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LEVINSON THEOREM FOR S-MATRIX THEORY
WITH INELASTICITY

D. Atkinson^{*)} and D. Morgan^{**)}

CERN - Geneva

A B S T R A C T

A form of Levinson's theorem is stated and proved for a certain class of relativistic, inelastic scattering amplitudes. The notion of the CDD class of an amplitude is introduced. This enables one to use the theorem to say how many elementary particles are associated with the scattering process in question. Lastly, these considerations are generalized, on a less rigorous basis, to include cases in which $\delta(\infty) \neq 0(\pi)$.

*) Present address: Physics Department, University of California, Berkeley, California, U.S.A.

***) Present address: Rutherford High Energy Laboratory, Chilton, Didcot, Berks, England.

1. INTRODUCTION

The main object of this paper is the statement and proof of certain sufficient conditions for a form of Levinson's theorem that is applicable to relativistic scattering with inelasticity.

Levinson's original theorem¹⁾ applied to non-relativistic potential scattering. For a certain class of potentials, he showed that $n_B = -\delta(\infty)/\pi$, where n_B is the number of bound states in a particular partial wave, and $\delta(s)$ is the phase shift, as a function of the energy squared. The phase shift is set equal to zero at the threshold, $s = s_1$. The result is a powerful one in potential theory, revealing the close relation between bound states and resonances: its postulation assumed a key role in the interpretation of S-matrix dispersion relations in relativistic scattering. This importance arose from the existence of the well-known CDD ambiguity²⁾, according to which the equations of dispersion dynamics have an infinity of solutions. This ambiguity was identified with the possibility of there being elementary particles coupled to the scattering channel (in a Lagrangian field theory formulation^{3,4)}). It was speculated that n_B in Levinson's theorem be replaced by $n_B - n_p$, where n_p denotes the number of elementary particles in the scattering channel. Thus, $n_B - n_p$ is the number of "non-elementary" states. The role of Levinson's theorem in this form, as the central technique of particle accountancy, is very clearly brought out in Ref. 5).

The relation to be proved is $n_B - n_p = -\delta(\infty)/\pi$. The first point to note is that this statement refers not only to the amplitude and its phase shift, but also to a particular scheme of dynamical equations, in order that n_B and n_p be given operational definitions. It has been shown that any amplitude, which has a bounded phase shift that has a finite number of oscillations, has an N/D decomposition⁶⁾, in which N and D have certain prescribed asymptotic bounds, depending on the kinematics. In this decomposition, it is natural to identify the existence of poles of D with the occurrence of elementary particles for the following reasons: Firstly, in certain simple cases the relation $\delta(\infty)/\pi = n_c - n_B$, where n_c is the number of poles of D, can be demonstrated. They are familiarly known

as CDD poles. Secondly, each CDD pole introduces two extra constants, which can be associated with the mass and coupling constant of an elementary particle. Lastly, consider an amplitude and a modified version of it, obtained by coupling in weakly an elementary particle. The modified amplitude will have a pole and a nearby zero that were not present in the original amplitude. Assuming analyticity in the coupling, this state of affairs must persist as the coupling is increased, unless the zero leaves the physical sheet of energy.

It will be shown that, in the purely elastic case, these CDD poles can always be associated with some of the zeros of the amplitude. In general, not all of the zeros need be CDD zeros, and any zero of an amplitude that has CDD zeros may serve as such. In certain models, it may be possible to associate particular zeros with particular particles (e.g. the continuation in coupling method); but from a strict N/D approach, all zeros are equivalent.

The first step in proving Levinson's theorem from S-matrix equations, still within the assumption of elastic unitarity, is to construct

$$\mathcal{D}(s) = \exp \left[-\frac{s}{\pi} \int_{s_0}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)} \right]$$

Then the most general D function is given by $D(s) = R_D(s) \mathcal{D}(s)$, where $R_D(s)$ is a rational function of s . Zeros and poles of R_D are identified as bound states and CDD poles, respectively; furthermore D has the asymptotic behaviour $s^{\delta(\infty)/\pi}$, apart from possible logarithm terms. The main effort is then deployed in establishing the asymptotic behaviour of D, which is to be inferred from the dynamical equations. It is at this point that the main restrictions on the input information, the forces and (where present) the inelasticity, have to be introduced.

R.L. Warnock⁷⁾ has generalized the considerations on elastic scattering to an inelastic system. He assumed that N and D satisfy once subtracted dispersion relations, and that the resultant integral equation for N is Fredholm.

In Section 2 below, a similar discussion is presented for the case in which N and D are constructed to have no subtractions, and explicit sufficient conditions are stated. Whereas in the elastic case it can be shown (see Appendix) that the number of zeros of the amplitude is always at least n_0 , this need not be so when inelasticity is admitted. In such a case, some of the poles of D do not correspond to zeros of the amplitude (since N must have a cancelling pole). This might appear to be reducing Levinson's theorem to a mere definition of n_0 . In fact, the crucial property is not the identification with zeros, but a more general concept, that of the CDD class, which will be defined in Section 2. The essential idea is to map the complete set of unitary amplitudes on to the set of possible N/D solutions. The index of the CDD class is then half of the number of parameters that must be supplied, in order to set up the N/D equations, in addition to the force and inelasticity.

2. THE CONCEPT OF A CDD CLASS

Suppose a scattering amplitude $A(s)$ to be given. Let $\delta(s)$ be the phase shift if one is concerned with an elastic system, the real part of this if one is to set up the inelastic equations of Frye and Warnock (FW)⁸⁾, or simply the phase of $A(s)$ if one is dealing with the Chew-Mandelstam (CM) equations⁹⁾. In any case, supposing $\delta(s)$ to have a limit as $s \rightarrow \infty$, and to be differentiable, except possibly on a set of measure zero, one can construct

$$\mathcal{D}(s) = \exp \left[-\frac{s}{\pi} \int_{s_0}^{\infty} \frac{\delta(s') ds'}{(s'-s)s'} \right] \quad (2.1)$$

It will be assumed for the present, and proved for a certain stated class of amplitudes in Section 3, that $\delta(\infty) = m\pi$, with m an integer. Two separate N/D decompositions will now be constructed with the aid of Eq. (2.1), in order to elucidate the concept of a CDD class. The first is

$$A = n/d \quad (2.2)$$

where d is constructed to tend to a real constant at infinity, and the second is

$$A = N/D \quad (2.3)$$

where D has no poles, but may require subtractions.

To define $d(s)$ in detail: it must have the given phase $\delta(s)$ for $s_0 \leq s < \infty$; it must have a zero corresponding to each of the particle poles of A (whether elementary or not); and it must tend to a constant at infinity. By the Riemann-Schwartz principle, this constant must be real, and it may be normalized to one. Thus $d(s)$ can be written, in the case $m + n_B \geq 0$,

$$d(s) = \frac{\prod_{i=1}^{n_B} (s - p_i)}{\prod_{j=1}^{m+n_B} (s - s_j)} \mathcal{D}(s) \quad (2.4)$$

The zeros at p_i ($i = 1 \dots n_B$) are the particle pole positions. In contradistinction to the usual practice, the s_j will be taken to be arbitrary: the reason for this departure from normalcy will become clear immediately. Since $n = Ad$, it will, in general, share all the poles of d . In the converse case $m+n_B < 0$, one writes, instead of Eq. (2.4):

$$d(s) = \prod_{i=1}^{n_B} (s - p_i) \prod_{j=1}^{|m+n_B|} (s - q_j) \mathcal{D}(s) \quad (2.5)$$

where the q_i are arbitrary. This has the effect of introducing $|m+n_B|$ arbitrary zeros into d , and consequently into n also. In this case, there are moment conditions on n and d , as discussed below.

In either case, the index of the CDD class to which the amplitude A belongs is defined to be $n_c = m+n_B$. It will be shown that, for a given force and a given inelasticity (if present), the general solution belonging to a particular class n_c has precisely $2n_c$ arbitrary parameters. It can be seen that all CDD classes of given order contain those of lower order. (This is equivalent to the notion of "unnecessary subtractions", as will be seen below.) The $2n_c$ parameters are the residues of n and d at each of the n_c poles. If an amplitude belongs to a CDD class n_c , but to no class of smaller index, then there are said to be n_c elementary particles in the problem. In general, it may not be possible to say which particles are elementary, and which are not. Indeed, in this context, elementarity applies not to a particle, but to a property of a scattering process.

It will now be shown that for the case $n_c > 0$, the pole positions are indeed arbitrary, and do not constitute additional parameters. Consider, for simplicity, the case $n_c = 1$, so that n and d have one pole each. The Cauchy representations of d and n are:

$$d(s) = 1 + \frac{a_1}{s-s_1} + \frac{1}{\pi} \int_{s_0}^{\infty} \frac{\rho(s') n(s')}{s'-s} ds' \quad (2.6)$$

$$n(s) = \frac{b_1}{s-s_1} + \frac{1}{\pi} \int_{-\infty}^0 \frac{\text{Im} A(s') ds'}{s'-s} \quad (2.7)$$

Here, $\text{Im} A(s)$ is the discontinuity of A on the left-hand cut. In the elastic case, $\rho(s)$ is the phase space factor, while in the CM case this is multiplied by the inelasticity factor. For the FW system, the equations must be reformulated; but the essential point remains unchanged. In Eqs. (2.6) and (2.7) a_1 and b_1 are the two bona fide parameters; and s is the arbitrary pole position. To verify that the above equations do represent n and d , note that, as has been mentioned above, d tends to a real constant at infinity. This implies $\text{Im} d \rightarrow 0$, so that an unsubtracted dispersion relation can be written for $d-1$. Since $\text{Im} d = -\rho n$ for $s > s_0$, and since $\rho(s) \sim 1$ (for spinless particles), it follows that $n(s) \rightarrow 0$ for $s \rightarrow +\infty$. If it is supposed that this limit is uniform in all directions (a point that will be taken up in the next section), then $n(s)$ satisfies an unsubtracted dispersion relation.

If one forms

$$\begin{aligned} d_2(s) &= \frac{s-s_1}{s-s_2} d(s) \\ n_2(s) &= \frac{s-s_1}{s-s_2} n(s) \end{aligned} \quad (2.8)$$

it is easy to see by direct substitution in Eqs. (2.6) and (2.7) that d_2 satisfies an equation exactly similar to Eq. (2.6), but with $n(s')$ replaced by $n_2(s')$ and a_1 replaced by a_2 , with:

$$a_2 = a_1 + (s_2 - s_1) + \frac{s_2 - s_1}{\pi} \int_{s_0}^{\infty} \frac{\rho(s') \eta_2(s')}{s' - s_1} ds' \quad (2.9)$$

A similar statement can be made for $\eta_2(s)$. Thus d_1 and d_2 do belong to the same CDD class, and s_1 is not a parameter.

At this point it will be useful to reconsider the analysis from the point of view of the decomposition (2.3), in which D has no poles, but can have any number of subtractions. It will be possible to give a unified treatment of both representations [Eqs. (2.2) and (2.3)], and the place of homogeneous equations will also be indicated.

First, consider the case $n_c \geq 0$. In order that D shall have no poles, one must write

$$D(s) = \prod_{j=1}^{n_c} (s - s_j) d(s) \quad (2.10)$$

Thus $\text{Re } D(s) \sim s^{n_c}$ and $\text{Im } D(s)/s^{n_c} \rightarrow 0$ as $s \rightarrow \infty$; so that there exists a dispersion relation for D with n_c subtractions. It can be seen directly from the equations for n and d that N and D satisfy the equations

$$D(s) = 1 + \alpha_1 (s - x_0) + \dots + \alpha_{n_c} (s - x_0)^{n_c} + \frac{(s - x_0)^{n_c}}{\pi} \int_{s_0}^{\infty} ds' \frac{\rho(s') N(s')}{(s' - s)(s' - x_0)^{n_c}} \quad (2.11)$$

$$N(s) = \beta_1 + \beta_2 (s - x_0) + \dots + \beta_{n_c} (s - x_0)^{n_c-1} + \frac{(s - x_0)^{n_c}}{\pi} \int_{s_0}^{\infty} ds' \frac{\text{Im } A(s') D(s')}{(s' - s)(s' - x_0)^{n_c}} \quad (2.12)$$

One can verify algebraically that the subtraction point x_0 is not a parameter, and that the $2n_c$ parameters α_i and β_i are determined by the residues a_i and b_i .

The case $n_c = -1$ will now be treated: the general class $n_c < 0$ will then be considered. Here $d(s)$ and $n(s)$ have one arbitrary zero each, which cancels in the ratio n/d . What are the candidates for N and D in this case? One possibility is $N = n$, $D = d$, when N and D satisfy unsubtracted relations, and have coincident zeros. This could be regarded as a dynamical solution with a bound state (since $D(q_1) = 0$ but with zero residue (since $N(q_1) = 0$), which one might term an extinct bound state. Situations in which such a possibility exists can be envisaged. For example, N and D might have nearby zeros that only coincide for particular values of some external parameters. However, a second possible choice is

$$D_R(s) = d(s) / (s - q_1) \quad , \quad N_R(s) = n(s) / (s - q_1) \quad (2.13)$$

Here $D_R(s) \sim 1/s$, so that the dispersion relations are homogeneous:

$$D_R(s) = -\frac{1}{\pi} \int_{s_0}^{\infty} \frac{\rho(s') N_R(s')}{(s' - s)} ds' \quad (2.14)$$

$$N_R(s) = \frac{1}{\pi} \int_{s_0}^{\infty} \frac{\text{Im} A(s') D_R(s')}{(s' - s)} ds'$$

and the following moment conditions must be satisfied:

$$\int_{s_0}^{\infty} \rho(s') N_R(s') ds' = 0 = \int_{s_0}^{\infty} \text{Im} A(s') D_R(s') ds' \quad (2.15)$$

The general case $n_c < 0$ will have the two candidates for N and D as above, and also a number of intermediate situations - B_0 extinct bound states and $|n_c| - B_0$ pairs of moment conditions. All cases with extinct bound states can be put into correspondence with cases with moment conditions, and this will be done for the purposes of classification.

This discussion illustrates a general point. Suppose an amplitude of class n_c were identified with an amplitude calculated with $n_c + 1$ poles in both n and d . Then two of the $2(n_c + 1)$ residues would not be arbitrary, as is the case for a bona fide class $(n_c + 1)$ amplitude. The same situation can also be traced through in terms of the N/D decomposition. Suppose that the same amplitude were to be extracted from N and D equations with $(n_c + 1)$ subtractions each. In this case, either $D(s) \sim s^{n_c + 1}$, in which case D and N have coincident zeros at some point $s = s_1$ (an extinct bound state), or $D(s) \sim s^{n_c}$, in which case two moment conditions like Eq. (2.14) must be satisfied. In this latter situation, one could say that the equations have one unnecessary subtraction.

To sum up, it is convenient to classify an amplitude, according to the lowest CDD class to which it belongs, in such a way that all parameters are arbitrary. If there is a positive number of pairs of parameters ($n_c > 0$), one says that n_c elementary particles are present. If $n_c < 0$, one speaks of n_c extinct bound states, or n_c moment conditions. The Levinson theorem enables one to determine n_c , given n_B and $\delta(\infty)$, and so to assign the amplitude to its CDD class, and to know how many elementary particles there are in the scattering channel.

3. PROOF OF LEVINSON'S THEOREM

In this section a proof will be given which applies to an amplitude that can be written in the form (2.2). The equations for n and d can be combined to give the following integral equation for n :

$$n(s) = \sum_{j=1}^{n_c} \frac{b_j}{s-s_j} + B(s) + \sum_{j=1}^{n_c} a_j \frac{B(s)-B(s_j)}{s-s_j} + \frac{i}{\pi} \int_{s_0}^{\infty} ds' \frac{B(s')-B(s)}{s'-s} \rho(s') n(s') \quad (3.1)$$

Then $d(s)$ is expressed in terms of $n(s)$ by

$$d(s) = 1 + \sum_{j=1}^{n_c} \frac{a_j}{s-s_j} - \frac{i}{\pi} \int_{s_0}^{\infty} ds' \frac{\rho(s') n(s')}{s'-s} \quad (3.2)$$

Here the amplitude is supposed to have a positive CDD index n_c and, in order to define the equations, the $2n_c$ constants a_j and b_j must be specified, in addition to the "Born term" $B(s)$. For an elastic system, the phase space factor $\rho(s)$, that occurs in Eqs. (3.1) and (3.2) is given by

$$\rho(s) = \left[\frac{s-4\mu^2}{s} \right]^{1/2} \quad (3.3)$$

where μ is the mass of each external particle. However, it will now be shown that by reinterpreting n , d and ρ , the same two equations can describe systems with inelasticity. The proof of Levinson's theorem applies therefore to all such systems.

To consider the Chew-Mandelstam (CM) system, in which inelasticity is described by the quantity

$$R(s) = \sigma_{\text{tot}}(s) / \sigma_{\text{el}}(s) \quad (3.4)$$

it is only necessary to set

$$\rho(s) = R(s) \left[\frac{s - 4\mu^2}{s} \right]^{1/2} \quad (3.5)$$

in place of Eq. (3.3). For the Frye-Warnock (FW) system, one assumes

$$\eta(s) = |e^{2i\delta(s)}| \quad (3.6)$$

to be given. Then, writing

$$A(s) = \bar{n}(s) / \bar{d}(s) \quad (3.7)$$

one introduces the following correspondences:

$$\begin{aligned} \frac{2\eta(s) \operatorname{Re} \bar{n}(s)}{1 + \eta(s)} &\longrightarrow n(s) \\ \left[\frac{s - 4\mu^2}{s} \right]^{1/2} \frac{1}{\eta(s)} &\longrightarrow \rho(s) \\ \bar{d}(s) &\longrightarrow d(s) \end{aligned} \quad (3.8)$$

With this interpretation, the Eqs. (3.1) and (3.2) are precisely those of Frye and Warnock, with $B(s)$ the corresponding FW Born term.

Before proceeding to the proof, the Born term will be expressed in terms of the left-hand discontinuity $\operatorname{Im} A(s)$. For the elastic and CM cases, this is

$$B(s) = \frac{1}{\pi} \int_{-\infty}^0 \frac{ds'}{s' - s} \operatorname{Im} A(s') \quad (3.9)$$

while for the FW case the relation is

$$B(s) = \frac{1}{\pi} \int_{-\infty}^0 \frac{ds'}{s'-s} \operatorname{Im} A(s') + \frac{P}{\pi} \int_{s_1}^{\infty} \frac{ds'}{s'-s} \left(1 - \frac{\eta(s')}{2}\right) \quad (3.10)$$

where s_1 is the inelastic threshold.

Sufficient conditions will now be given under which

$$\frac{\delta(\infty)}{\pi} = n_c - n_B \quad (3.11)$$

where $\delta(s)$ is the phase shift in the elastic case, the phase of the amplitude in the CM case, and the real part of the phase shift in the FW case. In each case n_c and n_B is the number of poles and zeros, respectively, of the corresponding function $d(s)$. The conditions are:

$$\begin{aligned} |P(s)| &< A s^{\alpha-\epsilon} && \text{for all } s_0 \leq s < \infty \\ |B(s)| &< B s^{\beta-\epsilon} && \text{and some } \epsilon > 0 \end{aligned}$$

with

$$0 \leq \alpha \leq 1, \quad \beta \leq -1/2, \quad \text{and} \quad \beta < -\frac{\alpha+1}{2} \quad (3.12)$$

and A, B any constants. A certain "asymptotic smoothness" is also required. More subtle conditions are doubtless possible, but their pragmatic value would be perhaps disproportionate to their complexity. In the elastic case, $\alpha = 0$, so $\beta \leq -1/2$. In the inelastic cases, it is possible for $R \rightarrow \infty$ or $\eta \rightarrow 0$, so long as $B(s)$ is sufficiently convergent.

The proof consists essentially in showing that Eq. (3.12) implies that Eq. (3.1) is Fredholm. Suppose Eq. (3.1) be symmetrized:

$$\left[\sqrt{\rho(s)} \eta(s) \right] = F(s) + \frac{1}{\pi} \int_{s_0}^{\infty} ds' \left[\frac{B(s') - B(s)}{s' - s} \sqrt{\rho(s) \rho(s')} \right] \cdot \left[\sqrt{\rho(s')} \eta(s') \right] \quad (3.13)$$

Here $F(s)$ stands for the inhomogeneous term of Eq. (3.1), multiplied by $\sqrt{\rho(s)}$. This satisfies

$$|F(s)| < C_1 s^{\alpha/2 + \beta - 3/2\epsilon} < C_1 s^{-1/2 - 3/2\epsilon} \quad (3.14)$$

where C_1 is a constant. The Born term $B(s)$ satisfies the same inequality. It will be supposed that $B(s)$ satisfies certain smoothness conditions, so that the derivative satisfies

$$|B'(s)| < C_2 s^{\alpha/2 + \beta - 1 - 3/2\epsilon} < C_2 s^{-3/2 - 3/2\epsilon} \quad (3.15)$$

This behaviour would be observed, for example, if

$$B(s) \sim C_2 s^{\alpha/2 + \beta - 3/2\epsilon} (\log s)^g (\log \log s)^h \dots (\log \dots \log s)^z$$

for any g, h, z , assuming that lower terms are of the same general form, and that differentiation in the limit is allowed. The smoothness conditions could be made explicit in terms of similar conditions on the spectral functions of $B(s)$, following the method of Lanz and Prosperi¹⁰⁾. However, this procedure is scarcely more enlightening than the direct assumption [Eq. (3.15)]. From Eq. (3.15) it follows, by the Mean Value Theorem, that:

$$\begin{aligned}
 \left| \frac{B(s) - B(s')}{s - s'} \right| &= |B'(cs + ds')| \quad \text{for some } 0 < c, d < 1 \\
 &\quad \text{where } c + d = 1 \\
 &< C_2 (cs + ds')^{-3/2(1+\varepsilon)} \quad (\text{by (3.15)}) \\
 &= C_2 (c^2 s^2 + 2cdss' + d^2 s'^2)^{-3/2(1+\varepsilon)} \quad (3.16) \\
 &< C_2 (2cdss')^{-3/2(1+\varepsilon)}
 \end{aligned}$$

Thus

since $cdss' > 0$

$$\left| \frac{B(s) - B(s')}{s - s'} \sqrt{\rho(s)\rho(s')} \right| < C_3 (ss')^{-3/4 + \alpha/2 - 5/4\varepsilon} < C_3 (ss')^{-1/2 - 5/4\varepsilon} \quad (3.17)$$

From Eqs. (3.14) and (3.17) it follows that Eq. (3.13) is a Fredholm equation, so that if there is no homogeneous solution, a unique square integrable solution $[\sqrt{\rho(s)} n(s)]$ exists. Hence

$$\begin{aligned}
 \left| \sqrt{\rho(s)} n(s) - F(s) \right| &\leq \frac{1}{\pi} \left\{ \int_{s_0}^{\infty} ds' \left[\frac{B(s') - B(s)}{s' - s} \sqrt{\rho(s')\rho(s)} \right]^2 \right. \\
 &\quad \left. \int_{s_0}^{\infty} ds' \left[\sqrt{\rho(s')} n(s') \right]^2 \right\}^{1/2} \quad (3.18) \\
 &< C_4 \left\{ \int_{s_0}^{\infty} ds' (ss')^{-1 - 5/2\varepsilon} \right\}^{1/2} = C_5 s^{-1/2 - 5/4\varepsilon}
 \end{aligned}$$

Thus

$$\begin{aligned}
 \left| \sqrt{\rho(s)} n(s) \right| &\leq \left| \sqrt{\rho(s)} n(s) - F(s) \right| + |F(s)| \\
 &< C_5 s^{-1/2 - 5/4\varepsilon} + C_1 s^{-1/2 - 3/2\varepsilon} \quad (3.19) \\
 &< C_6 s^{-1/2 - 5/4\varepsilon}
 \end{aligned}$$

Combining Eq. (3.19) with Eq. (3.12)

$$|\rho(s) n(s)| < C_7 s^{(\alpha-1)/2 - 7/4\epsilon} < C_8 s^{-7/4\epsilon} \quad (3.20)$$

If it is assumed in addition that the derivative of $\rho(s) n(s)$ is bounded, it follows from Ref. 10) that

$$\left| P \int_{s_0}^{\infty} ds' \frac{\rho(s') n(s')}{s' - s} \right| < C_9 s^{-7/4\epsilon} \quad (3.21)$$

Hence, according to Eq. (3.2)

$$\lim_{s \rightarrow \infty} d(s) = 1 \quad (3.22)$$

since none of the other terms on the right-hand side of Eq. (3.2) can cancel the first. Moreover, this limit is uniform in all directions.

Levinson's theorem is now a direct application of the theorem concerning a function $d(s)$, meromorphic in a domain $\mathcal{D}$, for which

$$\Delta \arg d(s) = 2\pi (n_c - n_p) \quad (3.23)$$

where the left-hand side is the increment of argument that obtains in a complete positive circuit within $\mathcal{D}$. Here n_c, n_p is the number of zeros and poles lying within the contour defining the circuit. Choosing the contour, as shown in Fig. 1, and defining $\delta(s_0) = \arg d(s_0) = 0$, it follows that $\Delta_\gamma \arg d(s) = 0$, and, as a consequence of Eq. (2.20), $\Delta_\Gamma \arg d(s) = 0$. Finally, $\Delta_{P+} \arg d(s) = \Delta_{P-} \arg d(s)$, where the decomposition

$$\Delta \equiv \Delta_{P+} + \Delta_\Gamma + \Delta_{P-} + \Delta_\gamma$$

has been made, the sense being defined as in the diagram. Thus Eq. (3.23) becomes:

$$\delta(\infty) = \pi(n_c - n_p) \quad (3.24)$$

where the zeros and poles are all those lying on the physical sheet.

This concludes the proof of Levinson's theorem for the class of amplitudes that satisfy Eq. (3.12). A simple corollary is that the phase shift $\delta(s)$ tends to a multiple of π .

4. EXTENSION OF THE NOTION OF A CDD CLASS

In this section more general possibilities than those envisaged in the preceding proof will be considered. In these cases it will not be possible to give a precise operational definition of what a CDD class means. Roughly, when the integral equation for $n(s)$ is not Fredholm, $\delta(\infty)$ need no longer be an integer. It is therefore appropriate to introduce a fractional CDD index n_0 , defined by Eq. (3.24). Some of the most interesting singular equations arise peculiarly in the FW system, of which a study will now be made.

It will be recalled that $B(s)$ is given, in the FW case, in terms of $\text{Im } A(s)$, $s < 0$, and $\eta(s)$, $s > s_1$, by Eq. (3.10), and that the theorem has been established for

$$|B(s)| < B s^\beta \quad |\eta(s)| < A s^{-\alpha} \quad (4.1)$$

with $0 < \alpha < 1$, $\beta < -\frac{\alpha+1}{2}$

An important delimitation of possibilities occurs through a theorem due to Martin⁽¹⁾, according to which if $\text{Im } A(s) \xrightarrow{s \rightarrow -\infty} 0$, then an unsubtracted dispersion relation holds for A . As a consequence, $\text{Im } A(s) \xrightarrow{s \rightarrow +\infty} 0$, so that $\eta(s) \xrightarrow{s \rightarrow +\infty} 1$ and $\delta(\infty) = 0(\pi)$. Thus the Fredholm class only has content for $\eta \rightarrow 1$, if $\text{Im } A(s) \xrightarrow{s \rightarrow -\infty} 0$ (at any rate like some inverse power). However, it is possible in general for $\eta \neq 1$, $\text{Im } A(s) \not\xrightarrow{s \rightarrow -\infty} 0$, in such a way that the divergences of the two integrals in Eq. (3.10) cancel. It is even possible for the $B(s)$ so defined to yield a Fredholm equation.

Before proceeding to the singular case, it is of interest to consider different models for $\eta(s)$ and examine their asymptotics. This is intended in no definitive spirit, but merely to gather suggestions as to which cases deserve attention.

i) The conventional assumption has usually been $\eta \rightarrow 0$, corresponding to the black absorber model. This is very plausible on a qualitative statistical view, in that with increasing energy more and more channels open, so that at infinite energy complete absorption would be expected.

ii) The experiments on high-energy scattering suggest a somewhat different view. The rigorous interpretation of these is spoilt by ones uncertainty concerning spin dependence and the phase of the scattering amplitude. However, if one takes the simplest model of no spin dependence and pure diffraction scattering, one can then assert that

$$1 - \eta_\ell = \frac{k}{2} \int_{-1}^1 d(\cos\Theta) P_\ell(\cos\Theta) \sqrt{\frac{d\sigma}{d\Omega}} \quad (4.2)$$

in the usual notation¹²⁾. The form of $d\sigma/d\Omega$ revealed by experiment¹³⁾ for the π -N system can be understood as the superposition of three effects: a large forward diffraction peak which dominates the total elastic cross-section, a considerably smaller but similarly narrow backward peak, and a roughly isotropic large angle scattering. The latter is well fitted by a statistical model in which it appears as compound elastic scattering¹⁴⁾. This contribution strongly decreases with energy, while the forward peak has the form $d\sigma/dt = a e^{-bt}$, with a and b independent of energy to within experimental error. The value of, say, η_0 determined from Eq. (4.2) is found to be dominated by the forward peak contribution and suggests a value of η intermediate between 0 and 1. Of course, one cannot be sure that available energies are asymptotic, but there is no clear requirement for $\eta \rightarrow 0$.

iii) The other extreme arises in the Regge pole model of diffraction scattering with constant total cross-sections and shrinking diffraction peaks¹⁵⁾. This gives explicitly $\eta \rightarrow 1$; in fact¹⁶⁾

$$|\operatorname{Im} A(s)| = O(1/\log|s|) \quad s \rightarrow -\infty$$

$$|1 - \eta(s)| = O(1/\log|s|) \quad s \rightarrow +\infty$$

and although the two integrals in Eq. (3.10) do not converge separately, the divergent parts cancel to give $|\operatorname{Re} A(s)| = O(1/(\log|s|)^2)$.

The left-hand discontinuity $\text{Im } A$ is much more remote from experiment than is η , but one is led to postulate a narrow range of asymptotic possibilities for it, from general theory. Halpern¹⁷⁾ has considered examples in which $\text{Im } A(s) \sim |s|^\alpha$ for $\alpha > 0$ as $s \rightarrow -\infty$. He found that $A(s)$ always has an infinity of poles. Likewise, it has been shown¹⁸⁾ that if $\text{Im } A(s) \rightarrow \lambda$ as $s \rightarrow -\infty$ there are either an infinite number of poles or other unphysical features unless $0 \leq \lambda \leq 1$. Roughly, in each of these cases, the only way to maintain the unitarity limit $\text{Im } A(s) \leq 1$ as $s \rightarrow +\infty$, in the face of worse behaviour as $s \rightarrow -\infty$, is by introducing an accumulation pathology. However, for $0 \leq \lambda \leq 1$, there is no such difficulty, although the equations for N and D must be subtracted, in order to define B . For such ansatz, the resulting integral equation for N is of a specific non-Fredholm type which has been studied by a number of authors¹⁹⁾. The solution has two arbitrary constants, namely the subtraction constant of the N equation, and an arbitrary multiplier of the homogeneous solution, which is always present (as it is not for a Fredholm equation). The situation is therefore rather like that of an amplitude of CDD class one. However, in general, $\delta(\infty) \neq 0(\pi)$. However, one has that $\delta(\infty)/\pi$ lies between $-n_B$ and $-n_B + 1$. When CDD poles are present as well as a singular B , the obvious generalization of Levinson's theorem would be to equate the integral part of $\delta(\infty)/\pi$ with $n_C - n_B$. Alternatively, one might define a (non-integral) index $\check{n}_C$ by

$$\check{n}_C = \delta(\infty)/\pi + n_B \quad (4.3)$$

Then the singular classes could be taken to be a generalization of the notion of a CDD class. Moreover, the above considerations lead to the speculation that the class of all CDD classes thus defined exhausts all non-pathological possibilities. Further, since a singular solution with no accumulation of poles satisfies

$$\frac{1}{2} [1 - \eta(\infty) \cos 2\delta(\infty)] = \text{Im } A(-\infty) \quad (4.4)$$

one might conjecture that the widest class for which the generalized Levinson's theorem holds is

$$1 \geq \frac{1 + \eta(\infty)}{2} \geq \text{Im } A(-\infty) \geq \frac{1 - \eta(\infty)}{2} \geq 0 \quad (4.5)$$

A situation for which $\eta \neq 0$ with $\delta(\infty) \neq 0(\pi/2)$ and thus for which $\text{Re } A_\ell / \text{Im } A_\ell \neq 0$ implies (barring subtle non-uniformities in ℓ) a break down of the diffraction picture $\text{Re } A(s, t) / \text{Im } A(s, t) \rightarrow 0$. Such a break-down is not required in order to fit the recent measurements of $\text{Re } A(s, 0)$ in high-energy p-p scattering²⁰⁾ and would, in fact, for many processes, contradict the usual assumption of $\sigma_{\text{tot}} \rightarrow \text{const. as } s \rightarrow \infty$ ²¹⁾. The diffraction picture is maintained if one requires either i) $\text{Im } A(-\infty) = (1 - \eta(\infty))/2$, or ii) $\text{Im } A(-\infty) = (1 + \eta(\infty))/2$. The FW method is peculiarly suited to the former case.

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APPENDIX

Although the point is not of central interest to the thesis of this paper, it is of interest to show that in the case of no inelasticity, and if no particle poles fall on the left-hand cut, one can always find at least n_0 zeros of the amplitude, so that each CDD pole can be associated with a zero of the amplitude. If either condition be relaxed, one can no longer assert the general statement.

Form

$$A_1 = A / N_1 \quad (A.1)$$

where N_1 has the left-hand cut of A , but has no zeros nor poles, and is positive definite for $s > 0$ (where it is real and zero free). Then A_1 has no left-hand cut. Suppose first that $\delta(\infty) = m\pi$, with m an integer, and that A has n_B poles. Then it is required to show that A_1 has at least $(m + n_B)$ zeros.

By conservation of probability, all the poles must have negative residue, so that there are certainly $n_B - 1$ zeros between the poles (if $n_B > 1$, and no poles lie on the left-hand cut of A). Between the rightmost pole and the threshold $s = s_0$, there will either be at least one more zero, or $\text{Re } A_1(s_0)$ will be negative. In the latter case

$$\left. \frac{d}{ds} \left[\sqrt{\frac{s - 4p^2}{s}} \cot \delta(s) \right] \right|_{s=s_0} < 0$$

Similarly, to the left of the leftmost pole A_1 either has one or more zeros, or $A_1(-\infty) > 0$. In the latter case, assuming

$$A_1(\infty) = A_1(-\infty) \quad (A.2)$$

$\delta(\infty)$ approaches $m\pi$ from above. Furthermore, there will be $(m-1)$ zeros at the points where intermediate multiples of π are attained. Finally, putting all these contingencies together (see Fig. 2), one can see that there are always at least $(n_B + m)$ zeros, as required.

The proof clearly breaks down:

- a) when there are poles along the left-hand cut, because the sign of A_1 at such points is governed by the sign of $\operatorname{Re} N_1$, which may change;

and

- b) where there is inelasticity, for then $\delta = 0(\pi)$ does not secure a zero of A .

In the singular case in which $\delta(\infty) \neq 0(\pi)$, but in which neither a) nor b) obtain, the proof holds, with minor modifications, if m now means the integral part of $\delta(\infty)/\pi$.

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FIGURE CAPTIONS

Figure 1 : Contour in complex s-plane for deducing Levinson's theorem.

Figure 2 : Some possibilities concerning the location of zeros of $A(s)$.

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Fig. 1

Fig. 2