Disk Instabilities and Cooling Fronts

Ethan T. Vishniac

Dept. of Astronomy, University of Texas, Austin, TX 78712

Abstract. Accretion disk outbursts, and their subsequent decline, offer a unique opportunity to constrain the physics of angular momentum transport in hot accretion disks. Recent work has centered on the claim by Cannizzo et al. that the exponential decay of luminosity following an outburst in black hole accretion disk systems is only consistent with a particular form for the dimensionless viscosity, \( \alpha = 35(c_s/r\Omega)^{3/2} \). This result can be understood in terms of a simple model of the evolution of cooling fronts in accretion disks. In particular, the cooling front speed during decline is \( \sim \alpha F c_s(F/c_sF/r\Omega)^n \), where \( F \) denotes the position of the cooling front, and the exact value of \( n \) depends on the hot state opacity, (although generally \( n \approx 1/2 \)). Setting this speed proportional to \( r \) constrains the functional form of \( \alpha \) in the hot phase of the disk, which sets it apart from previous arguments based on the relative durations of outburst and quiescence. However, it remains uncertain how well we know the exponent \( n \). In addition, more work is needed to clarify the role of irradiation in these systems and its effect on the cooling front evolution.

INTRODUCTION

The most popular model for soft X-ray transients and dwarf novae is that both are due to thermal instabilities in the accretion disks around collapsed objects. The specific mechanism thought to drive the thermal instability is the change in opacity associated with the ionization of hydrogen (for a recent review see Cannizzo 1993 [1], or Osaki 1996 [2]). In these systems the accretion disks can be modeled as geometrically thin, \( H \ll r \), and with vertical gravity supplied by the central object. As a result the sound speed to orbital velocity ratio, \( c_s/(r\Omega) \) scales with the thickness to radius ratio, \( H/r \), and the orbital frequency \( \Omega(r) \) is proportional to \( r^{-3/2} \).

The outburst cycle in these systems consists of alternating periods of quiescence and outburst. During quiescence the disk luminosity is low and gas is accreted from the companion without being transferred through the disk. In this state the disk is mostly neutral and cold. Eventually the accumulation of material at large radii leads to the runaway ionization of hydrogen and the disk material jumps to a hot state. The mass transfer rate increases dramatically...
and a heating wave propagates inward at the thermal speed, $\sim \alpha c_s$, where $\alpha$ is the dimensionless disk viscosity and $c_s$ is the midplane sound speed. In a short time the entire disk is ionized and hot, and remains in that state until the steady depletion of material makes it impossible to sustain the outburst. At this point the disk starts to fall back into the cold state, usually starting at some large radius, and a cooling wave sweeps inward while the luminosity of the disk decreases. The duration of the outburst and quiescent phases is consistent with an outburst value of $\alpha$ close to 0.1 and a cold state value several times smaller.

One of the more striking features of these systems is the extent to which the luminosity decline can be modeled by an exponential decay law, in extreme cases stretching of three orders of magnitude in X-ray luminosity (Chen et al. 1997 [3]) Mineshige et al. (1993) [4] noted that this implies an exponential decay of the disk hot phase, and that this can be achieved in the context of the disk instability model by requiring the cooling wave velocity to scale with the radius of the hot phase. More recently Cannizzo, Chen & Livio (1995) [5] have explored this argument through a series of high resolution simulations, and found that this scaling can be obtained only for a restricted class of viscosity laws, and that models with a constant $\alpha$ can be strongly excluded. Here we will review recent progress on the connection between cooling waves and models for angular momentum transport in accretion disks.

### THE COOLING FRONT SPEED

If the hot, inner parts of the accretion disk are evolving exponentially, then the simplest way to drive this evolution is to have the boundary of the hot phase of the disk evolve exponentially, that is

$$\frac{V_F}{r_F} = \text{constant},$$

(1)

where the subscript $F$ denotes the boundary of the hot phase, where rapid cooling sets in due to the thermal instability, and $V_F \equiv \dot{r}_F$. (Note that this may not be the only way to drive an exponential decay in $L_x$.) In any case, the requirement that $V_F$ be proportional to $r_F$ immediately forces a connection between the local physics of the cooling front, and the global evolution of the disk.

The earliest work on cooling front evolution (Mineshige 1987 [7], Cannizzo, Shafter, and Wheeler 1988 [6]) led to the conclusion that

$$V_F \sim \alpha_F c_s r \left( \frac{\Delta r}{r} \right).$$

(2)

In other words, that the cooling front speed is the local cooling rate, $\alpha \Omega$, times the local disk height modified by a universal spreading factor, thought to be
This result reflects the view that the cooling front should be thought of as a purely local phenomena.

However, recently CCL [5] showed instead that the cooling front speed can be approximated as

$$V_F \sim \alpha_F c_{s,F} \left( \frac{c_{s,F}}{r_F \Omega_F} \right)^{1/2}. \quad (3)$$

The significance of this result follows from its implications for $\alpha$. Since the midplane disk temperature at the cooling front is a very weak function of radius, equation (2) implies that $\alpha_F \propto r_F$, whereas equation (3) implies $\alpha_F \propto r_F^{3/4}$. Cannizzo et al. suggested adopting the rule

$$\alpha = \alpha_0 \left( \frac{c_s}{r \Omega} \right)^{3/2}, \quad (4)$$

following the form proposed by Meyer and Meyer-Hofmeister (1983) [8]. The time scale for the luminosity decay is then

$$\tau_F \approx 0.4 \frac{G M_{BH}}{\alpha_0 c_{s,F}^3}. \quad (5)$$

Matching the observations they found

$$\alpha_0 \approx 35 \frac{M_{BH}}{7M}. \quad (6)$$

They showed that this prescription leads to reasonable agreement with observations of X-ray novae, and were unable to find any other simple prescription that did as well. An added advantage of this model is apparent from equation (5). One immediately obtains dramatically shorter time scales for dwarf novae, in agreement with observations. In fact, the same prescription can be used to model the outburst cycle of SS Cygni (Cannizzo 1996 [9]).

These results raise several important questions. First, why is there a factor of $(c_s/r \Omega)^{1/2}$ (or $(H/r)^{1/2}$) in equation (3)? There are only two obvious velocities in the problem, the accretion velocity, $\alpha c_s^2/(r \Omega)$, and the thermal speed, $\alpha c_s$. Is this some sort of geometric average or is it totally unrelated? Second, is equation (4) another way of describing the drop in $\alpha$ from the hot state to the cold state, or is it a measurement of $\alpha$ in the hot state alone? Third, if this is the correct form for $\alpha$, why is the exponent $3/2$? Finally, the coefficient $\alpha_0$ is presumably a constant of order unity. However, 35 is a peculiarly large value for the number one. Where does this number come from?

To anticipate the rest of this presentation, we now have an analytic theory for cooling waves in a locally heated disk (Vishniac and Wheeler 1996 [10], Vishniac 1997 [11]) and the first two questions can be answered. The third and fourth questions remain confusing.
The analytic theory of cooling waves starts from the three equations which describe a vertically averaged disk model:

\[
\partial_t \Sigma = -\frac{1}{r} \partial_r (r \Sigma V_r),
\]

\[
V_r = \frac{2}{\Sigma \Omega r^2} \partial_r (r^3 \alpha c_s^2 \partial_r \ln \Omega),
\]

and an equation for thermal structure, which we take to be

\[
T = B \Sigma^a \alpha^b \Omega^\frac{3}{7},
\]

following CCL, where \(a = 3/7\), \(b = 1/7\), and \(c = 3/7\). We need to add the condition that below some critical temperature, which is a very weak function of radius, cooling becomes rapid. At this radius the radial thermal gradient becomes very large. This in turn implies that a large fraction of the angular momentum flux from smaller radii is deposited, following equation (8) within a radial scale length, \(r_T\). The resulting gas velocity is

\[
V_r \sim \alpha F c_s F \frac{\Omega F}{r_T}.
\]

Since the gas cools at a rate \(\sim \alpha F \Omega F\), we can estimate \(r_T\) as

\[
V_r \sim \alpha F c_s F \frac{\Omega F}{r_T}.
\]

We conclude that \(r_T\) is some constant times the disk height and \(V_r\) is directed outward.

This rapid outward motion at the cooling front implies that the cooling front is preceded by a rarefaction wave. This suggests that we can understand the progress of the cooling wave by considering the dynamics of the rarefaction wave. If we label a point just ahead of the rarefaction wave with a ‘p’, then conservation of mass implies that

\[
\Sigma_p V_F \approx \Sigma_F \alpha_F c_s F,
\]

or

\[
V_F \approx \alpha_F c_s F \frac{\Sigma_F}{\Sigma_p},
\]

assuming that the cooling front speed is more rapid than the radial motion of the gas at point ‘p’. The column density \(\Sigma_F\) is related to the critical cooling temperature through equation (9). The column density \(\Sigma_p\) needs to
be constrained separately to estimate \( V_F \). We can do this by setting \( V_F \) equal to the accretion velocity at the precursor point. In other words,

\[
V_F \approx \alpha_p \frac{c_{s,p}^2}{r \Omega} \sim \alpha_F c_{s,F} \frac{\Sigma_F}{\Sigma_p}.
\]

(14)

Our rationale for this choice is that no other choice is self-consistent. If \( V_F \) is much faster than this, then the accretion disk does not evolve significantly as the cooling wave plows through it. This decouples the cooling wave behavior from the evolution of \( M \), but it also implies that \( \Sigma_p \) climbs sharply as the cooling wave moves in. If we remember that \( T_F \) is nearly constant, then equation (9) implies that \( \Sigma_F \) actually drops as at smaller radii. This implies a steady decrease in \( \Sigma_F/\Sigma_p \), and a consequent drop in \( V_F \) (following equation (13)). On the other hand, if \( V_F \) is much smaller than the accretion velocity ahead of the rarefaction wave, then the disk evolves faster than the cooling wave can move. This is paradoxical since the draining of the inner disk automatically drops the disk into the cold state.

We can use equations (9) and (14) to solve for \( V_F \) for any assumed form for \( \alpha \). If

\[
\alpha = \alpha_0 \left( \frac{c_s}{r \Omega} \right)^n,
\]

(15)

then

\[
V_F \approx \alpha_F c_{s,F} \left( \frac{c_{s,F}}{r \Omega} \right)^q,
\]

(16)

where

\[
q^{-1} = 1 + \left( 1 + \frac{n}{2} \right) \frac{a}{1 - (n/2)b}.
\]

(17)

For a Kramers opacity law, the usual choice for the disk hot state, and \( n = 3/2 \) this gives \( q = 25/46 \). In general, it is very difficult to get a value of \( q \) which is very different from 1/2 for any of the usual opacity laws.

What have we learned from this scaling law argument? First, this is an observational test for the functional form of \( \alpha \) in the hot state, in spite of the fact that a cooling transition is involved. The dynamics of the wave are controlled by the rarefaction wave in the hot material, and do not depend on the structure of the rapid cooling region. There is a possible loophole here, which we will return to later. The cold material will pile up after it has finished cooling, with a column density not much below \( \Sigma_p \). We have assumed that this is not more than the maximum column density on the lower branch of the ‘S’ curve. If this condition is not satisfied, then gas is not actually ejected at a constant fraction of the thermal speed across the cooling front and we have, instead,
As the ‘S’ curve central branch becomes vertical this gives a cooling front velocity which is close to the accretion speed.

Second, as a test of the model we can ask how it behaves if the upper branch of the ‘S’ curve is almost horizontal, i.e. if the temperature is insensitive to the column density. In this limit, \( a = 0 \) and consequently \( q = 1 \). As expected, the cooling front moves at the accretion velocity. In the opposite limit, when the hot state is just barely thermally stable, \( a \to \infty \) and \( q \to 0 \). That is, the cooling front moves at close to the thermal speed. We can understand the intermediate behavior found by CCL as a reflection of the fact that realistic disks have a hot state which lies between these two extreme cases.

Third, we note that once we allow for the variation of the minimum hot state temperature with radius, a perfect exponential decline actually follows for \( n = 1.63 \). This has been confirmed by subsequent numerical tests (Cannizzo 1997 [12]). This does not imply that this is the correct value for the exponent \( n \), since the observations aren’t necessarily perfectly exponential. Instead, this should be regarded as test of the ability of the scaling law to predict the results of numerical simulations.

We can improve on this scaling law by constructing a similarity solution for the disk gas using the equations of continuity, torque, and thermal structure (Vishniac 1997 [11]). The result is only an approximate fit to the actual solution, since the problem itself has a built-in scale. The appropriate boundary condition for \( \dot{M} \) at the front is

$$
\dot{M}_F = 2\pi r_F \Sigma_F (\alpha_F \Sigma_{hot,\text{minimum}} / \Sigma_{cold,\text{maximum}})^{1-1/q} \left( \alpha_F c_{s,F} \Delta \right),
$$

(19)

where \( \Delta \) is a constant relating the outflow speed at the front to the thermal speed at the front. The analytic theory does not constrain the value of \( \Delta \) so we need to take it from the simulations of Cannizzo et al.. (This gives \( \Delta \approx 1/6 \)). The second boundary condition looks absurd. It is

$$
\Sigma_F = 0.
$$

(20)

In practice this means only that the density contrast through the rarefaction wave is large.

From this we derive a similarity solution with

$$
\frac{\partial_t \dot{M}_{\text{inner}}}{\partial_t \ln r_F} = 2.37,
$$

(21)

and

$$
V_F = 0.94 \alpha_F c_{s,F} \left( \frac{c_{s,F}}{r_F \Omega} \right) q (6\Delta)^{1/2}.
$$

(22)
Both of which are consistent with the simulation results. At any one time more than half the hot phase of the disk is actually moving outward, although the radial speed is close to the usual accretion velocity except in a thin annulus close to the cooling front. Comparing the similarity solution to the numerical work we find that the two differ by about 5-10% which gives us confidence that we understand both of them.

**IMPLICATIONS FOR A THEORY OF ANGULAR MOMENTUM TRANSPORT**

The model presented here for the evolution of the cooling wave is based on assuming that the disk is heated through the local dissipation of orbital energy and that $\alpha$ can be described in terms of local variables. Assuming for the moment that this is basically correct, what are the implications for a fundamental theory of $\alpha$? Knowing that a particular functional form of $\alpha$ fits the available observational evidence falls well short of a fundamental theory for $\alpha$. On the other hand, we can use our results to identify particularly promising models, if any exist. Since we are concerned here with angular momentum transport in the hot, ionized phase we can restrict our attention to models which are most efficient in this phase.

All currently popular models for this regime make use of the Velikhov-Chandrasekhar instability (Velikhov 1959 [13], Chandrasekhar 1961 [14]), first identified by Balbus and Hawley (1991) [15] as the most, and perhaps only, promising mechanism for angular momentum transport in conducting accretion disks. This is an instability of a magnetic field embedded in a strongly shearing flow. The linear instability has a typical growth rate $\sim \Omega$ and a typical length scale $\sim V_A/\Omega$ with a resultant $\alpha \propto V_A^2/c_s^2$. Numerical simulations (cf. the review by Gammie 1997 [16] and references therein) confirm the existence of the instability and its saturation in a turbulent state. Interestingly enough, they also indicate that vertical structure is largely irrelevant for the dynamo process, which allows us to exclude magnetic buoyancy as a driving force for the dynamo. Differences among the models based on these results are due entirely to different proposals for the underlying dynamo mechanism.

First, the turbulence might support an inverse cascade, in which a large scale field is generated spontaneously from the underlying turbulence without any appeal to the kind of symmetry breaking used in mean-field dynamo theory (Balbus and Hawley 1991 [15], see also Gammie 1997 [16] and references therein). Since this is a purely local process, it produces an $\alpha$ which is some constant of order unity, with a relaxation time which is some constant times the eddy turn over time ($\sim \Omega^{-1}$). The available simulations seem consistent with this model over a limited range of box sizes, but the resultant $\alpha$ is a function of grid resolution (e.g. Brandenburg et al. 1996 [17]). Periodic shearing box sizes have a length which varies between 1 and 4 times $2\pi$ times the box
height, which is probably a measure of the sensitivity to disk geometry over
the range $1 < (r/H) < 4$ in a thin disk. However, this model is inconsistent
with the cooling wave results. If the latter are correct, then the model needs
to be modified to allow some non-local effects.

Second, there is the internal wave-driven dynamo (Vishniac & Diamond
1992 [18], and references therein) in which tidally generated waves drive a
dynamo with a growth rate $\sim (H/r)^n$, where $n$ is a bit less than 1.5 in a
stationary disk. Balancing dissipation with growth gives $\alpha \sim \Gamma/\Omega$. This
is the right scaling, but this dynamo model is inherently nonlocal in a bad
sense. The dependence on internal waves propagating through the cool region
should make this dynamo very weak just inside the cooling front, wiping out
the ‘S’ curve effect and completely changing the evolution of the cooling front.
Although no detailed calculation of these effects exist, it seems likely that this
would lead to a broad cooling front, propagating at the viscous speed, with a
dependence on $(H/r)$ which is too steep.

Finally, there has been an attempt to construct a mean field dynamo theory
with no symmetry breaking, the incoherent mean field dynamo (Vishniac and
Brandenburg 1997 [19]). In this model the helicity necessary for generating
large scale radial magnetic field emerges from the random fluctuations pro-
vided by the small scale instabilities of the magnetic field. It can be shown
that this process will drive the exponential growth of large scale, axisymmet-
ric magnetic field domains in the presence of strong shearing. However, this
process becomes less efficient as the number of eddies per annulus increases,
i.e. this dynamo runs more slowly as $H/r$ decreases (or, in simulations, as
the height to length ratio of the periodic shearing box decreases). In a sense,
this is an attempt to implement a geometry dependent inverse cascade in the
context of the first model. Unfortunately, while this model seems qualitatively
promising, it predicts

$$\alpha \sim \left(\frac{V_A}{c_s}\right)^2 \propto \left(\frac{H}{r}\right)^2,$$

(23)

which is too steep. Moreover, the numerical simulations do not show the
expected sensitivity to geometry. This may be because the large value of $\alpha_0$
restricts the asymptotic regime, where the scaling law given in equation (23)
is valid, to boxes considerably longer and thinner than the range explored
to date. In any case, in order to reconcile this model to the cooling wave
calculations, we require dwarf novae and X-ray novae disks to be sufficiently
thick that small corrections to the dynamo growth soften the saturated value
of $\alpha$. More detailed modeling of this dynamo process will be necessary to
judge whether or not this is a realistic hope.

We conclude that no current model is fully consistent with the cooling wave
results, although there is some chance that a more realistic version of the
incoherent dynamo model will prove adequate.
FUTURE PROSPECTS

What can we expect by way of progress in this area? Our ability to set limits on the allowed forms of $\alpha$ is limited by the range of luminosities we can observe and the presence of non-exponential features in accretion disk light curves, e.g. reflares and secondary maxima. As long as these features remain poorly understood it will be difficult to pin down the required form of $\alpha$. However, even within the limits imposed by our current state of understanding the computational models could be made to yield significantly more information. For example, we have no precise idea of the appropriate error bars are on the parameters of the proposed functional form of $\alpha$. CCL originally estimated that the value of $n$ allowed by observational constraints was probably within 15% of the ‘exponential’ value of 1.5, but the value of $n$ which actually yields a perfect exponential decay is 10% higher, and the available grid of models has not yet nailed down the allowed range of $n$.

A more serious danger is the possibility is that the cooling wave model used here neglects some important physical effects. The relaxation of the magnetic field as the cooling wave approaches is difficult to calculate, since the actual dynamo mechanism is, as noted above, a controversial topic. However, we expect the typical rate for this relaxation to be comparable to the dynamo growth rate, which should be $\sim \alpha \Omega$. This is also the thermal relaxation rate in an optically thick disk. The disk structure will evolve at slower rates until the onset of rapid cooling, which is too late to affect the cooling wave arguments. Moreover, even if we assume that the magnetic field is frozen into the plasma from the very beginning of the rarefaction wave we can show that it has no perceptible effect on our results.

Another, and potentially more serious problem is that this treatment assumes that the disk temperature is determined by local dissipation, that is, it ignores irradiation. Attempts to account for the optical light from BH systems suggest that irradiation may be as significant as local energy dissipation (Canizzo 1997 [12]), although it probably does not dominate by a large factor. If we restrict our attention to dwarf novae systems, where irradiation is less important, then we find similar, but less precise, constraints on $\alpha$ (Canizzo 1996 [9]). In any case, the effect of moderate irradiation on the cooling wave structure needs to be explored.

Are there acceptable alternatives to the cooling wave model for these systems, perhaps ones that do not involve a varying $\alpha$? At present the answer is no, although a thorough search has not yet been done. If we wish to consider alternative explanations for the exponential decay of X-ray emission from black hole accretion disk systems, we need models that maintain a constant accretion time scale in the relevant mass reservoir, that is

$$\tau_{acc}^{-1} = \alpha \frac{c_s^2}{r^2 \Omega} = \text{constant}, \quad (24)$$
at the outermost radius which is feeding matter through the hot part of the
disk. In the cooling wave model this is the precursor point, and the somewhat
odd form for the cooling wave speed given in equation (16) is a consequence
of the fact that the precursor point is not at the cooling front. A model in
which irradiation maintains the entire disk in a hot state, so that $r$ is the
tidal boundary of the disk, will satisfy equation (24) only if $\alpha \propto T^{-1}$. A
model dominated by irradiation in which the outer edge of the hot phase is
defined by a constant temperature, and the dynamics of the cooling wave are
suppressed, will work only if $\alpha \propto (H/r)$, which differs from the cooling wave
constraint, but is just as problematic in terms of the fundamental physics of
$\alpha$. The only obvious way to combine an exponential decay law with a constant
$\alpha$ is to present a model in which both the temperature and radius at the disk
edge are nearly constant. This would require a transfer of energy from the
inner disk whose efficiency increases sharply during the decline, perhaps as
a result of coronal scattering through an inner disk corona whose size and
density increases smoothly as $L_x$ decreases. It remains to be seen whether or
not a physically reasonable model can be constructed along these lines.

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