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POLARIZATION CONTRIBUTION TO THE HYDROGEN HYPERFINE STRUCTURE AND
PHENOMENOLOGY OF THE ELECTRON-PROTON INTERACTION

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A B S T R A C T

In order to understand the significance of the well-known discrepancy of 36 ppm in the hydrogen hyperfine structure we have calculated the polarization correction using an isobar model for the proton, and compared the sensitivity to electron-proton forces of the analysis of the hyperfine splitting with that of the Lamb shift and of electron-proton scattering. A methodological innovation of the calculation is the avoidance of the Rarita-Schwinger formalism for spin $3/2$ by the construction of a covariant analytic amplitude from the familiar non-relativistic partial wave amplitude. It is found that the $3\text{-}\bar{3}$ isobar, and by extension higher resonances, are more than an order of magnitude too small to be significant and that the discrepancy is genuine. We consider the introduction of a phenomenological electron-proton interaction term, representing unknown or neglected forces, to account for the discrepancy, and find that this would not necessarily lead to a contradiction with Lamb shift or electron-proton scattering data. We point out that, whatever the cause of the discrepancy, it is likely to be proportional, in percent, to the electron mass, so that it would be roughly 200 times greater in μ mesic hydrogen (μ^- -p) than in hydrogen. We strongly encourage a measurement of the hyperfine splitting of μ mesic hydrogen, of sufficient accuracy to determine whether the discrepancy is common to both systems and proportional to the lepton mass.

1. INTRODUCTION AND PHENOMENOLOGY OF ELECTRON-PROTON INTERACTION

Prompted by the well-known discrepancy in the hydrogen hyperfine separation (hfs) we report here a calculation of the contribution to that quantity arising from proton polarization, using a simple isobaric model for excitation of the proton.

A methodological innovation of this calculation is the easy obtention of the analytic and covariant or spinorial form of the proton Compton scattering amplitude directly from the so-called non-relativistic or Pauli form. This is suitable for direct insertion into the Feynman integral which gives its effect on the electron-proton scattering amplitude. Our conclusion is that the discrepancy is genuine and the present work excludes the possibility of its being explained away by proton polarization.

In the remainder of this section we analyze the general significance of the discrepancy and its relation to other physical quantities. We first review the various known theoretical contributions and conclude that by now all seem to have been evaluated reliably. We then compare the sensitivity of the analysis of the hyperfine separation with that of the Lamb shift and of electron-proton scattering data, by considering examples of hypothetical phenomenological electron-proton interactions. It is found that some interactions would in fact be revealed in the former which are not yet detectable in either of the latter. Finally, it is pointed out that although the precise cause of the discrepancy is unknown, it is most likely to be roughly proportional (in % hfs) to mM^{-1} , where m is the mass of the electron and M^{-1} is some nuclear or mesonic length. Consequently, in μ mesic hydrogen (p, μ^-) the discrepancy is expected to be of order 0.7% instead of 36 ppm as for hydrogen (p, e^-). One thus gains a factor of the order of 200 for the onset of the interesting effect, and all other

experimental uncertainties, such as the uncertainty in α , are completely negligible by comparison. We hope that the hfs of μ mesic hydrogen can be measured in the near future with at least sufficient accuracy to verify the hypothesis that the discrepancy is roughly proportional to mM^{-1} . This would provide valuable insight into its nature.

In Section 2 we describe the isobaric model and the obtention of the analytic and covariant proton Compton scattering amplitude. This amplitude is obtained directly from the simple well-known non-relativistic or Pauli $3/2^+ \overline{M}(1) \rightarrow M(1) \overline{1}$ partial wave amplitude by a linear transformation. The comparatively intricate Rarita-Schwinger formalism for spin $3/2$ is thereby avoided without sacrificing analyticity and covariance: the form obtained is valid in all co-ordinate systems, and not just the γ -p centre of mass, so that it is suitable for covariant integration. In Section 3 the Feynman integral which contributes to the electron-proton scattering amplitude is set up and its contribution to the hfs is expressed in terms of a trace, in which manifest relativistic covariance is maintained. In Section 4 we give numerical results and display the effect of varying the parameters of the model.

The hyperfine separation of the ground state of atomic hydrogen (hfs) is one of the most precisely known quantities ^{1),2)}. The most recently reported measurement ³⁾ achieves an accuracy of one part in 10^{11} . To be useful for purposes other than calibration this experimental value must be compared with a reliable theoretical expression for it. The most recent calculations ^{4),5),6)}, however, result in a discrepancy of 36 ± 17 parts per million (ppm) ⁷⁾. Less than a year ago this discrepancy was attributed ⁵⁾ either to an incorrect value for the fine structure constant, α , in the theoretical formula for the hfs or to unevaluated corrections due to proton excitation in two-photon exchange processes, often called polarization corrections. However, in the meanwhile, a precise measurement of the hyperfine

structure of muonium ⁷⁾ has yielded an independent determination of α which is in excellent agreement with the previously accepted value obtained from fine structure measurements in deuterium. This apparently removes the first possible source of the discrepancy and it consequently seemed worth while to consider the second possibility, proton excitation.

In processes in which two photons are exchanged between the electron and the proton, which constitute the hydrogen atom, Fig. 1, the intermediate states include, besides the electron, the proton alone or accompanied by the mesons which it easily emits. When the proton appears alone there result recoil and finite size corrections which have been dealt with by various authors ⁸⁻¹¹⁾, and which contribute in total ¹¹⁾ -43 ppm. Now, since the proton is a strongly interacting particle we expect, a priori, the contribution from the states with the mesons present to be of the same order of magnitude as the recoil and finite size corrections from the single proton state. In fact, in proton Compton scattering the cross-section at the energy of the $\bar{3}-3$ resonance is more than ten times the value at threshold, which is given by the Born term or single proton intermediate state. Accordingly, we have undertaken a calculation, described in the following sections, of the contribution from the $\bar{3}-3$ resonance, indicated diagrammatically in Fig. 2.

This resonance can be excited electromagnetically by either magnetic dipole or electric quadrupole transitions. The former alone is well known ^{12),13),14)} to give a fairly adequate fit to scattering data and so we have neglected the electric quadrupole excitation. The result is -0.12 ppm, which is roughly 100 times smaller than the discrepancy (which does not justify making a more detailed calculation including the small electric quadrupole transition).

In the last section we discuss in detail the effect of varying the form factors and resonance shape and position. For no value does the correction become significant. In particular, we observe (Table 1) that the net contribution decreases very rapidly with increasing resonance energy. To understand why the contribution is so much smaller than the a priori expectation, we have, in the last section, made an estimate using a value of the N_{33}^* mass equal to the nucleon mass. This gives a correction of -18 ppm which agrees with the expected order of magnitude.

Our result agrees in order of magnitude with the various estimates of Iddings¹⁵⁾, which arrived in preprint form during the course of the present calculation, and also with that of Miss Guérin¹⁶⁾. An earlier estimate¹⁷⁾ is also of the same order of magnitude, but was not thought reliable because it is based on an angular dependence for the proton Compton amplitude that does not correspond to magnetic dipole excitation. This simply confirms, as do the various estimates of Ref. ¹⁵⁾, that the net contribution remains much too small to account for the discrepancy, regardless of which partial wave an enhancement in the γ -p system may occur in. For this reason and because, as we have noted, the contribution falls off very rapidly with increasing resonance energy, we may safely exclude the possibility that the discrepancy can be attributed to higher resonances in the γ -p system.

One should also consider the possible effect of resonances in the crossed channel of the γ -p scattering process (see Fig. 3):

$$2 \gamma \rightarrow \text{resonance} \rightarrow p \bar{p} .$$

If the resonance is in a $J = 0$ state, scalar or pseudoscalar, it can provide no spin correlation between the electron and proton at rest, and so cannot contribute to the hyperfine splitting. As is well known, no $J = 1$ state can decay into two γ rays. The effect of a $J = 2$ resonance on electron-proton

scattering has been considered by Flamm and Kummer¹⁸⁾. They find that a $J^P = 2^-$ resonance does not contribute to e-p scattering in the limit of vanishing electron mass. For $J^P = 2^+$ the contribution to the electron-proton scattering amplitude is given by Eq. (26) of Ref. ¹⁸⁾. It is easily seen upon inspection that this amplitude is spin independent for electron and proton at rest, and so does not contribute to the hyperfine splitting. We conclude (assuming the electron has only electromagnetic interactions) that only cross-channel enhancements of spin ≥ 3 can contribute to the hyperfine splitting. If any such exist we may safely suppose that their mass is sufficiently great that they give no significant contribution to the hydrogen hyperfine separation.

The excellent agreement between measured and calculated values of the hfs of muonium⁷⁾, may, if it is not fortuitous, be regarded with considerable satisfaction by both experimentalists and theoreticians. It apparently confirms our understanding as well as the numerical accuracy of the purely radiative^{19),20),5),6)} and recoil corrections^{8),9)} for these are common to both hydrogen and muonium.

The finite size corrections due to the proton form factors have been evaluated by two independent methods in Refs. ¹⁰⁾ and ¹¹⁾. The former is a calculation for infinite proton mass with a charge distribution of finite size, the later also includes the recoil corrections. As a check we have taken the infinite mass limit of the expression obtained by the latter authors and thereby retrieved the expression of the former which accounts for the bulk (80%) of the effect. One gets similar agreement from the calculation of Ref. ⁹⁾, which includes only recoil effects, if one simulates the form factors simply by cutting off all momentum integrals (not just the divergent ones!) at a particular, but reasonable momentum. Thus, the recoil and finite size corrections also appear to be on a firm basis.

There exists at present no other known theoretical correction which has not been mentioned above. All have apparently withstood verification of one kind or another. Furthermore, our conclusion is that the hfs is relatively insensitive (compared to the discrepancy) to the behaviour of the Compton amplitude beyond the Born term. For even something as gross as the 3-3 resonance in Compton scattering produces a negligible effect on the hfs. Consequently, it may be worth while to discuss the discrepancy from a general phenomenological point of view.

A direct way to express the order of magnitude of the discrepancy is simply to express it in megacycles

$$36 \text{ ppm} \times \text{hfs} \simeq 36 \times 10^{-6} \times 1.500 \text{ Mc} \simeq 0.054 \text{ Mc.} \quad (1.1)$$

This may be compared with the uncertainty in the Lamb shift ²¹⁾ (separation between the $2s_{1/2}$ and $2p_{1/2}$ states of hydrogen) of $\pm 0.15 \text{ Mc}$, which is about three times the discrepancy in the hyperfine splitting. In addition, as is well known, the Lamb shift is less sensitive to short range effects by a factor $1/8$, since $\psi^2(0)$, the relative probability that the electron be found at the proton is $1/8$ in the $2s$ state, where the Lamb shift is measured, of what it is in the ground state where the hfs is measured.

To gain some insight into the magnitude of the discrepancy from another point of view, let us suppose that unknown or neglected effects can be simulated by an effective short range coupling between the electron and the proton that is spin-dependent at low energy. We are not concerned with the precise form of the coupling, but only in its order of magnitude, and so take the simplest forms, say ²²⁾

$$H_{eff} = 8^{1/2} G \int \bar{\Psi}_p \gamma_{\mu\nu} \Psi_p \bar{\Psi}_e \gamma_{\mu\nu} \Psi_e d^3x , \quad (1.2a)$$

or

$$H_{eff} = 8^{1/2} G \int \bar{\Psi}_p \gamma_{\mu} \gamma_5 \Psi_p \bar{\Psi}_e \gamma_{\mu} \gamma_5 \Psi_e d^3x . \quad (1.2b)$$

If we evaluate either of these in the hydrogen atom 1s state we find a resulting energy shift

$$\Delta E = 8^{1/2} G \underline{G}_e \cdot \underline{G}_p \Psi^2(0) . \quad (1.3)$$

The lowest order expression for the hyperfine splitting is

$$h_{fs} = \frac{2\pi}{3} \alpha \mu_p m^{-1} M^{-1} \underline{G}_e \cdot \underline{G}_p \Psi^2(0) , \quad (1.4)$$

which, as we may note in passing, is simulated by a suitable value of G in Eqs. (1.2). Taking the ratio of the last two equations, we find

$$R = \Delta E (h_{fs})^{-1} = 8^{1/2} G M^2 \left(\frac{3}{2\pi} \right) (\alpha \mu_p)^{-1} \frac{m}{M} . \quad (1.5)$$

If we equate this to the discrepancy, 36 ppm, we find

$$G M^2 \approx 10^{-3}. \quad (1.6)$$

This may be compared with the Fermi coupling constant of weak decay (which is why we included $8^{1/2}$) given by $GM^2 = 10^{-5}$, which is 100 times smaller.

The relative magnitude of the interference term of the amplitude (1.2) with the one-photon Born amplitude in electron-proton scattering may be estimated very roughly to be $\sqrt{8^{1/2}G} \sqrt{4\pi\alpha/t}^{-1}$, where t is the square momentum transfer, on the assumption that the same form factor is present in both amplitudes. This ratio is proportional to t and, for example, at $t = 1 \text{ GeV}^2$ it is only 2% for the value of G given by Eq. (1.6). However, an amplitude of the form (1.2a) would, in fact, remain undetectable at all momentum transfers unless polarization were measured, because its presence would only simulate modifications in the apparent nucleon form factors now obtained by interpreting unpolarized cross-sections. For, even if the amplitude (1.2a) were present, the unpolarized cross-section would still conform to the phenomenological Rosenbluth formula which may be written

$$\left(\frac{d\sigma}{d\omega}\right)_{\text{lab.}} = \left[A(t) + B(t) \tan^2 \frac{\theta}{2} \right] \left(\frac{d\sigma}{d\omega}\right)_{\text{Mott}}(t, \theta). \quad (1.7)$$

Nor would it show up by comparing electron and positron cross-sections, for it is odd under charge conjugation, as is the one-photon Born term. We conclude that neither the Lamb shift measurement nor the electron-proton scattering data provide evidence against amplitudes of the form (2) of strength sufficient to account for the discrepancy.

Finally, it is worth observing that all the corrections mentioned in this section, apart from the purely radiative corrections whose validity is confirmed by the agreement in the case of muonium (μ^+e^-), are proportional to the electron mass m_e multiplied by some length M^{-1} which is characteristic of nuclear or mesonic distances or simply short distances compared to the electron Compton wavelength. If the discrepancy is due to any of these effects it should be proportional to $m_\mu M^{-1}$ in the case of μ mesic hydrogen, ($p\mu^-$), with the same coefficient, and so be of magnitude

$$36 \text{ ppm} \times 207 = 0.7 \%. \quad (1.8)$$

This extrapolation is not quite exact because terms $(m_e M^{-1})^2$ which are completely negligible in hydrogen may not be in ($p\mu^-$). Consequently, μ mesic hydrogen is experimentally very favourable for studying the discrepancy since all other uncertainties are very small by comparison. It certainly would be valuable to verify the suggestion that, whatever its cause, it is roughly proportional to the lepton mass.

Let us also note that the agreement ^{23),24)} to within the experimental accuracy of 0.3 ppm between measured and calculated values of the ratio of the hfs in the metastable 2s state of hydrogen to the hfs in the ground state, reassures us of the not very surprising truth that the discrepancy is proportional to $\psi^2(0)$, so that its source lies within a Bohr radius, $(m_e \alpha)^{-1}$, of the proton. It would be nice if this ratio could also be measured with say 0.1% accuracy in μ mesic hydrogen (assuming that the discrepancy is 0.7% there), to verify that its source also lies within a radius of the μ mesic Bohr radius $(m \alpha)^{-1}$.

2. OBTENTION OF COVARIANT PROTON COMPTON AMPLITUDE FROM THE NON-RELATIVISTIC FORM

We begin by stating our notational conventions. We adopt the Lorentz signature $(+,-,-,-)$ for $g^{\mu\nu}$, the alternating tensor $\varepsilon_{\mu\nu\lambda}$ with $\varepsilon_{0123} = -\varepsilon^{0123} = +1$, and the four-vector p with components $p^\mu = (E, \underline{p})$, $\mu, \nu = 0, 1, 2, 3$. On occasion, we shall make use of relativistic spinor calculus²⁵⁾ which we need only for spin $\frac{1}{2}$. Let $\underline{\sigma}_\mu$ and $\bar{\sigma}_\mu$ be the matrices with matrix components

$$(\underline{\sigma}_\mu)_{\alpha\beta} = \sigma_{\alpha\dot{\beta}\mu} = (\delta_{\alpha\beta}, \underline{\sigma}_{\alpha\beta}), \quad (2.1a)$$

$$(\bar{\sigma}_\mu)^{\alpha\beta} = \sigma^{\dot{\alpha}\beta}_\mu = (\delta_{\alpha\beta}, -\underline{\sigma}_{\alpha\beta}), \quad (2.1b)$$

where $\underline{\sigma}$ are the usual Pauli matrices and $\alpha, \beta = \pm \frac{1}{2}$. The Minkowski four-vector index on the σ_μ may be raised and lowered in the usual way using $g^{\mu\nu}$ and $g_{\mu\nu}$.

We shall write for any four-vector p^μ

$$\underline{p} = \underline{\sigma}_\mu p^\mu, \quad \bar{p} = \bar{\sigma}_\mu p^\mu, \quad (2.2a)$$

or in explicit spinorial notation

$$p_{\alpha\dot{\beta}} = \sigma_{\alpha\dot{\beta}\mu} p^\mu, \quad p^{\dot{\alpha}\beta} = \sigma^{\dot{\alpha}\beta}_\mu p^\mu. \quad (2.2b)$$

The raising and lowering operators are written

$$C^{\alpha\beta} = C_{\dot{\alpha}\dot{\beta}} = -C_{\alpha\beta} = -C^{\dot{\alpha}\dot{\beta}} = (-1)^{-1/2+\alpha} \delta_{\alpha, -\beta}, \quad (2.3)$$

which operate by matrix multiplication from the left. For Dirac matrices we use the representation

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}, \quad \gamma_5 = \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.4)$$

The one-particle phase space element is taken to be $(2E)^{-1} d^3p$ corresponding to a particle density $2E(2\pi)^{-3}$. The amplitude F is defined by

$$S = 1 + 2i \delta^4(p_f - p_i) F,$$

so that the centre of mass differential cross-section for elastic scattering from an initial pure spin state $\lambda_1 \lambda_2$ to a final pure spin state $\lambda_3 \lambda_4$ is

$$\frac{d\sigma}{d\omega} = \frac{\pi^2}{s} |F_{\lambda_3 \lambda_4, \lambda_1 \lambda_2}|^2, \quad (2.5)$$

where s is the centre of mass energy squared. We are using natural units ($\hbar = c = 1$) and thus F is a dimensionless quantity. By writing in lowest order

$$S = 1 - i \int_{-\infty}^{+\infty} H_{\text{eff}} dt ,$$

where H_{eff} is an effective Hamiltonian and, comparing with the definition of F , one finds a first order energy shift ²⁶⁾ in a state n of the hydrogen atom given by

$$\Delta E = \langle n | H_{\text{eff}} | n \rangle \simeq -2\pi^2 (mM)^{-1} F \psi_n^2(0) , \quad (2.6)$$

where m and M are the electron and proton masses, $\psi(0)$ is the non-relativistic Schrödinger electron wave function evaluated at the origin and F is the amplitude evaluated at threshold. The basic assumption underlying Eq. (2.6) is that

$$F(q_f, q_i) = F(0, 0) ,$$

with q_f, q_i the relative centre-of-mass momentum, for the values of q that are present in the hydrogen atom wave function, i.e., $q \sim \propto m$. The important point for us is that in the last equation F remains a matrix in spin space.

We now proceed to the main object of this section which is to find a suitable form for the proton Compton amplitude. We wish to use an isobaric model and thereby include only the effect of the $3-3$ resonance. One could make use of the Rarita-Schwinger spin $3/2$ propagator and vertex function. However, it is much simpler, and equally justified by our experimental knowledge of the resonance, to use the well-known spin $3/2$ partial wave amplitude in the so-called non-relativistic or Pauli form, and promote it in a way that will be seen, to a covariant analytic amplitude, suitable for insertion into a Feynman integral.

The resonance appears in the partial wave amplitudes $M(1) \rightarrow M(1)$, $M(1) \rightarrow E(2)$, $E(2) \rightarrow M(1)$ and $E(2) \rightarrow E(2)$, where $M(1)$ and $E(2)$ refer respectively to magnetic dipole and electric quadrupole $J^P = 3/2^+$ states. The magnetic dipole transitions are much larger^{12),13),14)} than the electric quadrupole transitions. We will retain only the former. The smallness of our final result suggests that it would not be worth the effort of retaining the smaller transition elements. The $J^P = 3/2^+$ $M(1) \rightarrow M(1)$ center-of-mass amplitude, which is the only one we retain, has the form in a Pauli basis

$$F = T(s) \left[3 \underline{S}_2 \cdot \underline{S}_1 - \underline{S}_2 \cdot \underline{S}_2 \underline{S}_1 \cdot \underline{S}_1 \right] . \quad (2.7)$$

Here $T(s)$ is a function of the total centre-of-mass energy and $\underline{S}_i = \underline{\epsilon}_i \times \underline{k}_i$, where $i = 1, 2$ refer respectively to the initial and final photons and $\underline{\epsilon}$ and $\underline{k}$ are polarization and momentum three-vectors of the photon in the centre-of-mass system.

We now wish to obtain the covariant form of this amplitude. Let us adopt the particle labels indicated in Fig. 4 (reserving p_1, p_3 and p_5 for the electron), and denote by Δ the centre-of-mass energy, so that $s^{1/2} = \Delta = \sqrt{M^2 + \underline{k}^2} + |\underline{k}|$, $\underline{k}^2 = \underline{k}_1^2 = \underline{k}_2^2$

$$P_6 = (\Delta, 0) , \quad P_2 = \left(\sqrt{M^2 + \underline{k}^2} , -\underline{k}_1 \right) ,$$

$$P_4 = \left(\sqrt{M^2 + \underline{k}^2} , -\underline{k}_2 \right) \quad \text{and} \quad \Delta^{-1} P_6 = \Delta^{-1} \overline{P}_6 = 1 ,$$

in spinorial notation.

Let us construct the relativistic four-vectors

$$S_1^\mu = \epsilon^\mu(k_1, \epsilon_1, p_1) = \epsilon^\mu(p_1, p_1, \epsilon_1), \quad (2.8a)$$

$$S_2^\mu = \epsilon^\mu(k_2, \epsilon_2, p_2) = \epsilon^\mu(p_2, p_2, \epsilon_2), \quad (2.8b)$$

where k_i, ϵ_i ($i = 1, 2$) are the momentum and polarization four-vectors of the photon, and $\epsilon^\nu(a, b, c)$ is defined by

$$\epsilon^\nu(a, b, c) = \epsilon_{\alpha\lambda\mu} a^\alpha b^\lambda c^\mu, \quad (2.9)$$

for four-vectors a, b, c . Then we may rewrite Eq. (2.7) as

$$F = T(S) \Delta^{-3} \left[-3 S_2^\mu S_1^\mu \bar{p}_c - \bar{S}_2^\mu \bar{p}_c \bar{S}_1^\mu \right], \quad (2.9)$$

where the first minus sign comes from the negative space matrix, and spinorial notation is used. The covariant amplitude is obtained from the non-relativistic amplitude by the formula²⁵⁾

$$\underline{A} = \sqrt{\frac{p_1}{M}} F \sqrt{\frac{p_2}{M}}. \quad (2.10)$$

We now observe that

$$\begin{aligned}
 \left[\frac{\underline{P}_6}{\Delta} + \frac{\underline{P}_2}{M} \right]^2 &= \left[1 + M^{-1} (E_2 + \underline{S} \cdot \underline{P}_2) \right]^2 \\
 &= 2 M^{-2} (E_2 + M) (E_2 + \underline{S} \cdot \underline{P}_2) \\
 &= 2 M^{-1} (E_2 + M) \left(\frac{\underline{P}_2}{M} \right),
 \end{aligned}$$

so

$$\begin{aligned}
 \sqrt{\frac{\underline{P}_2}{M}} &= \left[2(E_2 + M)/M \right]^{-1/2} \left[\frac{\underline{P}_6}{\Delta} + \frac{\underline{P}_2}{M} \right], \\
 \sqrt{\frac{\underline{P}_4}{M}} &= \left[2(E_4 + M)/M \right]^{-1/2} \left[\frac{\underline{P}_6}{\Delta} + \frac{\underline{P}_4}{M} \right].
 \end{aligned} \tag{2.11}$$

We thus obtain from Eqs. (2.9) and (2.10)

$$\begin{aligned}
 \underline{A} &= \left[2 \Delta^3 (E + M) \right]^{-1} M T(S) \cdot \\
 &\quad \cdot \left[M^{-1} \underline{P}_4 + \Delta^{-1} \underline{P}_6 \right] \left[-3 \underline{S}_2 \cdot \underline{S}_1 \bar{\underline{P}}_6 - \bar{\underline{S}}_2 \underline{P}_6 \bar{\underline{S}}_1 \right] \left[\Delta^{-1} \underline{P}_6 + M^{-1} \underline{P}_2 \right], \tag{2.12}
 \end{aligned}$$

for $E = E_2 = E_4$. This expression may be somewhat simplified using

$$\underline{A} \cdot \underline{\bar{B}} + \underline{B} \cdot \underline{\bar{A}} = 2 \underline{A} \cdot \underline{B} \quad , \quad P_6^2 = \Delta^2 \quad ,$$

$$P_2 \cdot S_3 = P_6 \cdot S_1 = P_6 \cdot S_2 = P_4 \cdot S_2 = 0 \quad .$$

One finds

$$\begin{aligned} \underline{A} = A(s) \left\{ -3 \underline{S}_2 \cdot \underline{S}_1 \left[M^{-1} (\underline{P}_2 + \underline{P}_4) + \Delta^{-1} (\underline{P}_6 + M^{-2} \underline{P}_4 \underline{\bar{P}}_6 \underline{P}_2) \right] - \right. \\ \left. - \underline{S}_2 \left[M^{-1} (\underline{\bar{P}}_2 + \underline{\bar{P}}_4) + \Delta^{-1} (\underline{\bar{P}}_6 + M^{-2} \underline{\bar{P}}_4 \underline{P}_6 \underline{\bar{P}}_2) \right] \underline{S}_1 \right\} \quad , \end{aligned} \quad (2.13)$$

where we have written

$$A(s) = \left[2 \Delta^3 (E + M) \right]^{-1} M T(s) \quad .$$

We have now obtained the covariant amplitude $\underline{A}$ equivalent to the non-relativistic amplitude F of Eq. (2.7). Since it is expressed as an explicitly covariant function of the momentum four-vectors, the expression (2.13), which we have obtained in the centre-of-mass frame, in fact represents the covariant amplitude in all frames.

We now must make some assumption about $A(s)$ and it is at this point that there is some arbitrariness. We are tempted to assume that $A(s)$ enjoy simple analyticity properties. However, there is a $\Delta = s^{1/2}$ appearing explicitly in Eq. (2.13) that cannot be factored out and which has a square root singularity. One could introduce two independent functions $A_1(s) \equiv M^{-1} A(s)$ and $A_2(s) \equiv \Delta^{-1} A(s)$, which enjoy simple analyticity properties, but then

Eq. (2.13) would not in general represent a pure $J^P = 3/2^+$ $M(1) \rightarrow M(1)$ amplitude since this would require $A_2(s) = M^{-1} \Delta A_1(s) = M^{-1} s^{1/2} A_1(s)$. Thus, we see that there exists no completely analytic amplitude that contributes in only one partial wave at all energies. However, it is in fact only necessary to have a pure partial wave at the energy of the resonance. By making the amplitude analytic, other partial wave are then necessarily introduced at other energies. This situation which might at first seem unexpected or undesirable, however, turns out to be exactly the one that obtains in a field theoretical formalism. As is well known, the propagator of a virtual particle carries several spin states, and not just the spin of the real particle. To make a determination of the amplitude we choose to let it be represented by Eq. (2.13), but suppose that Δ is a fixed number, the mass of the isobar (assuming for the moment it has zero width) and let $A(s)$ be a function free of kinematical singularities. Since we wish this amplitude to represent an isobar, we represent $A(s)$ by a one-pole formula and write

$$\underline{A} = -\frac{1}{\pi} \frac{\alpha g^2}{\Delta^2} \frac{1}{s - \Delta^2 + i\epsilon} \left\{ -3 \underline{S}_2 \cdot \underline{S}_1 \left[M^{-1} (\underline{P}_2 + \underline{P}_4) + \Delta^{-1} (\underline{P}_6 + M^{-2} \underline{P}_4 \underline{P}_6 \underline{P}_2) \right] - \right. \\ \left. - \underline{S}_2 \left[M^{-1} (\underline{\bar{P}}_2 + \underline{\bar{P}}_4) + \Delta^{-1} (\underline{P}_6 + M^{-2} \underline{\bar{P}}_4 \underline{P}_6 \underline{\bar{P}}_2) \right] \underline{S}_1 \right\}, \quad (2.14)$$

with $\underline{S}_1$ and $\underline{S}_2$ given by Eqs. (2.8). Any other determination of A which has no other singularities besides a pole at $s = \Delta^2$ in the $J^P = 3/2^+$ $M(1) \rightarrow M(1)$ partial wave can only differ by a polynomial in the four-vectors which represents a non-resonant background.

In Eq. (2.14) we have explicitly factored out an α , the fine structure constant, since the amplitude represents a two photon diagram, and also a factor of Δ^{-2} so that g^2 is a dimensionless quantity which we expect to

be of the order of unity. It must be real, so that the amplitude is real away from the pole (in the forward direction) and positive, so that at the pole the imaginary part of $\underline{A}$ in the forward direction is positive. The numerical value of g^2 must be obtained from experiment. To do this, we replaced ε in Eq. (2.14) by Γ , such that the half-width at half maximum of the $3/2 \ 3/2$ resonance, $\Gamma / \Delta = 125 \pm 15$ MeV. Then using Eq. (2.5) or using the formalism of Ref. ²⁵⁾ to express the cross-section directly in terms of the covariant amplitude, one finds for the non-polarized differential cross-section for Compton scattering at 90° , at the energy of the resonance, $s = \Delta^2$

$$\frac{d\sigma}{d\omega} = \left(\frac{\alpha}{M}\right)^2 \left[\frac{7}{32} g^4 \right] \left[\frac{\Delta^2 - M^2}{\Gamma} \right]^2 \frac{(M + \Delta)^4}{\Delta^8} (\Delta^2 - M^2)^2. \quad (2.15)$$

Inserting the values $M = 938$ MeV, $\Delta = 1238$ MeV and ²⁷⁾

$$\frac{d\sigma}{d\omega} = (1.6 \pm 0.2) \times 10^{-31} \text{ cm}^2,$$

into Eq. (2.15) one obtains

$$g^2 = 0.87 \pm 0.09. \quad (2.16)$$

We have also evaluated this quantity by making a fit to the resonant photo-production amplitude and calculating the residue of the resonance pole in the Compton scattering amplitude by unitarity. The result agrees with Eq. (2.16).

This completes our determination of the covariant analytic isobaric contribution to the Compton scattering amplitude. One should in principle multiply Eq. (2.14) by a weight function $\rho(\Delta^2)$ normalized to $\int \rho = 1$ and integrate over Δ^2 to account for the width. This may be done at the very end of our calculation after all other integrations have been performed. It will be seen then, however, that this will make relatively little change in the final result.

The advantage of the covariant expression (2.14) over the non-relativistic or Pauli form (2.7) is, of course, that it represents the amplitude in all reference frames and not just the centre of mass of the γ -p system. The crossed diagram may be obtained from it by simple substitution. Furthermore, it is suitable for an extrapolation off the photon mass shell to virtual photons. We do this straightforwardly (and perhaps brazenly) simply by retaining the same expression, but allowing the photon momentum four-vectors k_1 and k_2 to have any length, and multiplying each photon vertex by a form factor. This amplitude will be inserted directly into the Feynman integral which represents the two-photon exchange contribution to the electron-proton scattering amplitude.

3. SET-UP OF TRACES AND INTEGRALS

The proton Compton amplitude is obtained by adding the crossed diagram to the expression (2.14) of the last section. We write the sum in the form

$$\underline{A}(p_2 k_2 \varepsilon_2 \leftarrow p_1 k_1 \varepsilon_1) = \varepsilon_2^\nu \underline{A}_{\nu\mu}^p \varepsilon_1^\mu. \quad (3.1)$$

The electron Compton amplitude, indicated in Fig. 5, is written, with our normalization

$$\begin{aligned} A(p_3 k_1 \varepsilon_1 \leftarrow p_1 k_2 \varepsilon_2) &= \varepsilon_1^\mu A_{\mu\nu}^e \varepsilon_2^\nu \\ &= -\frac{1}{\pi} \alpha m \bar{u}(p_3) \left[\varepsilon_2 \frac{p_1 - k_1 - m}{(p_1 - k_1)^2 - m^2} \varepsilon_1 + \varepsilon_1 \frac{p_1 + k_2 - m}{(p_1 + k_2)^2 - m^2} \varepsilon_2 \right] u(p_1), \end{aligned} \quad (3.2)$$

in which $\not{a} = a_\mu \gamma^\mu$, for any four-vector a , and the Dirac spinors are normalized to $\bar{u}u = 1$. Then the contribution to the electron-proton scattering amplitude, Fig. 1, from isobar formation is given by

$$\underline{A}^{ep} = \frac{1}{4\pi^2 i} \int \frac{d^4 k_1}{k_1^2 k_2^2} \underline{A}_{\nu\mu}^p A^{e,\mu\nu}, \quad (3.3)$$

with $k_2 \equiv p_2 + k_1 - p_4$, and the integral extends over the Feynman contour. The normalization constant is easily verified by reducing the contribution from the two photon poles to the unitarity condition in the channel where the two photons appear as an intermediate state, keeping in mind the phase space appropriate to identical bosons.

We now substitute Eqs. (2.14) and (3.2) into this expression and obtain

$$\begin{aligned}
 \underline{A}^{ep} = & \left(\frac{1}{\pi} \frac{\alpha_0^2}{\Delta^2} \right) \left(\frac{1}{\pi} \alpha_m \right) \left(\frac{1}{2\pi^2 i} \right) \int \frac{d^4 k_i}{k_i^2 k_i^2} \\
 & \times \left\{ -3 S_{2\nu\lambda} S_{1\mu}{}^\lambda \left[M^{-1}(\underline{p}_2 + \underline{p}_4) + \Delta^{-1}(\underline{p}_6 + M^{-2} \underline{p}_4 \bar{p}_6 \underline{p}_2) \right] - \right. \\
 & \left. - S_{2\nu} \left[M^{-1}(\bar{p}_2 + \bar{p}_4) + \Delta^{-1}(\bar{p}_6 + M^{-2} \bar{p}_4 \underline{p}_6 \bar{p}_2) \right] S_{1\mu}{}^\mu \right\} \frac{F(k_1^2) F(k_2^2)}{(\underline{p}_2 + k_1)^2 - \Delta^2} \\
 & \times \bar{u}(p_3) \left[\gamma^\nu \frac{\underline{p}_1 - k_1 - m}{(\underline{p}_1 - k_1)^2 - m^2} \gamma^\mu + \gamma^\mu \frac{\underline{p}_1 + k_2 - m}{(\underline{p}_1 + k_2)^2 - m^2} \gamma^\nu \right] u(p), \quad (3.4)
 \end{aligned}$$

where

$$S_{2\nu\lambda} = \epsilon_{\mu\nu\lambda} p_4^\mu p_6^\mu, \quad S_{1\mu}{}^\lambda = \epsilon_{\mu\nu\lambda} p_2^\mu p_6^\nu,$$

$\underline{S}_{2\nu} = S_{2\nu\lambda} \underline{e}^\lambda$ and $\underline{S}_{1\mu} = S_{1\mu}{}^\lambda \underline{e}^\lambda$, and $F(k^2)$ is the form factor appropriate to the transition matrix element, with $F(0) = 1$.

We will now express the contribution to the hfs from this amplitude. The energy shift in an atomic state of hydrogen due to the electron-proton scattering amplitude is expressed in Eq. (2.6) in terms of the Pauli amplitude F , evaluated at threshold. Threshold means that

$$P_1^\mu = P_3^\mu = (m, 0, 0, 0) \quad , \quad P_2^\mu = P_4^\mu = (M, 0, 0, 0) \quad , \quad (3.5)$$

so that the transformation matrices of Eq. (2.10) and Ref. ²⁵⁾

$$\sqrt{P_1/m} \quad \text{and} \quad \sqrt{P_2/M} \quad ,$$

between F and the covariant amplitude $\underline{A}$ are unity and $F = \underline{A}$. The hyperfine separation is, by definition, the energy of the triplet state minus the energy of the singlet state. It is convenient to express the contribution to it ΔE_{hfs} in terms of a trace in spin space,

$$\Delta E_{\text{hfs}} = \text{tr} \left[\left(\frac{1}{3} P_t - P_s \right) H_{\text{eff}} \right] \quad , \quad (3.6)$$

where $P_t = \frac{1}{4} (\underline{\sigma}_1 \cdot \underline{\sigma}_2 + 3)$ and $P_s = \frac{1}{4} (1 - \underline{\sigma}_1 \cdot \underline{\sigma}_2)$ are the projection operators onto the triplet and singlet states respectively, and $\underline{\sigma}_1$ and $\underline{\sigma}_2$ are the electron and proton Pauli spin operators. Equation (3.6) is thus simply

$$\Delta E_{\text{hfs}} = \frac{1}{3} \text{tr} \left[\underline{\sigma}_1 \cdot \underline{\sigma}_2 H_{\text{eff}} \right] \quad , \quad (3.7)$$

which we may express in covariant form as

$$\Delta E_{hfs} = -\frac{1}{3} \text{tr} \left\{ \left(\bar{\sigma}_1^{\kappa} - m^{-2} \bar{p}_1 p_1^{\kappa} \right) \left(\bar{\sigma}_{2\kappa} - M^{-2} \bar{p}_2 p_{2\kappa} \right) H_{eff} \right\}, \quad (3.8)$$

in virtue of Eq. (3.5). The over-all minus sign again comes from the sign of the space metric. This last expression is meaningful when both the electron and proton spin basis is two-dimensional. However, our expression (3.4) is a hybrid which is two-dimensional, though covariant, in the proton spin, but which uses the Dirac four-dimensional basis for the electron. We could reduce the Dirac matrices to a two-by-two covariant representation, but it seems more convenient to promote the trace (3.8) into the trace over the Dirac matrices of the electron. For the representation given at the beginning of the last section, we have

$$U(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ m^{-1} \bar{p} \end{pmatrix}, \quad \bar{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} m^{-1} \bar{p} & 1 \end{pmatrix},$$

and hence

$$U(p) \left(\bar{\sigma}_1^{\kappa} - m^{-2} \bar{p}_1 p_1^{\kappa} \right) \bar{U}(p) = \frac{(p_1 + m)}{2m} \left(\gamma^{\kappa} - m^{-2} \bar{p}_1 p_1^{\kappa} \right) \gamma_5,$$

so Eq. (3.8) may be written

$$\Delta E_{hfs} = -\frac{1}{3} \text{tr} \left\{ \left[\left(\frac{p_1 + m}{2m} \right) \left(\gamma_1^{\kappa} - m^{-2} \bar{p}_1 p_1^{\kappa} \right) \gamma_5 \right] \left[\bar{\sigma}_{2\kappa} - M^{-2} \bar{p}_2 p_{2\kappa} \right] H_{eff} \right\}. \quad (3.9)$$

It is understood that the electron spin trace is four-dimensional, the proton spin trace two-dimensional. We now substitute Eq. (2.6) into this expression and find

$$\Delta E_{hfs} = \frac{2}{3} \pi^2 (mM)^{-1} \psi^2(0) \text{tr} \left\{ \left[\left(\frac{\vec{p}_1 + m}{2m} \right) (\gamma_1^K - m^2 \vec{p}_1 \cdot \vec{p}_1^K \gamma_5) \right] \cdot \right. \\ \left. \times \left[\vec{G}_{2K} - M^{-2} \vec{p}_2 \cdot \vec{p}_2^K \right] \underline{A} \right\} . \quad (3.10)$$

The trace on the right is a manifestly covariant quantity. Since it is a scalar it is a function of the scalar products of the four-vectors involved.

As a matter of convenience let us introduce the fractional correction R which is the ratio of the correction to the hyperfine separation itself

$$R = \frac{\Delta E_{hfs}}{hfs} . \quad (3.11)$$

For hfs we may take the lowest order expression for the hyperfine separation in an S state of hydrogen

$$hfs = \frac{8\pi}{3} \frac{\alpha \mu}{mM} \psi^2(0) , \quad (3.12)$$

where $\mu = 2.79$ is the proton magnetic moment in nuclear magnetons. The last three equations yield for R

$$R = \pi (4\alpha\mu)^{-1} \text{tr} \left\{ \left[\left(\frac{\vec{p}_1 + m}{2m} \right) (\gamma_1^K - m^2 \vec{p}_1 \cdot \vec{p}_1^K \gamma_5) \right] \left[\vec{G}_{2K} - M^{-2} \vec{p}_2 \cdot \vec{p}_2^K \right] \underline{A} \right\} . \quad (3.13)$$

We now substitute Eq. (3.4) into this expression and obtain

$$R = \left(\frac{1}{\pi} \frac{\alpha g^2 m}{4 \gamma \Delta^2} \right) \left(\frac{1}{2\pi^2 i} \right) \int \frac{d^4 k}{(k^2)^2} \frac{F^2(k^2)}{[(p_2+k)^2 - \Delta^2]} \text{tr}(e)^{\mu\nu} + \text{tr}(P)_{\mu\nu} . \quad (3.14)$$

We consider forward scattering so $p_1 = p_3$, $p_2 = p_4$ and $k_1 = k_2 = k$.

The traces are

$$\text{tr}(e)^{\mu\nu} = \text{tr} \left\{ \left[\left(\frac{p_1 + m}{2m} \right) (\gamma_1^\mu - m \gamma_1^\mu \gamma_5) \right] \left[\gamma^\nu \frac{(p_1 - k - m)}{(p_1 - k)^2 - m^2} \gamma^\mu + \gamma^\mu \frac{(p_1 + k - m)}{(p_1 + k)^2 - m^2} \gamma^\nu \right] \right\}, \quad (3.15)$$

and

$$\begin{aligned} \text{tr}(P)_{\mu\nu} = \text{tr} \left\{ \left(\bar{G}_{2\mu} - M^{-2} \bar{P}_2 P_{2\mu} \right) \left(-3 S_{\nu\lambda} S_\mu^\lambda \left[2M^{-1} \bar{P}_2 + \Delta^{-1} (\bar{P}_2 + k + M^{-2} \bar{P}_2 (\bar{P}_2 + k) \bar{P}_2) \right] - \right. \right. \\ \left. \left. - \underline{S}_\nu \left[2M^{-1} \bar{P}_2 + \Delta^{-1} (\bar{P}_2 + k + M^{-2} \bar{P}_2 (\bar{P}_2 + k) \bar{P}_2) \right] \underline{S}_\mu \right) \right\}, \quad (3.16) \end{aligned}$$

with $S_{\nu\lambda} = \varepsilon_{\mu\gamma\nu\lambda} p_2^\mu k^\gamma$ and $\underline{S}_\nu = S_{\nu\lambda} \underline{e}^\lambda$. All that remains to be done is the straightforward evaluation of these traces and integrals.

4. EVALUATION OF TRACES AND INTEGRALS AND NUMERICAL RESULTS

The trace on the electron spin, Eq. (3.15), is evaluated very easily and yields

$$\text{tr}(\epsilon)^{\kappa\mu\nu} = 2 \left(g_{\lambda}^{\kappa} - m^2 p_1^{\kappa} p_{1\lambda} \right) i \epsilon^{\tau\lambda\nu\mu} k_{\tau} \left(\frac{1}{k^2 + 2 p_1 \cdot k} + \frac{1}{k^2 - 2 p_1 \cdot k} \right) \quad (4.1)$$

The electron mass m is a very small quantity compared to the other parameters appearing in the integral (3.15). Since $p_1 = (m, 0, 0, 0)$ we will neglect it in the last factor of the last equation. This introduces no divergence as will be seen shortly when we evaluate the integral. Then, we have

$$\text{tr}(\epsilon)^{\kappa\mu\nu} = 4 \left(g_{\lambda}^{\kappa} - m^2 p_1^{\kappa} p_{1\lambda} \right) i \epsilon^{\tau\lambda\nu\mu} k_{\tau} k^{-2} \quad (4.2)$$

Using the commutation relation

$$\underline{a} \underline{b} \underline{c} = \underline{a} \cdot \underline{b} \underline{c} + \underline{b} \cdot \underline{c} \underline{a} - \underline{a} \cdot \underline{c} \underline{b} + i \underline{\epsilon}(\underline{a} \underline{b} \underline{c}) \quad , \quad (4.3)$$

with

$$\underline{\epsilon}(\underline{a} \underline{b} \underline{c}) = \epsilon_{\kappa\lambda\mu\nu} a^{\kappa} b^{\lambda} c^{\mu} \underline{g}^{\nu} \quad ,$$

we observe that

$$2M^{-1} \underline{p}_2 + \Delta^{-1} \left(\underline{p}_2 + \underline{k} + M^{-2} \underline{p}_2 (\bar{\underline{p}}_2 + \bar{\underline{k}}) \underline{p}_2 \right) = 2 \left[M^{-1} + \Delta^{-1} (1 + M^{-2} \underline{p}_2 \cdot \underline{k}) \right] \underline{p}_2 \quad ,$$

so the proton trace, Eq. (3.16), is

$$\text{tr}(P)_{\kappa\nu\mu} = 2 \left[M^{-1} + \Delta^{-1} (1 + M^{-2} P_2 \cdot k) \right] \text{tr} \left\{ (\bar{S}_{2\kappa} - M^{-2} \bar{P}_2 P_{2\kappa}) \right. \\ \left. + \left(-3 S_{\nu\lambda} S_{\mu}^{\lambda} \bar{P}_2 - \underline{S}_{\nu} \bar{P}_2 \underline{S}_{\mu} \right) \right\} ,$$

or

$$\text{tr}(P)_{\kappa\nu\mu} = 4 (M^2 \Delta)^{-1} (M \Delta + M^2 + P_2 \cdot k) i \epsilon_{\kappa}^{\rho\sigma\tau} S_{\nu\rho} P_{2\sigma} S_{\mu\tau} . \quad (4.4)$$

The equations (4.2) and (4.4) yield

$$\text{tr}(e)^{\kappa\rho\nu} \text{tr}(P)_{\kappa\nu\mu} = -\frac{32}{k^2} \left(\frac{M \Delta + M^2 + P_2 \cdot k}{M^2 \Delta} \right) P_2 \cdot k \left[P_2^2 k^2 - (P_2 \cdot k)^2 \right] . \quad (4.5)$$

This expression is now substituted into Eq. (3.14) and we find for the fractional correction

$$R = \left(\frac{4 \alpha g^2 m}{\pi^3 \mu M^2 \Delta^3} \right) i \int \frac{d^4 k}{(k^2)^3} \frac{F^2(k^2) [M \Delta + M^2 + P_2 \cdot k]}{[k^2 + 2 P_2 \cdot k + M^2 - \Delta^2]} P_2 \cdot k \left[P_2^2 k^2 - (P_2 \cdot k)^2 \right] . \quad (4.6)$$

This integral is easily effected. We choose the time axis parallel to the p_2 direction so $p_2 = (M, 0, 0, 0)$. The path of integration over k_0 is rotated in a familiar fashion from the real axis to the imaginary axis. Introducing the new variable k_4 according to $k_0 = ik_4$, and defining the vector $K = (k_1, k_2, k_3, k_4)$ in Euclidean space, $K^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2 = -k^2$ we find

$$R = \left(\frac{4\alpha g^2 m}{\pi^3 \mu \Delta^3} \right) (-i) \int \frac{dK}{(-K^2)^3} \frac{F^2(-K^2) [M\Delta + M^2 + iMK_4]}{[-K^2 + 2iMK_4 + M^2 - \Delta^2]} (iMK_4) (K_4^2 - K^2) \quad (4.7)$$

where the integration now extends over all of Euclidean four-space. It is now convenient to introduce four-dimensional spherical co-ordinates, so that R may be written

$$\begin{aligned} R &= - \left(\frac{4\alpha g^2 m M}{\pi^3 \mu \Delta^3} \right) (4\pi) \int_0^\infty dK K^3 \int_0^\pi d\theta \sin^2 \theta \frac{F^2(-K^2)}{K^6} \\ &\quad \times \frac{(M\Delta + M^2 + iMK \cos \theta)(i \cos \theta)(\sin^2 \theta) K^3}{[-K^2 + 2iMK \cos \theta + M^2 - \Delta^2]} \\ &= \left(\frac{8\alpha g^2 m M}{\pi^2 \mu \Delta^3} \right) \int_0^\infty dK F^2(-K^2) [2M\Delta + M^2 + K^2 + \Delta^2] \\ &\quad \times \int_0^\pi d\theta \frac{\sin^4 \theta (i \cos \theta)}{K^2 + \Delta^2 - M^2 - 2iMK \cos \theta} \end{aligned} \quad (4.8)$$

The integration over θ is easily performed with the result

$$R = - \left(\frac{2\alpha g^2 m}{\pi \mu \Delta^3} \right) \int_0^\infty dK F^2(-K^2) \frac{[(\Delta + M)^2 + K^2]}{K} \left[2b^4 + 3b^2 + \frac{3}{4} - 2b(b^2 + 1)^{\frac{3}{2}} \right], \quad (4.9)$$

where

$$b = (2MK)^{-1} [K^2 + (\Delta^2 - M^2)]$$

To simplify this integral we introduce the variable u defined by $K^2 = u^{-1}(u - \Delta^2)(u - M^2)$ and find after some elementary algebra

$$R = - \left(\frac{\alpha g^2 m M^2}{4\pi \mu \Delta^3} \right) \int_{\Delta^2}^{\infty} du F^2(-K^2) \frac{(u^2 - \Delta^2 M^2)}{u^2} \cdot \frac{(u + \Delta M)^2}{u^2} \cdot \frac{(2u^2 - M^2 u - M^2 \Delta^2)}{(u - M^2)^3} \quad (4.10)$$

The form factor F is a quantity which should be determined phenomenologically. For any fit to $F(t)$ by a rational function of t the integrand is a rational function of u and hence can be performed analytically. However, the integrand is a very well behaved positive apparently monotonic function and we have evaluated it numerically for various values of the resonance position Δ and form factors F_i . The results are displayed in the last four (i.e., all but the first) rows of Table 1, for various form factors.

What is actually required is the transition form factor $N \longrightarrow N^*$. This has not been determined experimentally and we have chosen various different form factors to study the resulting variation of R . The fit of Wilson and Levinger²⁸⁾ to the nucleon isovector magnetic form factor G_{MV} seemed reasonable since the resonance is excited by an isovector magnetic dipole transition and we have taken

$$F_1(t) = G_{MV} = \frac{59}{29.5 - t/M_\pi^2} - \frac{61}{61 - t/M_\pi^2} \quad (4.11)$$

The remaining form factors we have arbitrarily invented to study the possible variation of R ,

$$F_2(t) = \frac{M^2}{M^2 - t}, \quad F_3(t) = \frac{\frac{1}{2}M^2}{\frac{1}{2}M^2 - t}, \quad F_4(t) = \frac{\frac{1}{3}M^2}{\frac{1}{3}M^2 - t}, \quad (4.12)$$

and

$$F_5(t) = \theta(24\Delta^2 - M^2 + t) \quad (4.13)$$

The latter is the step function and describes a high and sharp cut-off at $t = -(24\Delta^2 - M^2)$ an arbitrary but numerically convenient value.

A reasonable value for R would presumably be $\Delta = \Delta_0$, $F = F_1$ corresponding to

$$R = -0.12 \text{ ppm} \quad (4.14)$$

We see that for Δ in the resonance region $\Delta \geq M + m_\pi$ and for all form factors R is bounded by $0 < -R \leq 1.6 \text{ ppm}$ with the maximum occurring at threshold.

The width of the resonance can be accounted for by multiplying $R(\Delta)$ by a weight function $\rho(\Delta)$ vanishing below threshold and normalized to unity, and then integrating

$$R = \int_{M+m_\pi}^{\infty} R(\Delta) \rho(\Delta) d\Delta \quad (4.15)$$

By the mean value theorem we have $0 < -R \leq 1.6$ ppm. Since the discrepancy is 36 ± 9 ppm we conclude that the effect of the 3-3 resonance though in the direction of increasing the discrepancy is completely negligible.

At first it seemed surprising that the contribution of the 3-3 resonance should be so small compared to the term corresponding to the proton intermediate state, when, as stated in the introduction, the a priori expectation is that they be of the same order of magnitude. Although it is actually difficult to say what is meant by the proton intermediate state, since the hydrogen atom itself is a bound state of the electron and the proton, it seems reasonable to compare in order of magnitude the resonant contribution, obtained by two photon exchange, with the total correction given in Ref. ¹¹⁾ that includes the recoil and two-transverse photon exchange modified by the form factors. Using the same form factors the comparison yields 0.12 ppm from the resonance vs. 43 ppm from the proton, a factor of 375 larger. However, a striking feature of Table 1 is the strong dependence of R on the resonance energy Δ with a rapid drop-off for increasing resonance energy. As Δ is moved below threshold $M+m_\pi$, from a resonance to a bound state position, $\Delta = M + \frac{1}{4}m_\pi$, R continues to increase. To make a fair comparison, it appears reasonable to set $\Delta = M$. However, in this case the integral for R , Eq. (4.9), diverges logarithmically at low K . To get a finite answer one

should use the exact expression Eq. (4.1) instead of the expression Eq. (4.2), obtained from it by setting $m = 0$. We have done so, but retained only the term proportional to $\ell_1 (M/m)^2 \approx 15$ compared to a remainder, expected to be of order unity. The result is

$$R = - \left(\frac{6}{\pi} \frac{\alpha g^2}{\mu} \frac{m}{M} \right) \ell_1 \frac{M}{m} \approx -18 \text{ ppm} , \quad (4.16)$$

independent of the form factor. This is the quantity displayed in the first row of Table 1. This value of -18 ppm is indeed of the order of magnitude expected a priori. The very small value obtained for a realistic value of Δ , in the resonance region $\Delta > M + m_\pi$, is an example of how the value of a Feynman integral depends on the location of its singularities; in the present case it decreases very rapidly as the singularity moves away. This may be understood in terms of the Drell-Fubini²⁹⁾ mechanism, introduced to explain why the $3-3$ resonance should not contribute significantly to the electron-proton scattering cross-section.

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Δ	Δ	F_1	F_2	F_3	F_4	F_5
6.72	M	-18	-18	-18	-18	-18
6.97	$M + \frac{1}{4} m_\pi$	- 1.82				- 3.61
7.72	$M + m_\pi$	- 0.22	- 0.65			- 1.57
8.85	Δ_o	- 0.12	- 0.20	- 0.14	- 0.11	- 0.72
12.52	$\sqrt{2} \Delta_o$	- 0.01	- 0.02			- 0.17

TABEL 1

Values of R , the fractional correction to the hydrogen hyperfine separation due to the $3/2 \ 3/2$ resonance, expressed in parts per million of the hyperfine separation, are displayed for various resonance positions Δ and form factors F_i . The first column express Δ in multiples of the pion mass m_π . The second column expresses the corresponding values of Δ in terms of the nucleon mass M the pion mass m_π and the actual resonance position Δ_o . The remaining columns display R for the form factors F_i defined in the text. The first two rows correspond to resonance positions that would be pion-nucleon bound states, the last three rows to resonance masses $\geq M+m_\pi$. The first row is the value of

$$-(6\alpha g^2 m)(\pi \mu M)^{-1} \ln(m M^{-1}),$$

the remaining rows are numerical evaluation of Eq. (4.10) with $\mu = 2.79$ and $g^2 = 0.87$.

REFERENCES

- 1) P. Kusch, Phys. Rev. 100, 1188 (1955) and
J.P. Wittke and R.H. Dicke, Phys. Rev. 103, 620 (1956).
- 2) L.W. Anderson, F.M. Pipkin and J.C. Baird, Phys. Rev. 120, 1279 (1960).
- 3) S.B. Crampton, D. Kleppner and N.F. Ramsey, Phys. Rev. Letters 11, 338 (1963).
- 4) A.J. Layzer, Nuovo Cimento 33, 1538 (1964).
- 5) D. Zwanziger, Nuovo Cimento 34, 77 (1964).
- 6) L. Brown, Yale University, preprint.
- 7) W.E. Cleland, J.M. Bailey, M. Eckhause, V.W. Hughes, R.M. Mobley,
R. Prepost and J.E. Rothberg, Phys. Rev. Letters 13, 202 (1964).
We refer to this letter for the comparison of various theoretical
and experimental quantities. Since the publication of this letter,
however, the calculation reported in Ref. 6) reduces the discrepancy
by 9 ppm to the value quoted in the text.
- 8) R. Arnowitt, Phys. Rev. 92, 1002 (1953).
- 9) W.A. Newcomb and E.E. Salpeter, Phys. Rev. 97, 1146 (1955).
- 10) C. Zemach, Phys. Rev. 104, 1771 (1956).
- 11) C.K. Iddings and P.M. Platzman, Phys. Rev. 113, 192 (1959).
- 12) M. Gell-Mann and K. Watson, Ann. Rev. Nucl. Sci. 4, 234 (1954).
- 13) M. Gourdin and Ph. Salin, Nuovo Cimento 27, 193 (1963).
- 14) M. Gourdin and Ph. Salin, Nuovo Cimento 27, 309 (1963).
- 15) C.K. Iddings, Phys. Rev., to be published.
- 16) Private communication from F. Guérin and C. Bouchiat.

- 17) C.K. Iddings and P.M. Platzman, Phys. Rev. 115, 919 (1959).
- 18) D. Flamm and W. Kummer, Nuovo Cimento 28, 33 (1963) [Eq. (26)].
We have not checked this work.
- 19) J. Schwinger, Phys. Rev. 73, 416 (1948);
A. Peterman, Helv. Phys. Acta 30, 407 (1957);
C. Sommerfeld, Phys. Rev. 107, 328 (1957).
- 20) N.M. Kroll and F. Pollock, Phys. Rev. 84, 594 (1951);
R. Karplus, A. Klein and J. Schwinger, Phys. Rev. 84, 597 (1951).
- 21) A.J. Layzer, J. Math. Phys. 2, 308 (1961).
- 22) We define all our symbols and conventions at the beginning of Section 2.
- 23) D. Zwanziger, Phys. Rev. 121, 1128 (1961).
- 24) M. Sternheim, Phys. Rev. 130, 211 (1963).
- 25) This is expounded in greater detail with the present conventions in
D. Zwanziger, Phys. Rev. 1965, to be published.
- 26) One could in principle also do higher order perturbation theory with the
effective Hamiltonian. However, the smallness of the first order
term suggests that this exercise would not be profitable.
- 27) R.F. Stiening, E. Loh and M. Deutsch, Phys. Rev. Letters 10, 536 (1963).
- 28) R.R. Wilson and J.S. Levinger, preprint, Cornell University.
- 29) S.D. Drell and S. Fubini, Phys. Rev. 113, 741 (1959).

FIGURE CAPTIONS

Figure 1 : Two-photon contribution to electron-proton scattering. The blob represents the amplitude for proton Compton scattering by virtual photons.

Figure 2 : Contribution of proton isobar to electron-proton scattering amplitude.

Figure 3 : Effect of crossed channel resonance on electron-proton scattering.

Figure 4 : Particle labels for the isobaric contribution to the proton Compton amplitude.

Figure 5 : Particle labels for the electron Compton amplitude.

FIG.1

FIG.2

FIG. 3

FIG. 4

FIG. 5