ON THE CONSTRUCTION OF HIGHER SYMMETRY GROUPS

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A B S T R A C T

After defining the translation operators as the generators of a certain invariant subgroup of a higher symmetry group, some necessary conditions for them are derived which concern their minimal number and possible commutation relations.

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I. INTRODUCTION

Recently a series of papers has been published with the aim of pointing out that in some respects it might be possible to combine internal symmetry groups and space symmetry groups into higher groups. One of the models proposed¹⁻⁹⁾ is the $SU(6)$ group, which is a generalization of a similar model used by Wigner in his theory of nuclear spectra¹⁰⁾. Further models have been put forward, e.g., by Barut¹¹⁾, Kursunoglu¹²⁾, and Gardiner¹³⁾. The group of the latter author includes the complete inhomogeneous Lorentz group. On the other hand, $SU(6)$ does not contain this group but as claimed in Ref. 1) it is to be understood as the little group of a still unknown group which contains the Poincaré group as a subgroup.

In this paper we study the problem of constructing a higher symmetry group G which contains the Poincaré group and an internal symmetry group as subgroups. In order to do this, we introduce the concept of a certain invariant subgroup which we shall call the group of translations T . We define this subgroup T of G as a set of transformations of G with the property that

- a) no invariant subgroup of G different from T is contained in T , we say: T is **minimal** *),
- b) there exists a subgroup of T which is made up by the translations in space and time.

We shall make no assumptions concerning the commutation properties of T ; we even allow for the possibility that the complete group G is contained in its invariant subgroup T .

Because of this general definition of the translations, their physical interpretation is **difficult**. Only in the case when T is abelian can the group elements be understood as translations in an abstract linear vector space containing the Minkowski space as a subspace.

*) We always exclude from the discussion the trivial invariant subgroup containing only the identity element.

We shall show that there exists at least a certain number of independent infinitesimal translations (Section II) and that the group structure of T is of a special kind (Section III). The developments of Section III are based on a theorem which is proved in Section IV. We apply the results to the case that G contains $SU(6)$ as a subgroup. Our results consist in necessary conditions on the higher symmetry group G and its subgroup T , respectively. They neither prove the existence of a group G or disprove it. But in the case of $SU(6)$ invariance one conclusion can be drawn: if a group G exists, its structure is so complicated that its applicability to physics is unlikely.

II. THE MINIMAL DIMENSION OF THE TRANSLATION SUBGROUP

Throughout this paper we shall make use of the following assumptions.

- 1) There exists a Lie group G which contains the inhomogeneous Lorentz group and the internal symmetry group as subgroups.

This hypothesis already implies that if one of the space-time translation operators belongs to an invariant subalgebra of the Lie algebra of G , all four operators do.

According to our definition given in the introduction the group of translations T represents the smallest invariant subgroup of G which contains the subgroup of space-time translations. Given a group G which obeys assumption 1), an invariant subgroup T of this kind can be constructed in any case, because G is an invariant subgroup of G which contains the space-time translations and the intersection of all such invariant subgroups of G is therefore not empty.

Now there are two cases to be discussed:

- a) T contains an invariant subgroup R of G , $R \neq T$, which necessarily cannot involve any space-time translation;
- b) T does not possess any subgroup different from T which is invariant with respect to G .

In case a) we can show that the intersection of R with the homogeneous Lorentz group L is the identity element e . This can be proved in the following way: due to a simple theorem of group theory^{*)}, L is either a subgroup of R or the intersection $R \cap L$ is the identity element, because L is a simple group. Since no element of the group of physical

*) With respect to this, see the footnote on page 9.

a further irreducible representation of $SU(6)$ which supplies us with the fourth component for the super-singlet part of the representation 35. For simplicity we take the one-dimensional representation of $SU(6)$. The octet $(8,1)$ can serve as the fourth component of the octet $(8,3)$.

In this way we are led to the conclusion that there are at least 36 independent translation operators. If we assume now that the space-time translation operators are indeed contained in the 36-dimensional representation of $SU(6)$, we can identify them either with the singlet $(1,3) + (1,1)$ or with the $I = Y = 0$ component of the octet $(8,3) + (8,1)$ or with a linear combination of both. A further interpretation is hardly possible until we know anything about the commutation relations between the translation operators. But there is already one physical consequence: we can derive mass relations.

i) P_μ are assumed to transform as $(1,3) + (1,1)$ of $35 + 1$.

Then we have (in the notation of Ref. 4):

$$\begin{aligned} M^2 &= P_\mu P^\mu = M_1^{(1)} + M_{189}^{(1)} + M_{405}^{(1)} \\ &= A + B \cdot J(J+1) + C \cdot C_2^{(3)} \end{aligned}$$

with A, B, C depending on the Casimir operators of $SU(6)$ and $C_2^{(3)}$ the quadratic Casimir operator of $SU(3)$. This mass relation contains three unknown parameters, therefore it leads to a non-trivial condition only in the case of a representation of $SU(6)$, which reduces to at least four supermultiplets under $SU(3)$. For instance, let us consider the 70-dimensional representation. It reduces according to

$$70 > (1,2) + (8,2) + (10,2) + (8,4)$$

and we get

$$M^2(1,2) = M^2(8,2) - 6C$$

$$M^2(8,2) = M^2(10,2) - 6C$$

$$M^2(8,4) = M^2(8,2) + 3B .$$

There is an equal spacing between the spin- $1/2$ -supermultiplets.

ii) P_μ are assumed to transform as $(8,3) + (8,1)$ of $35 + 1$.

If we neglect the part transforming as a member of a 27-plet under $SU(3)$, the mass relation becomes identical to that one derived in Ref. 4), which reduces to the Gell-Mann - Okubo formula if applied to one supermultiplet.

We shall not consider here the case when the operators P_μ transform as a linear combination of a singlet and an octet.

One comment concerning the mass relations derived above should be made: they hold strongly within the limits of validity of the symmetry and are a consequence of the symmetry, not of the symmetry breaking. They come about because the space-time translations do not belong to the centre of G .

III. THE COMMUTATION RELATIONS OF THE TRANSLATION OPERATORS

The discussion of Section II showed that the subgroup of translations must have a minimal number of dimensions. In this section we make some statements about the structure of this subgroup. Taking both information together, we reduce the number of candidates for T to a countable set, the simplest examples of which can immediately be given. Nevertheless we shall not discuss here the question of whether these examples are applicable to elementary particle physics. Indeed, they are so complicated that it seems rather unlikely that we can ascribe to them any physical sense.

If we apply the general theorem proved in Section IV to our symmetry group we are led to the following alternative: the subgroup of translations is either abelian or it is the direct product of m simple Lie groups, which are all locally isomorphic to one another.

Let us consider the first alternative. The canonical coordinates of the first kind of T span a linear vector space in which the translations operate. The Minkowski space is a subspace. Now we introduce a pseudo-euclidean metric into this space. The adjoint group of G induces a group of transformations of this space which are linear and form an irreducible representation of G . This means that G/T is locally homeomorphic to a group of homogeneous linear transformations of this space which leaves no subspace invariant. The simplest case is obviously when G/T is locally isomorphic to the rotation group of this space. In the example of $SU(6)$, when we assume that the space-time translations belong to the representation $36 = 35 + 1$, G is then locally isomorphic to the inhomogeneous rotation group in 36 dimensions with the metric

$$g_{\mu\nu} g_{\eta\lambda}^{(\sigma)} \delta_{\sigma}^{\sigma'} \quad \left\{ \begin{array}{ll} \mu, \nu & : 0, 1, 2, 3 \\ \eta, \lambda & : 1, \sigma = 1 \\ \eta, \lambda & : 1, 2, 3 \dots 8, \sigma = 8 \\ \sigma, \sigma' & : 1, 8 \end{array} \right.$$

where $g_{\mu\nu}$ is the space-time metric, $g_{\eta\lambda}^{(\sigma)}$ is the metric which reduces the product of two octets ($\sigma = 8$) or two singlets ($\sigma = 1$) to a singlet.

Now let us look at the other alternative, which is much more complicated. T is not a compact Lie group. The same is true for its simple constituents. The complex form of T is therefore a direct product of m locally isomorphic semisimple groups which again can be decomposed into simple subgroups. The latter have real compact forms which are well known and up to a local isomorphism given by Cartan's theorem.

In this way, we have many possibilities for the group T . Their number is reduced if one takes into account the following two restrictions.

- a) T must contain at least a four-dimensional abelian subgroup, the group of space time translations,
- b) the dimension of T is bigger than a certain given number (result of Section II).

Let us discuss the consequence of restriction b) in the simple case, when T contains in its complex form only one simple subgroup. Then the possibilities for T are (up to a local isomorphism)

A_n with $n \geq 6$
 B_n with $n \geq 4$
 C_n with $n \geq 4$
 D_n with $n \geq 5$
 $F_4, E_6, E_7, E_8,$

where the real form of T is then a non-compact form of these real compact groups.

Now before we look at one of these examples, let us make the following statement^{*)}.

*) This theorem was pointed out to the author by Dr. Bacry.

If G contains a simple subgroup F [e.g. $SU(6)$, the homogeneous Lorentz group] then there are the following two possibilities: $F \cap T = F$ or $F \cap T = Z$, where Z is a central normal subgroup of F . The proof is so elementary that we shall not give it here, only the invariance of T is needed for it.

Take for instance B_4 . In general it has enough commuting elements in order to allow for a four-dimensional abelian subgroup and a few more commuting generators, which could be identified with the hypercharge, the third component of isospin, and the Casimir operator of isospin. But it is not large enough to contain $SU(6)$ as a subgroup and the abelian subgroup of space-time translations. This means that $SU(6)$ and T have as common elements only central elements of $SU(6)$. G must be much larger than B_4 .

We shall not go further in constructing a group G .

IV. PROOF OF THE THEOREM*)

First, we give the theorem in its general mathematical form.

If there is a Lie group G and an invariant subgroup T of G which does not contain any invariant subgroup of G which is different from T , then T is either abelian or a direct product of m locally isomorphic simple groups.

Proof:

Let us first state some simple implications of our assumptions.

- 1) Every element $g \in G$ induces an automorphism of T through $t' = gtg^{-1}$, $t, t' \in T$.
- 2) If U is an invariant subgroup of T , $U' = gUg^{-1}$ has the same property.

Proof of the invariance:

$$\begin{aligned} t \in T, \quad t g U g^{-1} t^{-1} &= g \cdot g^{-1} t g \cdot U \cdot g^{-1} t^{-1} g \cdot g^{-1} \\ &= g \cdot t' U t'^{-1} g^{-1} = g U g^{-1}. \end{aligned}$$

- 3) For every pair U, V of invariant subgroups of T

$$V \cup U \not\subset T$$

there exists at least one element $g \in G$ so that $V' = gVg^{-1}$,
 $V \cap U \neq V'$.

Proof:

Consider the set W of all the products of a finite number of factors gvg^{-1} , $g \in G$, $v \in V$. If statement 3) is wrong, U contains W , because it contains gVg^{-1} for every $g \in G$ and is a group. W is an abstract ^{†)} subgroup of U , invariant with respect to G .

*) The author is aware of the fact that this theorem might be contained in the mathematical literature.

†) That means: not necessarily topological. For this question, see L. Pontrjagin, Topological groups, Princeton 1958. We make use of Paragraph 18, H in the proof of our statement 3).

$$g\{g_1 v_1 g_1^{-1} g_2 v_2 g_2^{-1} \dots g_n v_n g_n^{-1}\}g^{-1} = (gg_1)v_1(gg_1)^{-1}(gg_2)v_2(gg_2)^{-1} \dots (gg_n)v_n(gg_n)^{-1} \in W.$$

If U contains W , the same holds for the closure $\bar{W}$, which is an invariant subgroup of G . Therefore $\bar{W} \subset U \cap T$. But this contradicts the definition of T .

Corollary: As a special case one can choose $U = V$ in 3).

4) The commutator subgroup R of T is either equal to T or equal to $\{e\}$.

Proof:

The subgroup R is invariant with respect to every automorphism of T . Therefore 4) is a direct consequence of 3).

Proof of the theorem:

Let us assume that T is not semisimple (first alternative). Then it contains a maximal solvable invariant subgroup V^* . If V is different from T , according to 3) there exists a $V' \cong V, V' \neq V \cap V'$. It is also solvable. But this contradicts V being maximal. Therefore, necessarily $T = V$, T is solvable, $R = \{e\}$. In other words, T is abelian.

If T is semisimple it is a direct product of m simple Lie groups. For simplicity we assume that they do not contain discrete invariant subgroups. Take one of them U_1 and let us assume $U_1 \neq T$. Then 3) shows that there exists a $U_1' \cong U_1$ so that $U_1 \cap U_1' \neq U_1$. But U_1 and U_1' are invariant subgroups of T [compare 2)]; their intersection is also invariant. But both are simple, this implies $U_1 \cap U_1' = \{e\}$. Therefore, T contains the direct product of U_1 and $U_2 = U_1'$ and both are isomorphic. We complete the proof by induction. Let us assume that T contains the direct product of $U_1 \otimes U_2 \dots \otimes U_n$, all the U_i being isomorphic,

*) See L. Pontrjagin, Topological groups, Princeton 1958, theorem 77.

but,

$$\bigcap_{\nu=1}^n U_{\nu}$$

still different from T . According to 3) there exists a g , so that

$$gU_1g^{-1} = U_1 \cap \bigcap_{\nu=1}^n U_{\nu} \neq U_1.$$

Then the intersection is an invariant subgroup of T , therefore of U_1 ; this is a simple group, and the intersection is necessarily $\{e\}$. That means:

$$\bigcap_{\nu=1}^{n+1} U_{\nu}, U_{n+1} = U_1,$$

is contained in T . Furthermore $U_{n+1} \cong U_1$. Because T is finite dimensional, this completes our proof.

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R E F E R E N C E S

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F I G U R E

Alternatives for the group T