CM-P00057172

HELICITY REPRESENTATIONS FOR INHOMOGENEOUS
 $SL(6, \mathbb{C})$ GROUPS

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ABSTRACT

We generalize the concept of helicity operators known from the inhomogeneous Lorentz group to inhomogeneous $SL(6, \mathbb{C})$ groups. We make use of these operators to classify basis elements of unitary representations.

65/656/5--TH.551
6 April 1965

I. INTRODUCTION

As is well known the application of $SU(3)$ symmetry to meson-baryon scattering leads to a decomposition of the scattering amplitude into eight invariant contributions. On the other hand we can write down nine different trace expressions for the $SU(3)$ tensors. Similarly the Lorentz invariant S matrix element for scattering of spin-1 particles off spin- $\frac{1}{2}$ targets can be analyzed by means of invariant tensors made up of Dirac spinors and polarization vectors. In this case group theory shows that some of the invariant tensors are again redundant. These arguments indicate that we must expect a relativistic $SU(6)$ theory to contain more information than can be exhausted by decomposing the S matrix into invariant spinors. So far no model of relativistic $SU(6)$ invariance accounts for this fact.

One way of dealing with a theory of relativistic $SU(6)$ symmetry, which allows a unique decomposition of S matrix elements into contributions from irreducible parts, is to generalize the concept of helicity. It enables us to study from the outset only that amplitude which we are interested in so that a calculation of a huge number of invariant spinor expressions is no longer necessary.

In this paper we are concerned only with the construction of one-particle helicity states. The two-particle helicity states can be built from these in a well-known way. The inhomogeneous versions of $SL(6, C)$ to which we refer are the groups $G_<$ and $G_>$ introduced in an earlier paper¹⁾. They involve translations in a space $\{x^{ab}\}$ or a pair of spaces $\{x^{ab}\}$, $\{\tilde{x}_{ab}\}$ respectively. We can treat them in a completely equivalent manner by assuming that the two momentum operators p_{ab} , $\tilde{p}^{ab}$ obey the relation

$$p\tilde{p} = \tilde{p}p = 1$$

In the case of G , this equation means to hold only for a restricted class of representations, which we defined to correspond to the physical ones ¹⁾. We have normalized the momentum matrices for convenience. This normalization does not alter the operator properties of p and $\tilde{p}$.

II. THE LITTLE GROUP AND ITS GENERATORS

In this section we repeat the notations and some of the results contained in Ref. ²⁾.

A group element S of $SL(6, C)$ can be defined by its transformation property if applied to a six component "spinor" ψ^a :

$$\psi'^a = S^a_b \psi^b$$

If S is infinitesimal we write

$$S^a_b = \delta^a_b + i \gamma^A_B E^B_{A'} \psi^a_{b'}$$

where E^B_A is a 6×6 matrix of the form

$$A \left(\begin{array}{c|c|c} & B & \\ \hline & 0 & 0 \\ \hline & 0 & 1 & 0 \\ \hline & 0 & 0 & 0 \end{array} \right)$$

and γ^A_B is a complex number satisfying

$$\gamma^A_A = 0$$

The complex conjugate transformation can then be defined by

$$S^*{}_{\dot{b}}{}^{\dot{a}} = \delta_{\dot{b}}{}^{\dot{a}} - i \gamma_B{}^{\dot{A}} E_A{}^{\dot{B}}{}_{\dot{b}}{}^{\dot{a}}$$

with

$$\gamma_{\dot{B}}{}^{\dot{A}} = \overline{(\gamma^A{}_B)}$$

and

$$E_{\dot{A}}{}^{\dot{B}} = (E^B{}_A)^*$$

In order to parametrize a transformation S we use $\gamma^A{}_B$, $\gamma_{\dot{B}}{}^{\dot{A}}$ and treat $E^E{}_A$ and $E_{\dot{A}}{}^{\dot{B}}$ as independent generators. In the double spinor representation

$$\Psi = \begin{pmatrix} \psi^a \\ \psi_a \end{pmatrix}$$

an infinitesimal transformation has the form

$$\begin{pmatrix} 1 + i \gamma^A{}_B E^B{}_A & 0 \\ 0 & 1 + i \gamma_{\dot{A}}{}^{\dot{B}} E_{\dot{B}}{}^{\dot{A}} \end{pmatrix}$$

because ψ_a transforms contragrediently to $\psi^{\dot{a}}$ ^{*)}.

*) Note the slight difference in definition as compared with Ref. 2).

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Introducing the generators p_{ab} of translations in the space $\{x^{ab}\}$ we get the commutation relation:

$$[E^A_B, p_{cd}] = + p_{cB} \delta^A_d,$$

$$[E^{\dot{A}}_{\dot{B}}, p_{ab}] = + p_{b\dot{B}} \delta^{\dot{A}}_{\dot{a}},$$

$$[E^A_B, \tilde{p}^{cd}] = -\tilde{p}^{Ad} \delta^c_B,$$

$$[E^{\dot{A}}_{\dot{B}}, \tilde{p}^{ab}] = -\tilde{p}^{a\dot{B}} \delta^{\dot{A}}_{\dot{b}}.$$

We introduce the notion $\underline{\text{[see Ref. 2]}}$ of the general Pauli-Lubanski operators

$$W_{AB} = E^C_B p_{Ac} - E^{\dot{C}}_A p_{\dot{C}B}$$

and get

$$[W_{AB}, p_{cd}] = [W_{AB}, \tilde{p}^{cd}] = 0.$$

If applied to a momentum eigenstate, the operators W_{AB} generate the corresponding little group. In the case where the eigenvalues of p are positive this little group is an $SU(6)$ group.

To characterize a state we can therefore use the eigenvalues of:

- 1) the momentum operator p_{ab} ;
- 2) the operators $\Delta_2, \Delta_3, \dots, \Delta_6$ defined by ²⁾

$$\begin{aligned}
\Delta_2 &= \frac{1}{6!} \varepsilon^{a_1 \dots a_6} \varepsilon^{b_1 \dots b_6} W_{a_1 b_1} W_{a_2 b_2} p_{a_3 b_3} p_{a_4 b_4} p_{a_5 b_5} p_{a_6 b_6} \\
\Delta_3 &= \frac{1}{6!} \varepsilon^{a_1 \dots a_6} \varepsilon^{b_1 \dots b_6} W_{a_1 b_1} W_{a_2 b_2} W_{a_3 b_3} p_{a_4 b_4} p_{a_5 b_5} p_{a_6 b_6} \\
&\vdots \\
\Delta_6 &= \frac{1}{6!} \varepsilon^{a_1 \dots a_6} \varepsilon^{b_1 \dots b_6} W_{a_1 b_1} W_{a_2 b_2} W_{a_3 b_3} \dots W_{a_6 b_6}
\end{aligned}$$

These operators commute with all the generators of $SL(6, C)$.

- 3) five linear combinations of W_{AB} ;
- 4) a set of operators of second and higher order in W_{AB} which characterize representations of subgroups of the little group. We call these operators the Σ_i operators (Σ_i , $i = 1, \dots, 10$).

In the rest system of p_{ab} the operators Δ_i can easily be rewritten using traces of powers of W :

$$\begin{aligned}
\frac{\Delta_n}{(6-n)!} &= (n-1) \sum_{m=2}^{n-2} (-1)^{m+1} \text{Sp}(W^m) \frac{\Delta_{n-m}}{(6+m-n)!} \\
&\quad + (-1)^{n+1} (n-1)! \text{Sp}(W^n).
\end{aligned}$$

with

$$\Delta_0 = \Delta_1 = 0, \quad \text{Sp}(W^2) = \sum_{AB} W_{AB} W_{BA}$$

We shall prefer another equivalent set of operators $D_2, D_3, \dots, D_6$ depending linearly on the set Δ_i and defined by

$$D_i = W_{A_1 B_1} W_{A_2 B_2} \dots W_{A_i B_i} \tilde{p}^{B_1 A_1} \tilde{p}^{B_2 A_2} \dots \tilde{p}^{B_i A_i}$$

In the rest system we find

$$D_0 = D_1 = 0, \quad D_i = S_{ip}(w^i),$$

so that Δ_n and D_n are connected by the recursion relation given above.

III. THE HELICITY OPERATORS

The operators Δ_i , resp. D_i , are $SL(6, C)$ invariant. But we cannot expect that the remaining sets of operators 3) and 4) can be defined in an $SL(6, C)$ invariant way. Indeed the Lorentz group possesses only one operator D_2 , the Casimir operator of the spin of the particle, and one operator of set 3) which can be chosen to be the helicity. The set 4) is empty. The Casimir operator of the spin is Lorentz invariant. The helicity operator defined by ³⁾

$$\Lambda = \vec{w} \cdot \vec{p} \cdot \frac{p^0}{|\vec{p}|} - w^0 |\vec{p}|$$

is invariant under rotations and pure Lorentz transformations in the direction of the momentum $\vec{p}$. Generalizing this concept we define the following subgroups of $SL(6, C)$:

- a) The group $SU(6)$, which we get if γ^A_B is real. This group must not be confused with the little group which in terms of the γ^A_B , $\gamma^{\dot{A}}_{\dot{B}}$ is defined by

$$\gamma^{\dot{A}}_A p^{\dot{B}}_{\dot{C}} = \gamma^B_C p_{AB}$$

- b) Transformations in direction $\tilde{p}$. We define them to consist of diagonal positive definite matrices R in the system in which $\tilde{p}$ is also diagonal. This system can be reached by a transformation belonging to the subgroup a). We call this group $R(p)$.

Then we postulate that the helicity operators depend linearly on W_{AB} (they form the set 3) of operators) and are invariant under $SU(6)$ and $R(p)$.

We shall find that we can write them as

$$\Lambda(H_i) = W_{AB} b^{BA}(H_i),$$

where H_i are diagonal numerical 6×6 matrices

$$\{H_i\} = \{S_3, J_3, Y, S_3 J_3, S_3 Y\}$$

In the case of the Lorentz group we shall prove that $\Lambda(S_3)$ coincides with the usual definition. Let us assume the form for $\Lambda(H_i)$ as an ansatz and try to solve it for $b^{BA}(H_i)$. We decompose b^{BA} as

$$b = \sum_{i=0}^5 B_i \tilde{p}^i$$

We satisfy the invariance postulates by the following statements:

- α) The coefficients B_i depend only on the eigenvalues of $\tilde{p}$: $\tilde{\pi}_i$, and on numerical factors. This guarantees invariance under $SU(6)$.

- β) We impose the condition

$$\sum_{i=0}^5 B_i R \tilde{p}^i R = \sum B'_i (R \tilde{p} R)^i$$

$$B_i = B_i(\pi_j), \quad B'_i = B_i(\pi'_j)$$

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with

$$\pi_j' = r_j^2 \pi_j$$

where r_j is the j -th eigenvalue of R .

The problem of solving this equation is nearly trivial. Writing it out in components ($\tilde{p}$ diagonal) and using

$$r_n^2 = \frac{\pi_n'}{\pi_n}$$

we get

$$\frac{1}{\pi_n} \sum_{i=0}^5 B_i(\pi_j) \pi_n^i = \frac{1}{\pi_n'} \sum_{i=0}^5 B_i(\pi_j') \pi_n'^i$$

Both sides can therefore not depend on R , and with the numerical matrix H_i we get

$$\sum_{i=0}^5 B_i(\pi_j, H_e) \tilde{p}^i = H_e \tilde{p} = \tilde{p} H_e$$

If we are interested in the explicit form of B_i we need only solve these linear equations and get

$$B_i = D^{-1} \begin{matrix} \text{ith column} \\ \downarrow \\ \begin{vmatrix} 1 & \pi_1 & \dots & \pi_1^5 h_1 & \dots & \pi_1^5 \\ 1 & \pi_2 & & \pi_2^5 h_2 & & \vdots \\ 1 & \pi_3 & & \pi_3^5 h_3 & & \\ 1 & \pi_4 & & \pi_4^5 h_4 & & \\ 1 & \pi_5 & & \pi_5^5 h_5 & & \\ 1 & \pi_6 & \dots & \pi_6^5 h_6 & \dots & \pi_6^5 \end{vmatrix} \end{matrix}$$

D is Vandermonde's determinant of the eigenvalues of $\tilde{p}$, h_i are the eigenvalues of H .

If we would add the identity matrix to the set $\{H_i\}$ we would get

$$B_i(1) = \delta_{i1}$$

and

$$\Lambda(1) = W_{AB} \tilde{p}^{BA} = 0.$$

This last equation is a generalization of the relation

$$p^\mu W_\mu = 0.$$

which is known to hold in the case of the Lorentz group.

Using the same type of matrix $b(H_i)$ we see that we can define also the $\sum$ operators in a way invariant under $SU(6)$ and $R(p)$. We need only substitute the b matrices for the $\tilde{p}$ matrices in the operators D_i and then take a linearly independent set of them. We do not construct these operators explicitly here. Calling the eigenvalues of $\Lambda(H_i)$: $\lambda(H_i)$ and those of the D_i and $\sum_i$, d_i and σ_i respectively, we have constructed basis elements of the form

$$|p_{ab}, d_i, \sigma_i, \lambda(H_i)\rangle$$

Finally we consider $\Lambda(S_3)$ in the case of the Lorentz group

$$\Lambda(S_3) = W_{\alpha\beta} (\frac{1}{2} \sigma^3 \tilde{p})^{\beta\alpha}$$

where $\tilde{p}$ is diagonal

$$\tilde{p} = \begin{pmatrix} p^0 + |\vec{p}| & 0 \\ 0 & p^0 - |\vec{p}| \end{pmatrix}$$

We make use of

$$\sigma_\mu^{\alpha\beta} = \tilde{\sigma}_{\mu\dot{\alpha}\beta}, \quad \text{Sp}(\sigma_\mu \tilde{\sigma}_\nu) = -2g_{\mu\nu}$$

$$W_{\dot{\alpha}\beta} = W_\mu \sigma^\mu_{\dot{\alpha}\beta}, \quad \tilde{p} = p^0 \sigma^0 + |\vec{p}| \sigma^3$$

and get

$$\Lambda(S_3) = -W_0 |\vec{p}| + \frac{\vec{W} \cdot \vec{p}}{|\vec{p}|} p^0.$$

IV. AN ADDITIONAL RELATION FOR THE HELICITY OPERATORS

In the case of the Lorentz group we know that the helicity operator can be written as

$$\Lambda = \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

Only the spin part $\vec{S}$ of the rotation operator is contained in Λ . We derive the corresponding equation for $\Lambda(H_1)$ in $SL(6, \mathbb{C})$.

The generator of an $SL(6, C)$ transformation decomposes into spin and orbital parts like

$$L^A_B = F^A_B + x^{A\dot{C}} p_{\dot{C}B},$$

$$L_{\dot{A}}^{\dot{B}} = F_{\dot{A}}^{\dot{B}} + x^{C\dot{B}} p_{\dot{A}C}$$

In the case of the group G_7 we have to add terms depending on $\tilde{x}$, $\tilde{p}$. F is the spin part. Applied to a representation $\Psi^a(x)$ we have

$$F^A_B = E^A_B,$$

$$F_{\dot{A}}^{\dot{B}} = 0$$

Inserting the operators L^A_B and $L_{\dot{A}}^{\dot{B}}$ into the expression $W_{\dot{A}B}$ we are left with the spin contribution

$$W_{\dot{A}B} = F^C_B p_{\dot{A}C} - F_{\dot{A}}^{\dot{C}} p_{\dot{C}B}.$$

Multiplying with $b^{\dot{B}\dot{A}}$ ($\tilde{p}$ is now diagonal)

$$b^{\dot{B}\dot{A}} = \tilde{p}^{\dot{B}\dot{D}} H_{\dot{D}0}^{\dot{A}} = H_{\dot{D}0}^{\dot{B}} \tilde{p}^{\dot{D}\dot{A}}$$

we get

$$\Lambda(H_i) = F^C_B H_{\dot{C}0}^B - F_{\dot{A}}^{\dot{C}} H_{\dot{C}0}^{\dot{A}}$$

This is the desired result. It shows also that under parity reflections $\Lambda(H_i)$ changes sign.

R E F E R E N C E S

- 1) W. Rühl, CERN preprint TH.505, December 1964, to be published in Nuovo Cimento.
- 2) H. Bacry and A. Kihlberg, CERN preprint TH. 532, February 1965.
- 3) See, e.g., R. Hagedorn, Selected Topics on Scattering Theory, MPI f. Physik, München 1963, equation IV, 358 and the discussion following it.