problems governed by the exchange of vector mesons there is a new one-parameter arbitrariness in addition to the ordinary CDD ambiguity. We have interpreted this arbitrariness as due to the presence of quasi-elementary particles (quasons) and conjectured that the vector bosons might belong to this category. In addition, we have investigated the hypothesis that special left-hand cuts might exist such that all solutions contain a quason. It is hoped to take these questions further in subsequent publications.

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A P P E N D I X

Details Concerning the Integral Equations

We first discuss the solution of (4.18). The Fourier representation of the kernels (4.17) is easily shown to be

$$\hat{K}_1(t'-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{e^{ik(t-t')}}{\cosh^2 \pi R} \quad (\text{A.1})$$

We can now apply Fourier inversion to the Eq. (4.17) to obtain

$$F_-(k) + F_+(k) = \frac{A}{\cosh^2 \pi R} F_-(k) + F_0(k) \quad (\text{A.2})$$

Substituting $A = \cos^2 \pi \alpha$ [Eq. (4.23)] this can be rewritten as

$$F_-(k) \left[1 - \frac{\cos^2 \pi \alpha}{\cosh^2 \pi R} \right] + F_+(k) = F_0(k) \quad (\text{A.3})$$

One can then easily show that

$$\begin{aligned} 1 - \cosh^2 \pi \alpha / \cosh^2 \pi R &= \frac{\Gamma(\frac{1}{2} + iR)^2 \Gamma(\frac{1}{2} - iR)^2}{\Gamma(1 + iR - \alpha) \Gamma(iR + \alpha) \Gamma(1 - iR - \alpha) \Gamma(-iR + \alpha)} \\ &\equiv \Phi_+(R, \alpha) / \Phi_-(R, \alpha) \end{aligned} \quad (\text{A.4})$$

[see the definitions (4.21) (4.22)]

The construction given in (4.20) then yields a form for $F_{\pm}(k)$ with no singularities in the upper half k -plane and appropriate boundedness so that the Fourier transform $F_{\pm}(t)$ has the required property of vanishing for positive t . More details are given in Ref. ¹⁷⁾. (See also Ref. ²⁶⁾). We thus have our inhomogeneous solution.

We obtain the solution to the homogeneous problem through the forms for $F_{\pm}(k, \alpha)$ and $F_{\pm}(k, \alpha)$ given in Eqs. (4.24) and (4.25). These expressions are related to those for $\Phi_{\pm}$ through

$$F_{\pm}(k, \alpha) = \Phi_{\pm}(k, \alpha) / (-ik - \alpha) \quad (\text{A.5})$$

The poles of $F_{\pm}(k, \alpha)$ in the k plane are at

$$k = \pm i\alpha - in \quad (n = 0, 1, 2, \dots) \quad (\text{A.6})$$

The asymptotic behaviour of F_{+} , F_{-} deduced from Stirling's formula is

$$F_{\pm}(k, \alpha) = O\left(\frac{1}{k}\right) \quad (\text{A.7})$$

A pole in $F_{\pm}(k)$ at $k = k_0$ contributes to the asymptotic behaviour for large t of $F_{\pm}(t)$ a term $e^{ik_0 t}$. Recalling the transformations (4.11) and (4.16) we read back to the corresponding behaviour in v

$$e^{ik_0 t} \rightarrow z^{ik_0 - 1/2} \rightarrow v^{-ik_0 - 1/2} \quad (\text{A.8})$$

with $k_0 = \pm i\alpha - n$ and $\alpha = \frac{1}{2} - \delta A/\pi$ we obtain terms in the solution

$$N_2(v) \sim v^{-\delta A/\pi - n}, \quad v^{-1 + \delta A/\pi - n} \quad (\text{A.9})$$

We now discuss the structure of the operator $O \equiv (I - AK_1)^{-1}$ in z -space [cf. Eq. (4.12)]. We first make the Fourier transform to t -space, picking out the contributions from the various poles of $F_-(k, \alpha)$. We obtain

$$\begin{aligned} F_-(t, \alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikt} F_-(k, \alpha) \\ &= [e^{\alpha t} F_1(t, \alpha) + e^{-\alpha t} F_2(t, \alpha)] \Theta(-t) \end{aligned} \quad (\text{A.10})$$

Here $F_1(t, \alpha)$ and $F_2(t, \alpha)$ are expressible as Taylor series in e^t , for example

$$F_1(t, \alpha) = \sum_{n=0}^{\infty} \frac{(e^t)^n \Gamma(-n-2\alpha) (-)^n}{\Gamma(-n-\alpha+1/2)^2 \Gamma(n+1)} \quad (\text{A.11})$$

Using the Stirling approximation for Γ , we deduce that as $t \rightarrow 0_-$, F_1 has the asymptotic behaviour

$$\begin{aligned} F_1(t, \alpha) &\sim \text{const.} \cdot \log(1 - e^t) \\ &\sim \text{const.} \cdot \log t \end{aligned} \quad (\text{A.12})$$

The logarithmic factor is cancelled by a similar factor in F_2 . We now introduce $\Phi_-(t, \alpha)$ defined by

$$\begin{aligned} \Phi_-(t, \alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikt} \Phi_-(k, \alpha) \\ &= \left(-\frac{d}{dt} - \alpha\right) F_-(t, \alpha) \end{aligned} \quad (\text{A.13})$$

We deduce that

$$\begin{aligned}\Phi_-(t, \alpha) &= (e^{\alpha t} \phi_1(e^t) + e^{(1-\alpha)t} \phi_2(e^t)) \Theta(-t) + C \delta(t) \\ &\equiv \mathcal{P}_-(e^t, \alpha)\end{aligned}\tag{A.14}$$

Here ϕ_1 and ϕ_2 are expressible as power series in e^t and C is a constant. We now take our explicit representation (4.21) for Θ and make the Fourier transforms to t -space. We use the relations

$$\Phi_+(k, \alpha) = 1 / \Phi_-(-k, \alpha)\tag{A.15}$$

and

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} dk' \frac{1}{k' - k - i\epsilon} e^{ik'\tau} = e^{ik\tau} \Theta(\tau)\tag{A.16}$$

We obtain

$$F_-^{(0)}(t) = \int_{-\infty}^0 dt' \int_{t'}^0 dt'' \Phi_-(t+t''-t') \Phi_-(t'') F_0(t')\tag{A.17}$$

Finally, translating into z variables

$$O(z, z') = \int_{z'}^1 \frac{dz''}{z''^{1/2} z'^{1/2} z''} \mathcal{P}_-\left(\frac{z z''}{z'}\right) \mathcal{P}_-(z'')\tag{A.18}$$

with $P_-(z)$ defined by (A.14). We note that the dominating power of $O(z, z')$ as $z \rightarrow 0$ is $z^{\alpha - \frac{1}{2}}$ and for $z' \rightarrow 0$ is $z'^{-\alpha}$. Hence, for $0 < \alpha < \frac{1}{2}$, O is an $\mathcal{L}^2$ kernel.

We are now ready to formulate the full equation with K_2 included. We first note that from (4.14) and (4.15) we have

$$K_2(z, z') = \frac{1}{2\pi} \sqrt{\frac{1-z'}{1+z'}} \left[\frac{(1-z)b(z) - (1-z')b(z')}{z' - z} \right] - \frac{A}{\pi^2} \frac{[\log z' - \log z]}{z' - z} \quad (\text{A.19})$$

Here

$$b(z) = -\frac{2A}{\pi} \log z + A'(1 + z \log z) + \dots \quad (\text{A.20})$$

We deduce that K_2 has always at least one more term in z' or z than K_1 and so like O is an $\mathcal{L}^2$ kernel. Therefore, the products OK_2 and K_2O are likewise $\mathcal{L}^2$ kernels [cf. Ref. 1a) p. 13].

The inhomogeneous Eq. (4.30) can now be written down explicitly as

$$n_1(z) = \int_0^1 dz' O(z, z') b(z') + \int_0^1 dz' \int_0^1 dz'' O(z, z') K_2(z', z'') n_1(z'') \quad (\text{A.21})$$

The homogeneous equation is similarly obtained by replacing the first term on the right-hand side of (A.21) by

$$n_2^{(0)}(z) = z^{-1/2} F_-(\log z, \alpha) \quad (\text{A.22})$$

and of course $n_1(z)$ by $n_2(z)$.

It is now evident that $n_1(z)$ retains the asymptotic behaviour at $z = 0$ imposed on it by the operator O ; similarly for $n_2(z)$ except for the extra term in $z^{-\alpha-\frac{1}{2}}$ from $n_2^{(0)}(z)$.

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