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OCCURRENCE OF AN $SU(3)$ REPRESENTATION IN THE
DECOMPOSITION OF A DIRECT PRODUCT :
BOUNDS AND MULTIPLICITY

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I. The purpose of this note is twofold. First, we want to provide a particularly handy tool for deciding whether a representation $D(p, q)$ occurs in the decomposition of the direct product $D(p_1, q_1) \times D(p_2, q_2)$ of two $SU(3)$ representations and for computing its multiplicity without going through the complete reduction. Secondly, we want to show that the number of times $D(p, q)$ can occur in the decomposition is intrinsically bounded, that is, there is a maximal multiplicity for the occurrence of a given representation in whatever product.

In particular, $\underline{8} \equiv D(1, 1)$ and, more generally, any representation $D(p, 1)$ or $D(1, q)$ cannot occur more than twice, $\underline{10} \equiv D(3, 0)$, $\overline{10} \equiv D(0, 3)$ and any other representation $D(p, 0)$, $D(0, q)$ cannot occur more than once; to our knowledge, these results have been proved only for the particular cases $D(p_1, q_1) \times D(1, 1)$ ¹⁾ and $D(p_1, q_1) \times D(3, 0)$ ²⁾, $(D(p_1, q_1) \times D(0, 3))$ respectively.

For what concerns the first point, our purpose is merely practical, as far as the matter of principle is concerned, the answer is already implicitly contained in the graphical rules given by Littlewood³⁾ for the group of permutations and applied by Edmonds⁴⁾ to $U(3)$ and $SU(3)$.

Our prescription for computing the multiplicity K of the irreducible representation $D(p, q)$ in the direct product $D(p_1, q_1) \times D(p_2, q_2)$ is contained in the following theorems.

1) K is zero unless

$$m = \frac{1}{3} \left[2(q_1 + q_2 - q) + (p_1 + p_2 - p) \right] \quad (1)$$

is a non-negative integer satisfying the inequalities

$$p_1 + q_1 - p - q \leq m \leq \min(q_1 + q_2 - q + p_1 + p_2 - p; p_2 + q_2) \quad (2)$$

2.

- 2) If m , as defined in (1), exists, K is equal to the number of different non-negative integers n satisfying :

$$B \leq n \leq A \quad (3)$$

where

$$A = \min(q; q_2; m+q-q_1; m+p+q-p_1-q_1; q_1+q_2-m) \quad (4)$$

$$B = \max(m+q-q_1-p_1; q_2-m) \quad (5)$$

The method furnished by the above theorems can be used alternatively instead of the explicit formula for K given by Simoni and Vitale ⁵⁾; although formally less elegant, our method proves faster for practical purposes.

Equivalent forms of the above theorems can be obtained by using the symmetry properties of the direct product. Indeed, since the multiplicity of $D(p,q)$ in $D(p_1,q_1) \times D(p_2,q_2)$, that of $D(q_1,p_1)$ in $D(p_2,q_2) \times D(q,p)$ and that of $D(q_2,p_2)$ in $D(p_1,q_1) \times D(q,p)$ are all equal to each other and to the multiplicity of $D(0,0)$ in $D(p_1,q_1) \times D(p_2,q_2) \times D(q,p)$, theorems 1) and 2) remain valid under the simultaneous substitutions

$$\begin{aligned} q_1 &\rightarrow q_2 \rightarrow p \\ p_1 &\rightarrow p_2 \rightarrow q \end{aligned} \quad (6)$$

Our results concerning the intrinsic bounds on the multiplicity are now immediately derived as a corollary to the above theorems and are contained in the statement : the multiplicity K fulfils

$$K \leq \min(m, m', m'', p_1, q_1, p_2, q_2, p, q) + 1 \quad (7)$$

where

$$m' = \frac{1}{3} \left[2(q_1 + p - p_2) + (p_1 + q - q_2) \right]$$

$$m'' = \frac{1}{3} \left[2(q_2 + p - p_1) + (p_2 + q - q_1) \right]$$

These results sharpen the bounds given by Preziosi et al. ⁶⁾, which read

$$K \leq \min(p_1, q_1, p_2, q_2) + 1$$

in so far as it is proved that the minimum of $p+1, q+1$ is an intrinsic bound on the multiplicity of the representation $D(p, q)$ independent of the representations whose direct product is considered ^{*}). Of particular physical interest are the results concerning the representations 8 and 10 (10), for which it is proved that they cannot occur more than twice and once, respectively.

The simple reducibility of such products as $D(p_1, q_1) \times D(p_2, 0)$, $D(p_1, q_1) \times D(0, q_2)$ ²⁾ follows also at once.

II. We now briefly sketch the proof. The representation $D(p, q)$ is contained in $D(p_1, q_1) \times D(p_2, q_2)$ if we can construct the Young tableau

^{*}) Our results concerning the bounds $p+1, q+1$ could have been derived directly from the results by Preziosi et al. by using the symmetry under the second substitution (6). We wish to thank Professor N. Cabibbo for calling our attention to the above argument.

4.

by adding squares from the tableau for $D(p_2, q_2)$, say, to the tableau for $D(p_1, q_1)$ according to the known rules. Let us call a (b) the squares in the first (second) row of the Young tableau for $D(p_2, q_2)$. From the conservation of the total number of squares we must have

$$3m + 2q + p = 2q_1 + p_1 + 2q_2 + p_2$$

This is equality (1). In the first row only a's may be added. The number of squares added to the first row, $m+p+q-p_1-q_1$, is thus subject to the obvious limitations

$$0 \leq m + p + q - p_1 - q_1 \leq q_2 + p_2 \quad (8)$$

The second row may contain both a's and b's. Let us denote the number of b's by n ; n is subject to the obvious limitations

$$0 \leq n \leq q_2 \quad (9)$$

Besides, n cannot exceed the number of a's in the first row and cannot be smaller than the number of columns in which squares from the Young tableau for $D(p_2, q_2)$ were added to both the first and the second row, that is

$$m + q - q_1 - p_1 \leq n \leq m + p + q - p_1 - q_1 \quad (10)$$

Further, as neither an a nor a b can stay in the third line below a b, it must be

$$n \leq q \quad (11)$$

The number of a's added in the second row, $m+q-n-q_1$, is also subject to the limitations

$$0 \leq m + q - n - q_1 \leq p_2 + q_2 - p - q + p_1 + q_1 - m \quad (12)$$

since $p_2+q_2-p-q+p_1+q_1-m$ is the number of a's left over after filling the first row.

In the third row the number of $\underline{b}$'s cannot be larger than the number of $\underline{a}$'s in the first two rows and smaller than the number of columns in which squares were added to both the second and the third row

$$m - q_1 \leq q_2 - n \leq q_2 + p_2 - (m - q_2 + n) \quad (13)$$

Inequalities (9) to (13) yield (2) and (5). This completes the proof as for each value of n there is exactly one Young tableau corresponding to the irreducible representation $D(p, q)$.

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