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A RELATIVISTIC GENERALIZATION OF THE $SU(6)$ SYMMETRY GROUP

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A B S T R A C T

We study two different inhomogeneous versions of $SL(6, \mathbb{C})$. After a discussion of the representations of these groups we inquire into the possibility of introducing a parity transformation and of constructing currents which conserve the transversality of a field coupled to them.

I. INTRODUCTION

In this paper we are concerned with a group theoretical model which generalizes the $SU(6)$ symmetry group in such a way that the higher symmetry group contains the inhomogeneous Lorentz group and the group $SU(3)$ as subgroups and has the further property to possess $SU(6)$ as a little group. We introduce two versions of this group which we call $G_<$ and $G_>$.

The smaller group, $G_<$, is defined as a group of linear transformations of a real vector space of Hermitian 6×6 matrices x of the form

$$x' = S \times S^* + h$$

where S is an element of $SL(6, \mathbb{C})$, the unimodular complex linear group.

This group is known in this context since some time. It was mentioned in a paper by Sakita¹⁾. Recently, some results of a preliminary character concerning the application of this group to elementary particle physics have been published²⁾. In this paper we discuss the representations of the group $G_<$ and apply them to the problem of constructing covariant field equations and invariant multilinear forms.

One main problem remains unsolved in the model $G_<$, namely that of defining a parity transformation. It seems to be true that this can be done only by enlarging the invariant subgroup of translations to a larger one which includes also the translations in the dual space. We call this group $G_>$. A further difficulty of the group $G_<$ is connected with the notion of a conserved current which preserves a transversality condition imposed on a field coupled to this current. If we enlarge the translation subgroup and go over to the group $G_>$ this problem can also be solved.

With respect to the physical interpretation of the 36 translation operators of G we propose that for physical states, e.g., free one-particle states, all the components of the momentum vanish except the four components of the physical space-time momentum. This assumption means that we consider physical states as being in a rest system with respect to "isotopic motion". It implies that only the spinorial part of isotopic spin is observable for physical systems, the orbital contribution vanishes. In addition, the field functions have to obey the conventional equations, the Klein-Gordon and the Dirac equation if they describe a free physical particle. This assumption supplies us with the correct phase for the energy.

II. THE HOMOGENEOUS GROUP $SL(6, \mathbb{C})$

Let us call the vector space (with respect to the field of real numbers) of Hermitian 6×6 matrices $H(6, 6)$. It transforms under $S \in SL(6, \mathbb{C})$ as

$$h \xrightarrow{S} h' = S h S^*$$

Because of

$$\det S = \det S^* = 1$$

it follows that

$$\det h' = \det h$$

If $\det h \neq 0$, there exists a matrix $S \in SL(6, \mathbb{C})$ which transforms h into a diagonal form with eigenvalues $\pm \lambda$. Indeed, there exists a unitary transformation U , $\det U = 1$, which diagonalizes h , and a positive definite diagonal matrix R , $\det R = 1$, which normalizes the eigenvalues. We need only to choose $S = RU$. (We may define $\lambda = + \sqrt[6]{|\det h|}$.) The number of positive eigenvalues be n , the number of negative eigenvalues be m . We call the bracket

$$\left(\frac{1}{2}|n-m|, \text{sign}(n-m) \right)$$

the type of a certain matrix $h \in H(6, 6)$. This notation is certainly only convenient if $H(6, 6)$ can be decomposed into invariant classes with respect to $SL(6, \mathbb{C})$ each one belonging to a certain type and a set of exceptional elements with $\det h = 0$.

Indeed, we show that for every $S \in SL(6, \mathbb{C})$ the type of h , $\det h \neq 0$, is the same as the type of ShS^* .

Without loss of generality we may suppose that h is diagonal and has the eigenvalues ± 1 . Let us assume in addition that the type of $h' = ShS^*$ is different from the type of h for a certain S . Because of the connectedness of $SL(6, \mathbb{C})$ we can draw a continuous curve $S(t)$, so that $S(1) = S$, $S(0) = E$ is the identity. The identity preserves the type. Because the eigenvalues of $h'(t) = S(t)hS^*(t)$ are continuous functions of t , there exists a first point on the curve $S'(t')$, $0 < t' < 1$, where one eigenvalue changes sign, i.e., it vanishes. This means $\det h'(t') = 0$ in contradiction to

$$\det h'(t') = \det h = \pm 1.$$

The class of elements of $H(6,6)$ belonging to the type $(3, \pm)$ will be called "positive" and "negative timelike" respectively. The surfaces of the analytical manifold $H(6,6)$ made up of elements h with $\det h = 0$ may be named "generalized light cones".

We know already that $\det h$ is invariant with respect to transformations $S \in SL(6, \mathbb{C})$. There are no further invariants. To prove this take λ as a real number, E as the identity of $H(6,6)$ and $RU = S \in SL(6, \mathbb{C})$ where R is positive definite Hermitian with $\det R = 1$ and U is unitary. Then we have with $h' = ShS^*$

$$\det(h - \lambda E) = \det(h' - \lambda R^2)$$

In the system where R^2 is diagonal,

$$(R^2)_{ij} = r_i \delta_{ij}, \quad \prod_{i=1}^6 r_i = 1,$$

this means

$$\lambda^6 - \lambda^5 \sum_i h_{ii} + \lambda^4 \sum_{i,j} \frac{1}{2} \begin{vmatrix} h_{ii} & h_{ij} \\ h_{ji} & h_{jj} \end{vmatrix} - + \dots + \det h =$$

$$\lambda^6 - \lambda^5 \sum_i \frac{h'_{ii}}{\tau_i} + \lambda^4 \sum_{i,j} \frac{1}{2} \frac{\begin{vmatrix} h'_{ii} & h'_{ij} \\ h'_{ji} & h'_{jj} \end{vmatrix}}{\tau_i \tau_j} - + \dots + \det h'.$$

Another representation of $SL(6, \mathbb{C})$ is achieved in the following way. Decomposing the generators of $SL(6, \mathbb{C})$ into a real and an imaginary part

$$h = r + t$$

r and $-i \cdot t$ generate the group $SU(6)$ separately. We can therefore write symbolically

$$[r, r] = r,$$

$$[r, t] = t,$$

$$[t, t] = -r.$$

Taking any generator R of $SU(6)$, the same relations hold for

$$r = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \quad t = \begin{pmatrix} iR & 0 \\ 0 & -iR \end{pmatrix}.$$

III. REPRESENTATIONS OF $SL(6, \mathbb{C})$

The fundamental contravariant spinor representation is defined by

$$\psi'^a = S^a_b \psi^b.$$

Additionally, we introduce covariant spinors and the corresponding complex conjugate spinors

$$\begin{aligned}\psi'_a &= S^{-1\pi}{}_a{}^b \psi_b, \\ \psi'^{\dot{a}} &= \bar{S}^{\dot{a}}{}_b \psi^{\dot{b}}, \\ \psi'^{\dot{a}}_a &= S^{*-1}{}^{\dot{a}}{}_a{}^b \psi_b.\end{aligned}$$

Obviously $\psi_a \psi^a$ and $\psi_{\dot{a}} \psi^{\dot{a}}$ are invariant expressions. A spinor of higher degree may bear indices of all four kinds. It is an essential fact that lowering and raising of indices is possible only by means of the Levi-Civita symbol of six indices

$$\psi^a = \epsilon^{abcdef} \psi_{bcdef}.$$

In the case of spinors of second degree there are the following possibilities

$$h^{ab}, h_{ab}, h^{\dot{a}\dot{b}}, h_{\dot{a}\dot{b}}$$

symmetric of dimension 42,
antisymmetric of dimension 30.

$$h^a{}_b, h_a{}^b, h^{\dot{a}}{}_{\dot{b}}, h_{\dot{a}}{}^{\dot{b}}$$

traceless of dimension 70.

$$h^{\dot{a}}{}_b, h_a{}^{\dot{b}}, h_{\dot{a}}{}^b, h^{\dot{a}}{}_b$$

of dimension 72.

$$h^{\dot{a}b}, h^{\dot{a}\dot{b}}, h_{\dot{a}b}, h_{\dot{a}\dot{b}}$$

of dimension 36 if Hermitian or
antihermitian.

It is not at all possible to ascribe particle multiplets to these representations. Indeed, multiplets belong to representations of little groups. The superfluous components of one of the above representations, which becomes reducible under the little group, have to be removed by imposing covariant differential relations. This is a well-known fact for the Lorentz group where a spin-one particle can be described by a four-vector field, the fourth component of which is removed by the transversality condition.

The question of constructing such covariant differential relations, "transversality conditions", will be discussed in connection with covariant field equations.

IV. UNITARY REPRESENTATIONS OF THE INHOMOGENEOUS GROUP $G_<$

We make use of the well-known formalism to construct all the unitary representations of the group $G_<$. The composition law for two group elements (S_1, h_1) and (S_2, h_2) is

$$(S_2, h_2)(S_1, h_1) = (S_2 S_1, S_2 h_1 S_2^* + h_2)$$

The first step towards a construction of unitary representations consists in the statement that all the irreducible unitary representations of the invariant subgroup are given by

$$(\mathcal{U}(1, h)\Psi)(a) = e^{i \text{Sp}(ah)} \Psi(a)$$

"a" is a Hermitian 6×6 matrix.

Then we have to define an invariant measure in this space of eigenfunctions Ψ . The solution is

$$d\mu(a) = \prod_{ij} da_{ij} \delta(\det a - \alpha)$$

Generally a_{ij} is complex. Because a is Hermitian the whole expression is real. Invariance of the measure means invariance against transformations

$$a \rightarrow S^* a S, \quad S \in SL(6, \mathbb{C}).$$

We have therefore to show that the Jacobian of this transformation is one. A simple calculation gives

$$\det \left(\frac{\partial a'_{ij}}{\partial a_{ke}} \right) = \det(S^*_{ik} S_{ej}) = (\det S)^6 (\det S^*)^6 = 1.$$

As a next step, we split a given unitary representation in the following way

$$U(S, a) = Q(S) T(S)$$

where we define

$$(T(S) \psi)(a) = \psi(S^* a S)$$

In our metric T is unitary. As one can easily show $Q(S)$ and $U(1, h)$ commute, therefore we can diagonalize Q on the same basis

$$(Q(S) \psi)(a) = Q(S, a) \psi(a).$$

The composition law of the group gives

$$Q(S, a) Q(S', S^* a S) = Q(SS', a).$$

As a consequence, if S obeys

$$S^* a S = a$$

$Q(S, a)$ is a unitary representation of the corresponding subgroup of $SL(6, \mathbb{C})$. These subgroups are called little groups. They are determined by the type of "a". If "a" is timelike the little group is $SU(6)$. In the other non-exceptional cases ($\det a \neq 0$) the little groups are real non-compact forms of $SU(6)$ which leave invariant a pseudohermitian metric with n positive and m negative eigenvalues.

Resuming our results we can say that every unitary representation of the inhomogeneous group G is equivalent to the following one. The Hilbert space H decomposes into a direct integral

$$H = \int^{\oplus} d\mu(a) H(a),$$

and in the subspaces $H(a)$ the projections are determined by

$$\begin{aligned} (U(s, h) \psi)(a) &= e^{i \text{Sp}(ah)} Q(s, a) \psi(s^* a s), \\ Q(s, a) Q(s', s^* a s) &= Q(ss', a). \end{aligned}$$

Finally we remark that the spinor properties of "a" are defined as a_{bc} , whereas "h" transforms as h^{ab} . The infinitesimal translation operator bears the same indices as "a". Physical translations have the submatrix form

$$P = \begin{pmatrix} \hat{p} & 0 & 0 \\ 0 & \hat{p} & 0 \\ 0 & 0 & \hat{p} \end{pmatrix},$$

where we define

$$\hat{p} = p^\mu \sigma_\mu, \quad \sigma^k = \text{Pauli matrices}, \quad \sigma^0 = 1$$

and the two-spinor properties [corresponding to $SL(2, \mathbb{C})$] are defined as

$$\hat{p}_{\alpha\beta}$$

This representation justifies our earlier notations "timelike", "positive" and "negative". Spacelike p_μ correspond to the type (0), lightlike p_μ to a submanifold of the generalized light cones with three vanishing eigenvalues.

V. FIELD OPERATORS AND FIELD FUNCTIONS

We shall write down only some transformation formulae for field operators and field functions without trying to define commutation relations for the operators.

If we interpret the function $\Psi(a)$ introduced in the last Section as a one-particle wave function

$$\Psi(a) = \langle a | \Psi \rangle$$

with

$$|a\rangle = u^*(a) |0\rangle,$$

where the vacuum state $|0\rangle$ transforms as

$$U(s, h) |0\rangle = |0\rangle,$$

then the field operators have the property

$$\begin{aligned} U(s, h) u(a) U^{-1}(s, h) &= e^{-iSp(a'h)} Q^{-1}(s, a) u(a'), \\ U(s, h) u^*(a) U^{-1}(s, h) &= e^{+iSp(a'h)} u^*(a') Q^{-1*}(s, a), \end{aligned}$$

$$a' = S^{*-1} a S^{-1}.$$

Fourier transforming these field operators by

$$\Psi_{\pm}(x) = (2\pi)^{-\frac{35}{2}} \int e^{-iSp(ax)} u(a) \delta(\det a - \alpha) d^3a, \\ a \in (3, \pm)$$

we get

$$u(s, h) \psi(x) u^{-1}(s, h) = Q^{-1}(s, \alpha) \psi(sx s^* + h)$$

and for the corresponding classical fields

$$\psi(x) \xrightarrow{s, h} Q^{-1}(s, \alpha) \psi(sx s^* + h).$$

VI. FIELD EQUATIONS

Before discussing the problem of constructing multilinear invariants of a given set of fields we are concerned with field equations. The field equations are written as covariant differential equations for field functions which transform as a spinorial representation of $SL(6, \mathbb{C})$. This spinorial basis does not form a unitary orthogonal basis for the representation space of the inhomogeneous group. On the contrary it is well known that the field equations serve to connect a set of spinorial bases in such a manner that they can be transformed into an unitary orthogonal basis simultaneously.

One of the covariant differential operators is identical with the infinitesimal translation operator p_{ab} . The other one is defined by

$$\tilde{p}^{ab} p_{bc} = \det p \cdot \delta^a_c$$

There are no other possibilities as long as we do not enlarge the space of translations by including the dual space.

Because the determinant of p is of sixth order in the derivatives, $\tilde{p}$ is of fifth order, whereas p defined as a translation operator in a dual space would be only of first order. In the case of the Lorentz group this problem does not arise, for the Levi-Civita symbol allows to map both spaces onto another, isomorphically with respect to the vector operations in both spaces. Furthermore we have

$$p_{ab} \tilde{p}^{cd} = \det p \cdot \delta_a^c \delta_b^d$$

(A) The scalar field obeys the equation

$$(\det p - \kappa^6) \chi = 0$$

(B) The spinor field obeys

$$\tilde{p}^{ab} \psi_b = \kappa^5 \psi^a,$$

$$p_{ab} \psi^b = \kappa \psi_a$$

or the conjugate equations.

(C) There are many possibilities for the spinor fields of second degree

$$\tilde{p}^{ab} \psi_b^c = \kappa^5 \psi^{ac},$$

$$p_{ab} \psi^{bc} = \kappa \psi_a^c \quad (1)$$

In this case the spinors ψ^{ab} and ψ_a^b are subjected to irreducibility conditions. ψ^{ab} is either Hermitian or antihermitian, ψ_a^b is traceless.

These restrictions imply compatibility conditions

$$\tilde{p}^{ab} \psi_b^c = \pm \tilde{\psi}_b^a \tilde{p}^{bc}$$

$$p_{ab} \psi^{ba} = 0$$

The latter equation has the character of a transversality condition. Its contents will become clearer in the next Section. These conditions leave independent 35 components of the fields ψ^{ac}, ψ_a^c which just fill a representation of the little group

$$\tilde{p}^{ab} \psi_{bc} = \kappa^5 \psi^a_c,$$

$$p_{ab} \psi^b_c = \kappa \psi_{ac} \quad (2)$$

This case is similar to (1).

$$\tilde{p}^{ab} \psi_{bc} = x^5 \psi^a_c,$$

$$p_{ab} \psi^b_c = x \psi_{ac}. \quad (3)$$

The symmetry of ψ_{ab} implies

$$p_{ab} \psi^b_c = \pm p_{cb} \psi^b_a$$

$$\tilde{p}^{ab} \psi_{bc} = x^5 \psi^a_c, \quad (4)$$

$$p_{ab} \psi^{bc} = x \psi_a^c.$$

These equations are similar to (3).

All the other equations for spinors of second degree can be obtained by conjugating and transposing the four types given above.

We shall not discuss field equations for spinors of higher degree, because the number of cases increases very rapidly.

VII. INVARIANT MULTILINEAR FORMS

We call a product of a set of fields an invariant (density) if it is invariant with respect to $SL(6, C)$ and if all factors depend on the same point in co-ordinate space. We shall not distinguish between two invariant densities, if they can be transformed into another by integrating by parts and neglecting those terms which lead to surface integrals. The generalized Dirac equation given in Section VI. (B) showed that we need two spinors in order to arrive at a covariant spinor equation. The same is true if we want to construct a bilinear invariant from two spinors. Let

$$\psi = \begin{pmatrix} \psi_{\dot{a}} \\ \psi_a \end{pmatrix}$$

be a 12-component spinor ^{*)}. The two matrices

$$\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

can serve as metric matrices. The expressions

$$\bar{\psi}_1 \rho \psi_2 = \bar{\psi}_1^{\dot{a}} \psi_{2\dot{a}} + \bar{\psi}_{1a} \psi_2^a,$$

$$\bar{\psi}_1 \eta \psi_2 = i \bar{\psi}_1^{\dot{a}} \psi_{2\dot{a}} - i \bar{\psi}_{1a} \psi_2^a$$

are both invariant. We define as usually

$$\bar{\psi} \rho = \tilde{\psi}$$

*) With respect to this, see also the end of Section II.

and introduce the notation

$$\gamma_5 = \gamma_5 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then both invariants can be written as

$$\bar{\psi}_1 \psi_2, \quad \bar{\psi}_1 \gamma_5 \psi_2$$

respectively. A transformation $\psi \rightarrow \varphi \psi$ leaves the first invariant unaltered but changes the sign of the second. This leads us to the definition of a parity operation by applying φ

$$\varphi = \begin{pmatrix} 0 & \delta_{ab} \\ \delta_{ab} & 0 \end{pmatrix}.$$

Using this matrix notation, the differential operators in the field equations can be comprehended as

$$P = \begin{pmatrix} 0 & \frac{p_{ab}}{\kappa} \\ \frac{\tilde{p}_{ab}}{\kappa} & 0 \end{pmatrix}, \quad p^2 = 1,$$

and the field functions as

$$\psi = \begin{pmatrix} \psi_b^{(\dots)} \\ \psi_b^{(\dots)} \end{pmatrix}.$$

so that

$$p\psi = \psi$$

But it turns out that the field equations can in no way be made invariant under the parity operation. Indeed, we get

$$\rho P \rho = \begin{pmatrix} 0 & \tilde{\rho}_{ab} \\ p^{ab} & 0 \end{pmatrix}$$

which is not defined.

If we do not enlarge the translation space but study only the group G_+ , we have to expect that our results mix parity in a certain way.

Let us study the coupling of two spinor fields of first degree to a spinor field of second degree. In general there are the following four possibilities to form a spinor of second degree from two 12-component spinors, $\tilde{\psi}_1, \psi_2$

$$\tilde{\psi}_1^a \psi_{2b} \pm \tilde{\psi}_{1a} \psi_2^b \quad (a)$$

$$\tilde{\psi}_{1a} \psi_{2b} \pm \tilde{\psi}_1^a \psi_2^b \quad (b)$$

In the case (a) the product can be coupled to the spinors of second degree h_1 and h_2 in the following way

$$\tilde{\psi}_1^a \psi_{2b} h_{1a}^b + \tilde{\psi}_{1a} \psi_2^b h_2^a_b$$

Generally the fields h_1 and h_2 should represent the same field. Therefore, they must be interrelated. Let us assume

$$h_1 = \alpha \cdot h_2^*$$

According to the subsidiary conditions of Eqs. (1) and (2) we have

$$h_1 \rho = \rho h_1^*, \quad h_2 \tilde{\rho} = \tilde{\rho} h_2^*$$

which means

$$\alpha^6 h_1 = \rho h_1^* \tilde{\rho} = \alpha^* \rho h_2 \tilde{\rho} = \alpha^* \alpha^6 h_2^*$$

or that α is real.

Now the question arises whether the tracelessness of h_1 and h_2 is conserved through this form of coupling, in other words, whether the coupling is independent against a change

$$\begin{aligned} h_{1a}{}^b &\rightarrow h_{1a}{}^b + h_{1c}{}^c \delta_a{}^b \\ h_{2a}{}^b &\rightarrow h_{2a}{}^b + h_{2c}{}^c \delta_a{}^b \end{aligned}$$

It can easily be shown that this is not true. It should be added that there seems to be no other possibility to connect h_1 and h_2 , if the field equations are taken into account.

Let us study now the more interesting cases (b). We have

$$\bar{\Psi}_1 \alpha \Psi_2 h_1^{ab} + \bar{\Psi}_1 \alpha \Psi_2 h_2^{ab}$$

and interrelate h_1 and h_2 by

$$h_1 = \alpha \cdot \tilde{p} h_2 \tilde{p}.$$

If h_1 and h_2 are Hermitian, α can be shown to be real. Then we postulate that the transversality of h_1 and h_2 is to be conserved. This assumption can also be expressed in the following way. The auxiliary fields

$$g_1 = \frac{p}{\kappa} h_1$$

$$g_2 = \frac{\tilde{p}}{\kappa^5} h_2$$

defined by the field equations are traceless. Adding a trace term of the form

$$g_1^0 \delta_a^b, \quad g_2^0 \delta_a^b$$

to them, where g_1^0 and g_2^0 are related by

$$g_1^0 = \frac{p}{\kappa} h_1^0 = \alpha \kappa^5 h_2^0 \tilde{p} = \alpha \kappa^4 p g_2^0 \tilde{p} = \alpha \kappa^{10} g_2^0$$

the additional term induced in the invariant form must vanish

$$[\alpha \kappa^6 \bar{\Psi}_{1a} \Psi_{2b} \tilde{p}^{ab} + \bar{\Psi}_1^a \Psi_2^b p_{ab}] g_2^0 = 0$$

If we integrate by parts (K is the common mass of the Ψ fields) we get

$$(\alpha \kappa^6 K^5 - K)(\bar{\Psi}_{1a} \Psi_2^a - \bar{\Psi}_1^a \Psi_{2a}) + \text{further terms} = 0$$

The "further terms" appear because $\tilde{p}$ is of fifth order, they contain derivatives of $\bar{\Psi}_1$ multiplied with derivatives of Ψ_2 . These peculiar terms cannot be

made to vanish. It seems to be impossible to define a notion "conserved current" in connection with a conservation of transversality. Nevertheless, let us assume the special value

$$\alpha = K^{-4} \cdot \kappa^{-6}$$

Then the invariant can be written in the compact notation

$$\tilde{\Psi}_1 H \Psi_2$$

with

$$H = \begin{pmatrix} 0 & h \\ \left(\frac{\kappa^4}{K}\right) \frac{\tilde{p} h \tilde{p}}{\kappa^{10}} & 0 \end{pmatrix}, \quad h \equiv h_2.$$

To get a clearer insight in this expression, we transform it into a more convenient form. We write

$$h_{ab} = k^{\mu\nu} \sigma_{\mu\alpha\beta}$$

where the index pairs $(i, \alpha), (j, \beta)$ correspond to the indices a and b respectively. Then we introduce vector fields V_A^i (A is a polarization index) and scalar fields S_j^i by

$$k^\mu = \frac{p^\mu}{\kappa} \cdot S + \sum_{A=1}^3 n_A^\mu V_A$$

with

$$n_A^\mu p_\mu = 0, \quad n_A^\mu n_{B\mu} = -1.$$

p_μ is the positive timelike four momentum of the field h , which is now assumed to be free and physical. Then the Klein-Gordon equation

$$(p_\mu p^\mu - \kappa^2) k = 0$$

holds additionally. $\tilde{p}$ has the matrix form

$$\tilde{p} = (p_\mu p^\mu)^2 \begin{pmatrix} \tilde{p} & 0 & 0 \\ 0 & \tilde{p} & 0 \\ 0 & 0 & \tilde{p} \end{pmatrix}$$

with

$$\tilde{p}^{\alpha\beta} = (p^\mu \sigma_\mu)^{\alpha\beta} = (p^\mu \tilde{\sigma}_\mu)_{\alpha\beta},$$

$$\tilde{\sigma}_\mu = g_{\mu\mu} \sigma_\mu.$$

We get

$$\kappa^{-10} \tilde{p} h \tilde{p} = \frac{p \tilde{\sigma}}{\kappa} S - \sum_A (n_A \tilde{\sigma}) V_A$$

where all two-spinor indices have been lowered. The matrix H can then be written in terms of the matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma_\mu \gamma_5 = i \begin{pmatrix} 0 & -\sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}$$

If we suppose that the Ψ fields obey the Dirac equation with the masses K_1 and K_2 we get

$$\begin{aligned} & + \frac{1}{2} \left(1 - \left(\frac{\kappa}{K}\right)^4\right) \tilde{\Psi}_{1i} \gamma_\mu \Psi_2^j \sum_A n_A^\mu V_A^i{}_j \\ & + \frac{1}{2} i \left(1 + \left(\frac{\kappa}{K}\right)^4\right) \tilde{\Psi}_{1i} \gamma_\mu \gamma_5 \Psi_2^j \sum_A n_A^\mu V_A^i{}_j \\ & + \frac{K_1 - K_2}{2\kappa} \left(1 + \left(\frac{\kappa}{K}\right)^4\right) \tilde{\Psi}_{1i} \Psi_2^j S^i{}_j \\ & + i \frac{K_1 + K_2}{2\kappa} \left(1 - \left(\frac{\kappa}{K}\right)^4\right) \tilde{\Psi}_{1i} \gamma_5 \Psi_2^j S^i{}_j. \end{aligned}$$

The subsidiary condition

$$Sp(\tilde{p}h) = 0$$

reduces directly to

$$S^i_c = 0.$$

VIII. THE GROUP $G_>$

In this Section we discuss to some extent the problem of enlarging the group $G_<$ to the group $G_>$. We define the group $G_>$ as the group of transformations of a pair of mutual dual spaces $H(6,6)$ and $\tilde{H}(6,6)$. We write an element of the pair of spaces as

$$(x_1, \tilde{x}_2), \quad x_1 \in H(6,6), \quad \tilde{x}_2 \in \tilde{H}(6,6).$$

The group of transformations is defined by

$$\begin{aligned} x_1' &= S x_1 S^* + h_1, \\ \tilde{x}_2' &= S^{*-1} \tilde{x}_2 S^{-1} + \tilde{h}_2, \end{aligned}$$

with $h_1 \in H(6,6)$, $\tilde{h}_2 \in \tilde{H}(6,6)$, $S \in SL(6, \mathbb{C})$. A group element can be described by $(S, h_1, \tilde{h}_2)$. The representation space of the inhomogeneous group depends on two sets of variables

$$\psi(a_1, \tilde{a}_2),$$

and the measure is now

$$d\mu(a_1, \tilde{a}_2) = d\mu(a_1) d\mu(\tilde{a}_2).$$

The operators $Q(S, a_1, \tilde{a}_2)$ obey the relation

$$Q(S, a_1, \tilde{a}_2) \cdot Q(S', S^* a_1 S, S'^{-1} \tilde{a}_2 S'^{-1*}) = Q(SS', a_1, \tilde{a}_2)$$

The definition of little groups remains unaltered. They have to leave invariant simultaneously a pair of matrices $(a_1, \tilde{a}_2)$ in the following way

$$a_1 = S^* a_1 S, \quad \tilde{a}_2 = S^{-1} \tilde{a}_2 S^{-1*}.$$

For general pairs $(a_1, \tilde{a}_2)$ the little groups contain only the identity element. But for the following class of pairs we get the same little groups as for the group $G_<$. Let a_1 be equal to

$$a_1 = S_1^* a_1' S_1, \quad S_1 \in SL(6, \mathbb{C}),$$

where a_1' is diagonal with eigenvalues $\pm \lambda$. If and only if $\tilde{a}_2$ can be written as

$$\tilde{a}_2 = \mu \cdot S_1^{-1} a_1' S_1^{-1*}, \quad \mu \text{ real},$$

the little group belonging to the pair $(a_1, \tilde{a}_2)$ is the same little group as defined by the pair $(a_1, 0)$. Another way to express this relation between a_1 and $\tilde{a}_2$ is

$$\alpha_1 \tilde{\alpha}_2 \text{ is a real multiple of the identity.}$$

The class of representations of $G_>$ characterized in this way plays a particular rôle in physics (in our model). In the language of field equations they correspond to fields which obey a second order differential equation of the Klein-Gordon type

$$(p\tilde{p} - \kappa^2) \psi = 0.$$

From now on we are only concerned with these representations.

Physical momenta are described by one four-momentum vector p_μ :

$$p = \begin{pmatrix} \hat{p} & 0 & 0 \\ 0 & \hat{p} & 0 \\ 0 & 0 & \hat{p} \end{pmatrix}, \quad \hat{p}^{\alpha\beta} = p_\mu \sigma^\mu_{\alpha\beta},$$

$$\tilde{p} = \begin{pmatrix} \tilde{\hat{p}} & 0 & 0 \\ 0 & \tilde{\hat{p}} & 0 \\ 0 & 0 & \tilde{\hat{p}} \end{pmatrix}, \quad \tilde{p}^{\alpha\beta} = p_\mu \sigma^{\mu\alpha\beta} = p_\mu \tilde{\sigma}^{\mu}_{\alpha\beta}.$$

Obviously it holds $p\tilde{p} = \tilde{p}p = p^2 \cdot E$. The field equations given by (A), (B), (C.1-4) remain essentially unaltered. The operator $\tilde{p}$ has to be interpreted as a first order differential operator of the dual space. The fifth power of the mass can accordingly be replaced by the first. The parity transformation is now well defined. It interchanges the co-ordinates

$$(x_1, \tilde{x}_2) \rightarrow (x_2, \tilde{x}_1)$$

For physical momenta the correspondence to the usual parity reflexion is obvious,

$$\tilde{p}^{\alpha\beta} = (p\tilde{\sigma})_{\alpha\beta} \quad \text{is replaced by} \quad \hat{p}_{\alpha\beta} = (p\sigma)_{\alpha\beta} \quad \text{or} \quad p_\mu \quad \text{by} \quad g_{\mu\mu} p_\mu.$$

Let us now consider a parity conserving multilinear form, the example of two spinor fields of degree one coupled to a spinor field of degree two. We write the invariant coupling as

$$\bar{\Psi}_1 \psi_2 h_1^{ab} + \bar{\Psi}_1^a \psi_2^b h_2 ab.$$

We know that a parity transformation implies

$$\psi_a \rightarrow \left(\frac{\tilde{p}\psi}{\kappa} \right)^a$$

Analogously we transform the meson field

$$h_2 \rightarrow h_1 = \frac{\tilde{p} h_2 \tilde{p}}{\kappa^2}$$

In the case of the Lorentz group the relation between the fields h_1 and h_2 can be simplified. Using the fact that h_1 and h_2 belong to equivalent representations and p and $\tilde{p}$ are equivalent for the same reason, the derivatives can be contracted to a d'Alembertian operator. We get h_1 if we apply the Levi-Civita symbol to h_2 .

This is impossible in our case. The derivatives cannot be eliminated. They have a very important physical consequence. We show that a meson field h_2 reduces under $SL(2,0) \otimes SU(3)$ either to a scalar octet and an axial vector nonet or to a pseudoscalar octet and a vector nonet.

Indeed, the choice

$$h_1 = + \frac{\tilde{p} h_2 \tilde{p}}{\kappa^2}$$

implies that the transversality of the meson field is conserved. Introducing the same notations as in the foregoing Section we get the invariant coupling

$$\tilde{\Psi}_1 H \Psi_2,$$

$$H = \begin{pmatrix} 0 & \frac{p\sigma}{\kappa} S + \sum_B (n_B \sigma) A_B \\ \frac{p\tilde{\sigma}}{\kappa} S - \sum_B (n_B \tilde{\sigma}) A_B & 0 \end{pmatrix}$$

where A is an axial vector field and S a scalar field. If we change the relative sign between h_1 and h_2

$$h'_1 = - \frac{\tilde{p} h'_2 \tilde{p}}{\kappa^2}$$

the transversality of the meson field is no longer conserved. We get

$$\tilde{\Psi}_1 H' \Psi_2,$$

$$H' = \begin{pmatrix} 0 & + \frac{p\sigma}{\kappa} P + \sum_A (n_A \sigma) V_A \\ - \frac{p\tilde{\sigma}}{\kappa} P + \sum_A (n_A \tilde{\sigma}) V_A & 0 \end{pmatrix}$$

P is a pseudoscalar and V a vector field. If we make use of γ matrices and assume that the Dirac equation is valid for the fields ψ_1 and ψ_2 with masses K_1 and K_2 respectively, we can write these invariants as

$$+i \sum_B \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2^j n_B^\mu A_{Bj}^i + \frac{K_1 - K_2}{\kappa} \bar{\psi}_1 \psi_2^j S_j^i$$

and

$$+ \sum_B \bar{\psi}_1 \gamma_\mu \psi_2^j n_B^\mu V_{Bj}^i + i \frac{K_1 + K_2}{\kappa} \bar{\psi}_1 \gamma_5 \psi_2^j P_j^i.$$

It seems to be worth while to study the model of the group G_7 with respect to the meson-baryon coupling. This will be done in a forthcoming paper.

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