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ON THE RELATION BETWEEN CHIRAL $SU(3) \times SU(3)$
CURRENT ALGEBRA AND $SU(6)$

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ABSTRACT

The general structure of the covariant sum rules derived on the basis of chiral $SU(3) \times SU(3)$ algebra is investigated in the case in which they are saturated by a finite number of intermediate states. We discuss first the consistency of the saturation with states belonging to an $SU(6)$ supermultiplet. We find that consistency is ensured and that the $SU(6)$ results for the renormalization ratios are obtained if mass differences between $SU(3)$ multiplets are neglected. This is due to the fact that for equal masses and parities the matrix elements of the divergences of the axial currents are proportional to the matrix elements of s_3 (third component of the spin operator). We then discuss the situation in which an arbitrary but limited set of states saturates all the matrix elements of the chiral $SU(3) \times SU(3)$ algebra taken among them. It is shown that the matrix elements of the chiral algebra coincide with the matrix elements of a collinear $SU(3) \times SU(3)$ algebra, so that the states that saturate the chiral algebra are a representation of the collinear one, on whose basis the coupling constants can be calculated. The collinear group is not the same as the one generally considered as a subgroup of the static $SU(6)$. It coincides with it only in the case of equal parities and masses of all the members of an irreducible representation.

INTRODUCTION

It has recently been shown that the hypothesis of the time components of the vector and axial currents satisfying the algebra of (chiral) $SU(3) \times SU(3)$ together with the PCAC (Goldberger-Treiman relation) leads to sum rules which allow to calculate g_A/g_V and the F/D ratio of the axial coupling ¹⁾.

On the other hand, $SU(6)$ makes specific predictions for these quantities. We shall see that for the particular case in which only spin $\frac{1}{2}$ baryon octuplet (B) and spin $\frac{3}{2}$ decuplet (Δ) (without imposing equal masses for B and Δ) are considered, either as initial, final or intermediate states in the current matrix elements, the $SU(6)$ results are indeed obtained ^{*}). We shall then show the reason of this coincidence and point out that, in general, if mass differences between $SU(3)$ multiplets are allowed, $SU(6)$ results for coupling constants are not uniquely obtained even by restricting the intermediate, initial and final states to be the members of an $SU(6)$ supermultiplet. This difference is intimately related to the possibility of several invariant forms for the axial current matrix elements or, equivalently, to the presence of more partial waves in the decay of a high spin particle into another one and a zero mass pseudoscalar. The relation between the system of equations for coupling constants obtained on the basis of chiral $SU(3) \times SU(3)$ algebra and of collinear $SU(3) \times SU(3)$ symmetry is then discussed.

In order to derive sum rules from the chiral $SU(3) \times SU(3)$ algebra, we shall use the relativistic invariant method recently proposed ⁴⁾ generalizing it to the case in which matrix elements of the commutators between states of different mass are also considered. Moreover, we shall limit the discussion to the case of zero momentum transfer for which the sum rules can be expressed in terms of the divergences of the currents ⁵⁾.

^{*}) This problem has been investigated to a certain extent by Gerstein, and Bergia and Lannoy ²⁾, and more recently by Gerstein and B. Lee ³⁾.

2.

II. SUM RULES

We start from the equal time commutation relations

$$[\bar{Q}_{(\mu)}(t), \bar{Q}_{(\nu)}(t)] = -\sqrt{3} \begin{vmatrix} 8 & 8 & 8_a \\ \mu & \nu & \lambda \end{vmatrix} Q_{(\lambda)}(t) \quad (1)$$

where $Q_{(\mu)}$ and $\bar{Q}_{(\mu)}$ are the vector and axial "charges" given by

$$\begin{aligned} Q_{(\mu)}(t) &= -i \int d\vec{x} V_4^{(\mu)}(\vec{x}, t) \\ \bar{Q}_{(\mu)}(t) &= -i \int d\vec{x} A_4^{(\mu)}(\vec{x}, t) \end{aligned} \quad (2)$$

(μ) being the SU_3 index in a spherical basis and V_4, A_4 the time components of the vector and axial currents, respectively.

We shall use the method of Refs. 4), 5) for obtaining relativistic invariant sum rules and we shall refer to these papers for the details.

We begin by considering the matrix elements of Eq. (1) between a particle $a_{\alpha 1}(p_1)$ and a particle $b_{\beta 1'}(p_2)$, where α and β are the $SU(3)$ indices and 1 and 1' the possible spin indices. The matrix elements of the left-hand side of Eq. (1) can be analyzed in at most three independent $SU(3)$ scalar amplitudes. Indeed, in the decomposition of an antisymmetric product of two octets, we find $8_a, 10$ and $\bar{1}\bar{0}$ and therefore the three independent scalar amplitudes correspond to the amplitudes $(8_a)_{a\bar{b}} \rightarrow (8_a)_{88}, (8_s)_{a\bar{b}} \rightarrow (8_s)_{88}$ and $(10)_{a\bar{b}} \rightarrow (10)_{88}$. Due to time reversal invariance the $(\bar{1}\bar{0})_{a\bar{b}} \rightarrow (\bar{1}\bar{0})_{88}$ amplitude is just equal to the $(10)_{a\bar{b}} \rightarrow (10)_{88}$ one up to a phase factor which depends

on the phase of the Clebsch-Gordan coefficients appearing in the definition of invariant $SU(3)$ amplitudes. In the notation of de Swart ⁶⁾, this phase is simply $-\frac{1}{3} \begin{Bmatrix} a & \bar{b} & 10 \\ 1 & 1 & 1 \end{Bmatrix}$. The number of independent amplitudes for every pair a, b (let us call this number $F_{a\bar{b}}$) shall therefore be given by the number of times we shall find 8 and 10 or $\bar{10}$ (if 10 and $\bar{10}$ both appear, they count as one) in the decomposition of the product. With the method of Refs. ^{4), 5)}, the matrix elements of the left-hand side of Eq. (1) can be reduced to the expression

$$\langle \bar{a}_{\alpha 1}(p_1) | [\bar{Q}_{(\mu)}, \bar{Q}_{(\nu)}] | b_{\beta 1}(p_2) \rangle = -i(2\pi)^3 \delta(\vec{p}_1 - \vec{p}_2) \frac{\phi(K=0) - \phi(K=p_1 - p_2)}{E_1 - E_2} \quad (3)$$

where ϕ is given by

$$\phi = i \int d^4z e^{iKz} \langle \bar{a}_{\alpha 1}(p_1) | [D_{(\mu)}(z), D_{(\nu)}(0)] | b_{\beta 1}(p_2) \rangle \Theta(z_0) \quad (4)$$

$$D_{(\mu)}(z) = \partial_\alpha A_\alpha^{(\mu)}(z) \quad (5)$$

ϕ , however, is an invariant function and therefore, apart from spin indices, it shall depend on kp, kp_2 and $p_1 p_2$. Let us define

$$\begin{aligned} K &= p_1 - p_2 + K \\ t &= -(p_1 - p_2)^2 \\ s &= -(K + p_1)^2 \end{aligned} \quad (6)$$

4.

We see that, if we impose $k^2 = 0$, $k'^2 = -t$, then $kp_1 = kp_2$ and $kk' = 0$. Kinematically, the condition $k = 0$ coincides with the condition $s = m_1^2$, while $k = p_1 - p_2$ implies $s = m_2^2$ independently of t . We can visualize ϕ as the scattering amplitude for a zero mass pseudoscalar particle on particle a giving b and another pseudoscalar of "mass" t at energy s and square momentum transfer t ; t is therefore just a parameter which we shall consider at zero value. For $m_1 \neq m_2$ the value $t = 0$ is not kinematically accessible and must be considered as reached through an analytic continuation. In the case $m_1 = m_2$, Eq. (3) reduces to

$$\langle a_{\alpha 1}(p_1) | [\bar{Q}_{(\nu)}, \bar{Q}_{(\nu)}] | b_{\beta 1}(p_2) \rangle = i(2\pi)^3 \int (\vec{p}_1 - \vec{p}_2) 2E_1 \left(\frac{d\phi}{d\lambda} \right)_{\lambda=m_1^2} \quad (7)$$

The matrix elements of the charges $Q_{(\lambda)}(t)$ on the right-hand side vanish if particle a differs from particle b , whereas

$$\langle a_{\alpha 1}(p_1) | Q_{(\lambda)} | a_{\beta 1}(p_2) \rangle = (2\pi)^3 \int (\vec{p}_1 - \vec{p}_2) 2E_1 \int_{-1}^1 \sqrt{3} \begin{vmatrix} \beta & 0 & \beta_\alpha \\ \beta & \lambda & \alpha \end{vmatrix} \quad (8)$$

as follows from the fact that the $Q_{(\lambda)}$ are the generators of $SU(3)$.

The next step is to develop $\phi(s)$ in invariants, the number of them depending on the spins of particles a and b . This is done by saturating the spinorial indices among themselves and with $p_{1\mu}$, $p_{2\mu}$, k_μ in all possible ways. We write this decomposition symbolically as

$$\phi(s) = \sum_i \phi_i(s) I_i \quad (9)$$

One then assumes that the $\phi_i(s)$ satisfy unsubtracted dispersion relations in the variable s , so that they are expressed in terms of integrals over their absorptive parts

$$\phi_i(s) = \frac{1}{\pi} \int \frac{A_I(s')}{s' - s} ds' + \frac{1}{\pi} \int \frac{A_{II}(u')}{u' - u} du' \quad (10)$$

The absorptive amplitudes will be given by the projection over the corresponding invariants of

$$A_I = \frac{1}{2} \int dz e^{ikz} \sum_n \langle a_{\alpha 1}(p_1) | D_{(n)}(z) | n(p') \rangle \langle n(p') | D_{(n)}(z) | b_{\beta 1'}(p_2) \rangle \quad (11)$$

$$A_{II} = \frac{1}{2} \int dz e^{ikz} \sum_n \langle a_{\alpha 1}(p_1) | D_{(n)}(z) | n(p') \rangle \langle n(p') | D_{(n)}(z) | b_{\beta 1'}(p_2) \rangle$$

where $\sum_n$ indicates the sum over all intermediate physical states (sum over the discrete ones and integral over the continuum). We can therefore write in a symbolic way our sum rules as

$$\sum_n \frac{\langle a_{\alpha 1}(p_1) | D_{(n)} | n(p_1+k) \rangle \langle n(p_1+k) | D_{(n)} | a_{\beta 1'}(p_2) \rangle - \langle a_{\alpha 1}(p_1) | D_{(n)} | n(p_2-k) \rangle \langle n(p_2-k) | D_{(n)} | a_{\beta 1'}(p_2) \rangle}{(m_n^2 - m^2)^2} \quad (12.a)$$

$$= 3 \int_{11'} \left| \begin{array}{ccc} p & p & p_a \\ \mu & \nu & \lambda \end{array} \right| \left| \begin{array}{ccc} p & p & p_a \\ \beta & \lambda & \alpha \end{array} \right|$$

for particles $a_{\alpha 1}$ and $a_{\beta 1'}$ with $m_1 = m_2 = m$;

$$\sum_n \frac{\langle a_{\alpha 1}(p_1) | D_{(n)} | n(p_1+k) \rangle \langle n(p_1+k) | D_{(n)} | b_{\beta 1'}(p_2) \rangle - \langle a_{\alpha 1}(p_1) | D_{(n)} | n(p_2-k) \rangle \langle n(p_2-k) | D_{(n)} | b_{\beta 1'}(p_2) \rangle}{m_n^2 - m_a^2} \quad (12.b)$$

$$= \sum_N \frac{\langle n \rightarrow N \rangle}{m_N^2 - m_b^2}$$

for particles a and b with $m_a \neq m_b$, where m_n and m_N are the invariant masses of the intermediate states n and N and the matrix elements of the divergences are calculated for vanishing momentum transfer ($k^2 \equiv (p' - p_1)^2 = (p' - p_2)^2 = 0$).

Equations (12.a,b) must be intended as equalities holding between all the scalar functions coming from the development of both sides in invariants.

A first problem arises in the choice of the invariants I_i ; indeed one can combine them arbitrarily with scalar coefficients. The choice shall be dictated by the requirement that the coefficients $\phi_i(s)$ in Eq. (9) have no poles for $s = m_a^2$ and m_b^2 or, in the case of equal masses $m_a = m_b = m$, the development of $d\phi/ds$ has no pole for $s = m^2$. This implies that, when developed in the right invariants, the numerator of Eqs. (12.a,b) shall cancel the mass difference in the denominator for the intermediate states which coincide with the initial or final ones.

3. APPLICATION TO THE B - Δ SYSTEM

As a first application we shall show that, if the axial charges (or the divergences) of Eqs. (12.a,b) connect the spin $\frac{1}{2}$ octuplet (B) and spin $\frac{3}{2}$ decuplet (Δ) among themselves and not to other states, one obtains a set of equations involving the renormalization ratios for β decay (g_A/g_V and D/F ratio), for $B\Delta$ transitions G^* , and for $\Delta\Delta$ transitions G^{**} which have the solutions ($m = m_B$, $M = m_\Delta$)

$$g_A/g_V = \mp \frac{5}{3}, \quad D/F = \frac{3}{2}, \quad \frac{(M+m)^2}{6M^2} G^{*2} = \frac{8}{3}, \quad G^{**} = \mp 3 \quad (13)$$

D/F, the magnitudes of g_A/g_V and G^{**} and their relative sign are those predicted by $SU(6)_{71}$; notice that these results are obtained without requiring B and Δ to have the same mass; if we add this condition, we have also $|G^*| = 2$, which is again the $SU(6)$ result. Indeed, let us first write the general form of the matrix elements of Eqs. (12.a,b), i.e.,

$$\begin{aligned} \langle B_{\alpha 1}(p) | D_{(\mu)} | B_{\beta 1}(p') \rangle &= -i(m+m') \sqrt{2m} \sqrt{2m'} \bar{u}^1(p) \gamma_5 U^{1'}(p') \sum_{\xi} \begin{vmatrix} 8 & 8 & 8 \\ \beta & \mu & \alpha \end{vmatrix} G_{\xi} \\ \langle B_{\alpha 1}(p) | D_{(\mu)} | \Delta_{\beta 1}(p') \rangle &= i \sqrt{2m} \sqrt{2M'} \bar{u}^1(p) \gamma_5 U^{1'}(p') (p-p')_{\mu} \begin{vmatrix} 10 & 8 & 8 \\ \beta & \mu & \alpha \end{vmatrix} G^* \\ \langle \Delta_{\alpha 1}(p) | D_{(\mu)} | \Delta_{\beta 1}(p') \rangle &= \end{aligned} \quad (14)$$

$$= -i(M+M') \sqrt{2M} \sqrt{2M'} \left[\bar{U}_{\mu}^1(p) \gamma_5 U^{1'}(p') + (p-p')_{\nu} \bar{U}_{\nu}^1(p) \gamma_5 U^{1'}(p') \right] \begin{vmatrix} 10 & 8 & 10 \\ \beta & \mu & \alpha \end{vmatrix} G^{**}$$

$$\xi = a, s \quad ; \quad -g_A/g_V = \left(\frac{1}{3}\right)^{1/2} \left[G_a + \left(\frac{9}{5}\right)^{1/2} G_s \right] ; \quad D/F = \left(\frac{9}{5}\right)^{1/2} \frac{G_s}{G_a} \quad (15)$$

8.

where U_μ is a Rarita-Schwinger⁸⁾ spinor representing the Δ . We shall decompose ϕ into invariants in the following way

$$\phi(BB) = \bar{u}(p_1) \left\{ \phi_1(BB) + \left[i\gamma \frac{k+k'}{2}, i\gamma \frac{k-k'}{2} \right] \phi_2(BB) \right\} u(p_2) \quad (16)$$

for BB matrix elements

$$\begin{aligned} \phi(B\Delta) = & \bar{u}(p_1) \left\{ \phi_1(B\Delta) + \left[i\gamma \frac{k+k'}{2}, i\gamma \frac{k-k'}{2} \right] \phi_2(B\Delta) \right\} U_\mu(p_2) \left(\frac{k_\mu}{k p_2} + \frac{k'_\mu}{k' p_2} \right) \\ & + \bar{u}(p_1) \left\{ \phi'_1(B\Delta) + \left[i\gamma \frac{k+k'}{2}, i\gamma \frac{k-k'}{2} \right] \phi'_2(B\Delta) \right\} U_\mu(p_2) \left(\frac{k_\mu}{k p_2} - \frac{k'_\mu}{k' p_2} \right) \end{aligned} \quad (17)$$

for B Δ matrix elements, and

$$\begin{aligned} \phi(\Delta\Delta) = & \bar{U}_\mu(p_1) \left\{ \phi_1(\Delta\Delta) + \left[i\gamma \frac{k+k'}{2}, i\gamma \frac{k-k'}{2} \right] \phi_2(\Delta\Delta) \right\} U_\mu(p_2) \\ & + \frac{\bar{U}_\mu(p_1) k_\mu}{k p_1} \left\{ \phi'_1(\Delta\Delta) + \left[i\gamma \frac{k-k'}{2}, i\gamma \frac{k-k'}{2} \right] \phi'_2(\Delta\Delta) \right\} \frac{U_\nu(p_2) k'_\nu}{k' p_2} \end{aligned} \quad (18)$$

for $\Delta\Delta$ matrix elements.

Inserting Eqs. (14) into Eqs. (12.a,b) and performing explicitly the sums over SU(3) indices and spins of the intermediate particles, using for the last one the spin $\frac{1}{2}$ and $\frac{3}{2}$ projection operators

$$\frac{M' - i\gamma p'}{2M'}, \quad \frac{M' - i\gamma p'}{2M'} \left\{ \delta_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu + \frac{1}{3M'^2} (k_\mu p'_\nu - k'_\nu p_\mu) + \frac{2}{3M'^2} p'_\mu p'_\nu \right\}$$

we can verify that the coefficients ϕ_i of the development into invariants of Eqs. (12.a,b) do not have any singularity. Besides, for the case $t = 0$ we are considering, we have $k^2 = k'^2 = kk' = 0$. It is

then easy to verify that all the terms containing ϕ_2 as well as the one with $\phi_1(B\Delta)$ do not contribute. On the other hand, we see that the δ_{11} , on the right-hand side of Eq. (12.a) is simply $\bar{u}^1 u^1$ for the BB case $\bar{U}_\mu^1 U_\mu^1$ for the $\Delta\Delta$ one, so that we must equate to zero the $\phi_1(\Delta\Delta)$ term obtained. Indeed it is easy to realize that the invariant $\bar{U}_\mu^1 k_\mu / k p_1 U_\nu^1 k'_\nu / k' p_2$, of which $\phi_1(\Delta\Delta)$ is the coefficient, contributes a $\delta_{11} \int |1| \frac{1}{2} \int |1'| \frac{1}{2}$ and gives therefore a dependence on the third component of the spin which is not allowed by the right-hand side of Eq. (12.a). In conclusion, from the point of view of space-time invariants, we obtain one equation for the BB, one for the $B\Delta$ and two for the $\Delta\Delta$ cases.

As discussed previously, the number of independent equations shall be 3 for BB ($3 \otimes 8$ contains $8_s, 8_a$ and 10), two for $B\Delta$ ($8 \otimes 10$ contains 8 and $\bar{10}$) and one for $\Delta\Delta$ ($10 \otimes \bar{10}$ contains 8). Therefore we obtain three equations for the BB, two for the $B\Delta$ and two for the $\Delta\Delta$ cases altogether; explicitly, these are

$$\begin{aligned} \frac{1}{3} G_a^2 \left(1 + \frac{D}{F}\right)^2 - \frac{2}{3} \frac{(M+m)^2}{6M^2} G^{*2} &= 1 \\ \frac{2}{3} G_a^2 \left[1 + \frac{1}{3} \left(\frac{D}{F}\right)^2\right] + \frac{1}{6} \frac{(M+m)^2}{M^2} G^{*2} &= 2 \\ \frac{1}{3} G_a^2 \left(1 - \frac{D}{F}\right)^2 + \frac{1}{3} \frac{(M+m)^2}{6M^2} G^{*2} &= 1 \end{aligned} \quad (19)$$

for BB matrix elements, and

$$\begin{aligned} \frac{1}{\sqrt{3}} G_a \left(1 + \frac{D}{F}\right) - \frac{5}{9} G^{**} &= 0 \\ \frac{1}{\sqrt{3}} G_a \left(1 - \frac{D}{F}\right) + \frac{1}{9} G^{**} &= 0 \end{aligned}$$

for $B\Delta$ matrix elements, and

10.

$$\frac{(M+m)^2}{4M^2} G^{*2} - \frac{4}{9} G^{**2} = 0$$

$$G^{**2} = g$$

for $\Delta\Delta$ matrix elements. As it is easily verified, Eqs. (19) give indeed the results stated in (13).

IV. DISCUSSION OF THE COMPATIBILITY WITH SU(6)

At first sight, the fact that B and Δ could exhaust the intermediate states of all the matrix elements of the commutators of Eq. (1) taken among them, would seem surprising. Indeed, we are dealing with matrix elements of vector and axial charges (i.e., Q_0 and $\bar{Q}_0$) which, being not generators of SU(6), would leak out from the SU(6) multiplet to which they are applied.

Let us, therefore, ask, firstly the reason for this compatibility of chiral SU(3) $\times$ SU(3) algebra with saturation with SU(6) multiplets - even allowing for mass differences - and, secondly, if this is general, i.e., if we expect this to happen not only for the representation 56 but for every other one.

Let us try first to understand the SU(6) result for the case we have treated (B and Δ). After the sum rules were reduced to the form of Eqs. (12), we restricted the intermediate states to B and Δ and we computed the sum over spin and SU(3) indices using the expressions of Eq. (14). These expressions are indeed pseudoscalar and therefore we can calculate them in any frame of reference with no lack of generality. Let us therefore calculate them in a reference frame for which p_1, p' - and therefore k - are collinear and let us quantize the spin in their common space direction. By using the explicit form of the plane wave Dirac and Rarita-Schwinger spinors, it is easy to verify that

$$\begin{aligned} \lim_{m' \rightarrow m} \frac{1}{m'^2 - m^2} \bar{u}_{(\alpha)}^1(p_i) \lambda^{(\mu)} \gamma_5 u_{(\beta)}^{1'}(p') &= \frac{1}{2m^2} \chi_{(\alpha)}^{*1} \lambda^{(\mu)} s_3 \chi_{(\beta)}^{1'} = \\ &= \frac{1}{2m^2} 1 \delta_{11'} \sum_{\xi} \begin{vmatrix} 8 & 8 & 8_{\xi} \\ \beta & \mu & \alpha \end{vmatrix} G_{\xi} \end{aligned} \quad (20.a)$$

$$\begin{aligned} \frac{1}{M^2 - m^2} K_{\mu} \bar{U}_{\mu(\alpha)}^1(p_i) \lambda^{(\mu)} u_{(\beta)}^{1'}(p') &= \frac{1}{\sqrt{3}M} \chi_{(\alpha)}^{*1} \lambda^{(\mu)} s_3 \chi_{(\beta)}^{1'} = \\ &= \frac{1}{\sqrt{3}M} 1 \delta_{11'} \begin{vmatrix} 10 & 8 & 8 \\ \beta & \mu & \alpha \end{vmatrix} G^* \end{aligned} \quad (20.b)$$

$$\begin{aligned} \lim_{M' \rightarrow M} \frac{1}{M'^2 - M^2} \bar{U}_{\mu(\alpha)}^1(p_i) \lambda^{(\mu)} \gamma_5 U_{\mu(\beta)}^{1'}(p') &= \frac{1}{6M^2} \chi_{(\alpha)}^{*1} \lambda^{(\mu)} s_3 \chi_{(\beta)}^{1'} = \\ &= \frac{1}{6M^2} 1 \delta_{11'} \begin{vmatrix} 10 & 8 & 10 \\ \beta & \mu & \alpha \end{vmatrix} G^{**} \end{aligned} \quad (20.c)$$

$$\begin{aligned} \lim_{M' \rightarrow M} \frac{1}{M'^2 - M^2} K_{\mu} \bar{U}_{\mu(\alpha)}^1(p_i) \lambda^{(\mu)} \gamma_5 K_{\nu} U_{\nu(\beta)}^{1'}(p') &= \frac{(M^2 - M'^2)^2}{6M^4} \chi_{(\alpha)}^{*1} \lambda^{(\mu)} s_3 \chi_{(\beta)}^{1'} \int_{11, \frac{1}{2}} = \\ &= \frac{(M^2 - M'^2)^2}{6M^4} 1 \delta_{11'} \int_{11, \frac{1}{2}} \begin{vmatrix} 10 & 8 & 10 \\ \beta & \mu & \alpha \end{vmatrix} \eta G^{**} \end{aligned} \quad (20.d)$$

where $\lambda^{(\mu)}$ are the (spherical) $SU(3)$ generators. s_3 is the third component of the spin operator and χ, χ' are the Pauli spinors of the corresponding particles. We see that the first three matrix elements are just proportional to the matrix elements of σ_3 , while the fourth one has not this proportionality due to the $\int_{11, \frac{1}{2}}$. If we insert expressions (20.a to d) into Eqs. (12) we see that, due to the $(M^2 - M'^2)^2$ factor of Eq. (20.d), this matrix element does not contribute to the sum

rules. Notice that at $t = 0$ we have k' collinear with k . Therefore, when calculating the matrix elements of the divergences taken between intermediate and final states of momenta p' and p_2 in a frame in which p' , p_2 and k' are collinear, this frame will coincide with the one considered for the matrix elements between initial and intermediate states. We are then left with just the same sum rules we would have obtained by taking the matrix elements of $\sigma_3 \lambda^{(\mu)}$, which are indeed generators of $SU(6)$. This fact immediately clarifies why we obtained consistency by saturating the sum rules with $SU(3)$ multiplets which belong to an $SU(6)$ supermultiplet. The fact that the $SU(6)$ results are founded just reflects the known fact that the collinear $SU(3) \times SU(3)$ group defined by the generators $(1 \pm \sigma_3) \lambda^{(\mu)}$ is sufficient to establish the results we quoted from $SU(6)$.

On the basis of the example shown, it is easy to understand the general situation. In general, if we consider matrix elements of our commutators between states that would belong to an $SU(6)$ representation without imposing, however, different $SU(3)$ submultiplets to have the same mass, we shall not obtain the $SU(6)$ results for the coupling constants as a unique solution. The reason is that the matrix elements of the axial divergence between different particles with spin higher than $\frac{1}{2}$ for fermions, (higher than 0 for bosons) have in general more than one invariant form. This is related to the possibility of having different angular momentum in the decay of one of the particles into the other one and a zero mass pseudoscalar. We would obtain the $SU(6)$ results for the coupling constants only if we impose all the angular momenta except the p wave (we are discussing equal parity particles due to the association with $SU(6)$ multiplets) to be absent. This implies to impose only one invariant form for the matrix element of D if $|\Delta J| \leq 1$ (ΔJ = difference of spin of the two states among which the divergence is considered) and a vanishing matrix element if $|\Delta J| > 1$. A way of having automatically this situation is to have all

particles with the same mass ; in that case, all invariant matrix elements of D , having more than one k_μ in their definition would not contribute to Eqs. (12). Or, in other words, higher partial waves in the decay are just absent due to angular momentum barrier.

Of course, in the case of equal masses or of no other than p waves, again the matrix elements of D reduce to matrix elements of σ_3 in the particular frame in which the momenta are collinear ^{*)} and, therefore, the consistency with the $SU(6)$ result for coupling constants is ensured.

On the basis of the preceding discussion, let us summarize which shall be the general structure of the sum rules obtained from the chiral $SU(3) \times SU(3)$ algebra considering a finite number of discrete intermediate states to saturate the commutation relations. The only symmetry we shall suppose is $SU(3)$, so that our states shall be $SU(3)$ multiplets ^{**)}. We shall obtain the sum rules by considering the matrix elements of Eq. (1) between every pair of states, writing down the most general form for the axial divergences, summing over the spins and $SU(3)$ indices of the intermediate states of Eqs. (12), developing the product into invariants and equating to the right-hand side the invariant which gives no dependence on the helicity and to zero the other ones.

Every form for the divergence matrix elements between the states a and b at $t = 0$ shall contain zero, one or two parameters (axial coupling constants), depending on whether the octet appears zero times, once or twice in the $SU(3)$ decomposition of the product

^{*)} This fact has been noted by Cabibbo and Radicati ⁹⁾ and Gerstein and Lee ³⁾.

^{**)} The same discussion can be done for chiral $SU(2) \times SU(2)$, classifying the states as isospin multiplets.

$a \times \bar{b}$. Let us call this factor $f_{a\bar{b}}$. On the other hand, the number of possible invariant forms is equal to $s_{a\bar{b}} + \frac{1}{2}$ (we are considering fermions) where $s_{a\bar{b}}$ is the smaller of particles a and b spins ^{*}). However, for matrix elements between equal mass particles, only one of these invariant forms shall contribute to Eqs. (12). Therefore, the number of independent parameters entering in the sum rules shall be given by

$$N = \sum_{a,b \neq a} \left(s_{a\bar{b}} + \frac{1}{2} \right) f_{a\bar{b}} + \sum_{a,b=a} f_{a\bar{b}} \quad (21)$$

where the sums are extended to all pairs of states considered (independently of whether they are external or intermediate) and where by $b = a$ ($b \neq a$) on the sum indices we intend multiplets having equal (different) masses.

Now let us count how many equations we would have for every pair of $SU(3)$ multiplets a and b between which we consider the matrix elements of our commutators in Eq. (1). In developing the left-hand side of Eq. (1) into invariants, we clearly obtain for every $SU(3)$ scalar equation $s_{a\bar{b}} + \frac{1}{2}$ sum rules; the number of $SU(3)$ scalar equations on the other hand is the number $F_{a\bar{b}}$ discussed before.

^{*}) This implies that there is a parameter for every absolute value of the helicity 1 and that

$$\langle a_1 | D | b_1 \rangle = \text{sign } l \delta_{11'} g_{ab}^{|1|} \quad (i)$$

if a and b have the same parity, and

$$\langle a_1 | D | b_1 \rangle = \delta_{11'} g_{ab}^{|1|} \quad (ii)$$

if their parity is opposite.

For the equal mass case, the only invariant that remains is just the one whose matrix elements are proportional to the helicity, for equal a and b parity (i.e., $g_{ab}^{|1|} = |1| g_{ab}$ in Eq. (i)) and independent of l for opposite a and b parity (i.e., $g_{ab}^{|1|} = g_{ab}$ in Eq. (ii)).

Therefore, we shall have

$$M = \sum_{a,b} \left(1_{a\bar{b}} + \frac{1}{2} \right) F_{a\bar{b}} \quad (22)$$

equations, where $\sum$ ' sums over all pairs of external particles considered.

As discussed earlier, we shall obtain the $SU(6)$ results for the coupling constants if we restrict the external and intermediate states to be those that belong to an $SU(6)$ supermultiplet and if we impose all the $s_{a\bar{b}} + \frac{1}{2}$ parameters except one (that is the one describing p wave decay and which gives $g_{ab}^{||1} = |1|g_{ab}$ in Eq. (i)) if $|s_a - s_b| \leq 1$ and to be all zero if $|s_a - s_b| > 1$.

The question now arises, which sort of limitations, or predictions, we should obtain if it happens that all matrix elements of the commutators in Eq. (1) taken between every pair of particles of a given set are just saturated by the particles of that set. Let us then investigate, similarly to what we have done in Section 3 with B and Δ , the situation in which the same particles act as a complete set of both external and intermediate states in our sum rules. In order to have some hope that this situation is not entirely academic, we must allow for different masses in different $SU(3)$ multiplets. In this case the sums in Eqs. (21) and (22) both run over every pair of the set of particles considered. Due to the fact that $F_{ab} \geq f_{ab}$, it is clear that we have in general more equations than parameters. This is not surprising: if we want to saturate an algebra with certain states, we must require these states to form a representation of such an algebra (not necessarily irreducible). Indeed, we shall show that the set of states in question shall just be a representation of a collinear $SU(3) \times SU(3)$ group, this collinear group being defined as the one whose generators are given by $Q_{(\mu)} \pm \bar{Q}_{(\mu)}^3$ acting on one particle states of momentum $p \rightarrow$ in the z direction, being

$$\bar{Q}_{(\mu)}^3 = -i \int d\vec{x} A_3^{(\mu)}(\vec{x}, t) \quad (23)$$

In order to show this, we first notice that

$$\begin{aligned} \langle a(p_1) | D_{(\mu)}(0) | b(p_2) \rangle \delta(\vec{p}_1 - \vec{p}_2) &= -i(E_1 - E_2) \langle a(p_1) | A_4^{(\mu)}(0) | b(p_2) \rangle \delta(\vec{p}_1 - \vec{p}_2) \\ &= (E_1 - E_2) \langle a(p_1) | \bar{Q}_{(\mu)}(0) | b(p_2) \rangle \end{aligned} \quad (24)$$

where particles a and b have masses m_1 and m_2 and a common momentum p , so that $E_1 = (p^2 + m_1^2)^{\frac{1}{2}}$, $E_2 = (p^2 + m_2^2)^{\frac{1}{2}}$. The square momentum transfer corresponding to the divergence matrix element shall be given by $t = (E_1 - E_2)^2$, which depends on the frame of reference and tends to zero as $(m_1^2 - m_2^2/p)^2$ when $p \rightarrow \infty$.

On the other hand, we can show that, when $p \rightarrow \infty$

$$\langle a(p_1) | \bar{Q}_{(\mu)} | b(p_2) \rangle = \langle a(p_1) | \bar{Q}_{(\mu)}^3 | b(p_2) \rangle \quad (25)$$

A simple way to prove it is to consider a Lorentz transformation with β given by

$$\beta = \frac{p}{\sqrt{p^2 + m_1^2}} \quad (26)$$

which brings the system we are dealing with ($\vec{p}_1 = \vec{p}_2$), to a system in which $p_1' = 0$, $E_1' = m_1$, $p_2' = (m_1^2 - m_2^2)/2m_1$ in the z direction, $E_2' = (m_1^2 + m_2^2)/2m_1$. Then

$$\langle a(p_1) | \bar{Q}_{(\mu)} | b(p_2) \rangle = \langle a(p_1') | \frac{\bar{Q}_{(\mu)} - \beta \bar{Q}_{(\mu)}^3}{\sqrt{1 - \beta^2}} | b(p_2') \rangle \quad (27)$$

$$\langle a(p_1) | \bar{Q}_{(\mu)}^3 | b(p_2) \rangle = \langle a(p_1') | \frac{\beta \bar{Q}_{(\mu)} - \bar{Q}_{(\mu)}^3}{\sqrt{1 - \beta^2}} | b(p_2') \rangle$$

It is easy to show that the matrix elements $\langle a(p'_1) | \bar{Q}(\mu) | b(p'_2) \rangle$ and $\langle a(p'_1) | \bar{Q}^3(\mu) | b(p'_2) \rangle$ are in a ratio which in general is different from 1. Indeed $\langle a(p'_1) | \bar{Q}(\mu) | b(p'_2) \rangle$ contains an extra factor $(m_1 - m_2)$ with respect to $\langle a(p'_1) | \bar{Q}^3(\mu) | b(p'_2) \rangle$ if $P_1 P_2 (-1)^{|J_1 - J_2|} = 1$ (or vice versa if $P_1 P_2 (-1)^{|J_1 - J_2|} = -1$), where P_1 and J_1 (P_2 and J_2) are the parity and spin of particle 1 (particle 2).

If $p \rightarrow \infty$, then $\beta \rightarrow 1$ [cf., Eq. (26)] and therefore, from Eqs. (27), the matrix elements of $\bar{Q}(\mu)$ and $\bar{Q}^3(\mu)$, even for different mass particles, are equal in that system. Therefore, from Eq. (24), also the matrix elements of the axial divergence for $t = 0$ are proportional to the matrix elements of $\bar{Q}^3(\mu)$ in the system $p \rightarrow \infty$.

This fact proves the equivalence we have stated above between chiral and collinear $SU(3) \times SU(3)$. Of course, the results for the axial renormalization constants (for every invariant or, equivalently, for every helicity), shall be those given by the collinear group.

We want to stress, however, that due to the different masses, the collinear $SU(3) \times SU(3)$ we have discussed is the one which contains $\bar{Q}^3(\mu)$ (intended as an operator acting on states with $p \rightarrow \infty$) and not $s_z \lambda(\mu)$, where s_z is the z component of the spin (or of the total angular momentum 10)).

If the masses were equal, then the two states $|a(p'_1)\rangle$ and $|b(p'_2)\rangle$ of Eq. (27) would both be at rest and - for equal parity particles - the matrix element $\langle a(p'_1) | \bar{Q}(\mu) | b(p'_2) \rangle$ would vanish. As a consequence, the matrix elements of $\bar{Q}(\mu)$ for $p \rightarrow \infty$, $\bar{Q}^3(\mu)$ for $p \rightarrow \infty$ and $p = 0$ would just be proportional to each other and to the matrix element of s_z . This is a more formal explanation of the fact already mentioned before that only if the masses are the same the matrix elements appearing in the sum rules are proportional (for equal parity particles) to the matrix elements of s_z .

Summarizing, we found that, if a set of $SU(3)$ multiplets saturates all matrix elements of the commutators of Eq. (1) taken between them, then these particles - independently of their masses and parities - form a representation of a collinear $SU(3) \times SU(3)$ group. This collinear group, however, is defined by the generators $Q_{(\mu)} \pm \bar{Q}_{(\mu)}^3$ applied to one-particle states in the z direction and with $p \rightarrow \infty$. It coincides with the collinear subgroup of the $SU(6)$ static group defined by the generators $Q_{(\mu)}, \bar{Q}_{(\mu)}^i$ operating over particles at rest, only if the parities are the same and if mass differences between different $SU(3)$ multiplets are neglected.

On the other hand, if we try to avoid the use of currents as generators of groups, we see that a grouping of particles in representations of collinear $SU(3) \times SU(3)$ is implied by the requirement that the matrix elements of the algebra of Eq. (1) between any pair of a set of states be saturated by these states.

This point of view, even if it has no strong theoretical background to support it, avoids some of the difficulties encountered in defining a relativistic group on the basis of $SU(6)$ ¹¹⁾.

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