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EXPONENTIAL ASYMPTOTIC BEHAVIOUR AND
LOW ENERGY PION-PION SCATTERING

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A B S T R A C T

A new approach to low energy $\pi\pi$ scattering is described in which it is assumed that, because of the very short range of the important forces in this problem, the distant singularity structure is relatively important. It is remarked that the exponential decrease of the amplitude at fixed angle observed experimentally, which also corresponds to the maximum rate of decrease allowed theoretically in a reasonable S matrix theory, gives information about this structure. The high energy and low energy phenomena are connected by using the fixed angle method to calculate the partial wave amplitudes at low energy. A representation is conjectured which contains the correct asymptotic behaviour as well as the nearby part of the left-hand cut which is known from crossing symmetry. A formal solution, analogous to the full determinantal N/D method, is presented. In a simple, but fairly realistic model an additional attractive potential is produced by the asymptotic left-hand cut; it is suggested that this may be responsible for the missing attraction in the ρ bootstrap mechanism.

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1. INTRODUCTION

Many calculations of low energy pion-pion scattering have been made in the last few years attempting to give quantitative support to the clear qualitative picture that has been developed, in which the strong $T = 1$ ρ resonance is ascribed principally to the strong attractive force generated in that state by its exchange. The result of all these calculations has been to show, beyond all reasonable doubt, that in these models the forces produced by ρ exchange, whose coupling strength is known from the experimental width of the ρ , are too weak to give the resonance by a factor that varies from 1.5 to 5 according to the details of the model ^{*}). All these calculations have been based on the assumption that the low energy scattering is controlled by the low energy singularities; we shall call this assumption the Dogma for ease of reference. The second equally serious problem concerns the calculated width of the resonances (the output width); it is difficult without special assumptions to explain dynamical resonances whose width is very small compared to the kinetic energy available for their decay, a feature characteristic of the higher resonances. We shall find here a possible solution to the first problem in an additional attractive potential that arises from the asymptotic behaviour.

The calculations made so far have been based on at least three distinct approaches, all of which should work in a situation in which the Dogma is justified, though none is without internal troubles.

1) Partial wave amplitudes

These were used by Chew and Mandelstam ¹⁾ in their original treatment of the problem and by many others since. The calculations run into trouble in practice because of the difficulty of computing the spectral functions, and thus the left-hand cut discontinuity,

^{*}) In self-consistent calculations of the "bootstrap" type the equivalent statement is "we found a solution with width three times the experimental value".

2.

once the partial wave expansions in the t and u channels have broken down ($s < -32$). Simple approximations lead to singular equations showing that the far spectral region must be important.

2) Differential approximation 2)

This approximates the partial waves by linear combinations of the forward scattering amplitude and its derivatives ; it has been used in a detailed study of the $\pi-\pi$ problem by Shirkov and his collaborators 3). While there is no problem of analytic continuation, since all quantities lie on the edge of a physical region, convergence difficulties force the introduction of arbitrary cut-offs in practical calculations.

3) Fixed-angle method 4)

This discusses linear combinations of fixed angle amplitudes which are chosen to resemble, to any specified accuracy, the partial wave amplitudes in the low energy physical region. It suffers from the same problems of analytic continuation into the far spectral region, $s < -32$, as does 1).

With all these methods the same conclusion emerges, namely that within the bounds of present ingenuity, and including only low energy singularities, the force from ρ exchange is not nearly strong enough to give a resonant $T = 1$, $J = 1$ state. This situation is in stark contrast with dispersion calculations of pion-nucleon and nucleon-nucleon scattering. The calculations of Hamilton and his co-workers have shown detailed quantitative agreement between theory and experiment for the pion-nucleon system, entirely within the Dogma. No comparably detailed treatment of the nucleon-nucleon system has been made, but simpler models give fairly good agreement.

It is not difficult to understand why there should be this difference. The pion-pion system has a number of very special features which make low energy calculations based on the Dogma much less reliable.

The forces in the ρ problem are of very short range. Table I shows the distance of the nearest important singularity in the crossed channels and the associated range of the forces in units of the reduced mass for the various scattering systems.

System	Exchanged particle	Distance from phys. reg.	Range of force
N-N	π	0.08	3.5
π -N	N	7	0.4
π - π	ρ	120	0.1

T A B L E I

Thus the dynamical equivalent in N-N scattering of the ρ exchange force in π - π scattering would be a principal force with a range of $0.04 \cdot 10^{-13}$ cms. The scattering particles at resonance in such a situation are in a highly relativistic region and it is not surprising if what are fundamentally low energy models do not work.

Secondly, and closely related to the above, the π double spectral function begins to contribute to the left-hand cuts at about the same place as the ρ exchange itself and will be important before the attractive part of that contribution is reached. The latter is mostly about and beyond $s = -200$. Attempts to compute the spectral

functions ⁵⁾, have run into problems of convergence which are closely related to the asymptotic behaviour that we shall be discussing. Thus it is fair to say that we have no reasonable guide to the form of the left-hand cut discontinuity in the region of the attractive ρ contribution and beyond.

If one accepts the above as a true picture of the present situation there are still a number of ways out of the difficulty. The simplest assumption is that the ρ is a CDD singularity as has been firmly suggested by Shirkov ³⁾; this explanation is firmly based on the Dogma, but it too has difficulties. Firstly, Shirkov's analysis seems to require a width for the ρ of less than 50 MeV, which in spite of the uncertainties that arise from the indirectness of the measurements which Shirkov stresses, seems incompatible with experiment. Secondly, this assumption involves either the abandonment of the attractive bootstrap philosophy which excludes arbitrary CDD poles or of the assumption that the problem is a truly multichannel one and that the resonance is a CDD pole only in the inelastic one-channel approximation; this is entirely possible but unattractive since the ρ seems superficially a highly elastic resonance. Alternatively, we may abandon the Dogma in its stated form and recognize that the distant singularities must be taken more seriously into account; in this paper we want to suggest a way in which this last approach may be investigated, and to some extent implemented.

In Section 2 we shall discuss how we may obtain information on the distant singularities from the experimental high-energy behaviour and the advantages for this of the fixed angle method. In Section 3 the dynamical equations are formulated; the key assumptions are discussed in more detail in Section 4. Section 5 is devoted to a brief discussion of general features of a simple model; a detailed discussion of the calculations for low energy pion-pion scattering will be given in a subsequent paper.

2. THE STRUCTURE OF THE DISTANT SINGULARITIES

We shall now see to what extent one can determine the amplitude in the spectral region ($s < 0$, t and $u > 0$) which in turn determines the left-hand cut for our one dimensional dispersion relations. We have, of course, no prescription that will provide as much detail in the far spectral region as does crossing in the low energy left-hand region, but because it is qualitatively different, even a little high-energy information can have important effects.

Firstly we want to discuss the three main approaches so far used and to show that the fixed angle method has a clear advantage when the asymptotic region is to be taken seriously. This arises because both the forward and partial wave amplitudes are dominated by the forward diffraction peak on the right and the associated strip regions on the left and so have power law asymptotic behaviour, almost certainly of the form

$$A_e(s) \sim \text{const} \quad (2.1)$$

$$\frac{\partial A(s, z)}{\partial z^n} \sim s^{n+1} \quad s \rightarrow \infty \quad (2.2)$$

which contains no real features of interest. In contrast, experiment ⁶⁾ has shown that, at fixed angle, the asymptotic behaviour of the amplitude is exponential and in particular

$$A(s, z) \sim e^{-\sqrt{\frac{s}{s_A(z)}}} \quad (2.3)$$

We do not here want to discuss the z dependence of this behaviour, although it too shows a remarkable simplicity and universality and must surely be included later. For the moment we want to stress the very special properties of the s dependence (2.3)

6.

Firstly (2.3) is the fastest rate of fall that still permits a dispersion relation in s . This has been rigorously discussed by Cerulus and Martin ⁷⁾, who also take care of the logarithms, but can be easily seen as follows. If, in an s plane cut from $-\infty$ to s_L , the amplitude has a definite asymptotic phase and behaves as

$$A(s, z) \sim e^{-s^\alpha} \quad (2.4)$$

along the positive real axis, it will behave as

$$\sim e^{-|s|^\alpha (\cos \theta + i \sin \theta)} \quad (2.5)$$

in a direction θ in the complex plane. If an exponentially rising amplitude is to be avoided in all directions on the top (physical) sheet, $|\theta| \leq \pi$, then

$$\alpha \leq \frac{1}{2} \quad (2.6)$$

There is, of course, still an essential singularity at $s = \infty$ but its most awkward features are confined to unphysical left-hand sheets and the usual dispersion relations can still be used. Thus we have a second situation (the Froissart ⁸⁾ bound for $z = 0$ being the first) in which the experiments approach a theoretical limit very closely, though here it is a lower limit on the amplitude. However, unlike the Froissart bound this behaviour does not follow from the usual simple models. It also implies an intimate collaboration between partial wave amplitudes at high energy. Merely putting $A_\ell = i/2$ gives

$$A(s, z) = \frac{1}{2} \sum_{\ell=0}^{\ell_{\max}} (2\ell+1) P_\ell(z) \quad \ell_{\max} \sim s^{1/2} \quad (2.7)$$

and at general angles away from the forward direction

$$A(s, z) \sim (\ell_{\max} + 1) |P_{\ell_{\max}}(z)| \sim s^{1/4} \quad (2.8)$$

Secondly, the behaviour (2.3) is extremely restrictive. We may illustrate this in the following way. Suppose the amplitude obeys an unsubtracted dispersion relation

$$a(s) = \frac{1}{\pi} \int_{-\infty}^{s_L} \frac{p_L(s') ds'}{s' - s} + \frac{1}{\pi} \int_{s_R}^{\infty} \frac{p_R(s') ds'}{s' - s} \equiv \frac{1}{\pi} \int_C \frac{p(s') ds'}{s' - s} \quad (2.9)$$

as we should expect from the Mandelstam representation and our earlier discussion. The further requirement that

$$a(s) \sim s^{-n-1} \quad s \rightarrow \infty \quad (2.10)$$

implies n moment conditions of the form

$$\int_C \frac{p(s') ds'}{s'^r} = 0 \quad r = 1, 2, \dots, n \quad (2.11)$$

on the discontinuity function p . The exponential behaviour (2.3) implies an infinite number of such conditions.

A more precise illustration of this has been provided by Balachandran⁹⁾. He has shown that the left-hand discontinuity function p_L alone determines the solution without the use of unitarity on the right and with no CDD ambiguity. He hopes that this may ultimately provide a way to by-pass the awkward features of the unitarity condition. We feel that a reasonable treatment of unitarity on the right is

much closer than the continuation procedure which is needed to give the left-hand cut discontinuity function. We shall thus establish a formalism which makes use of unitarity and crossing symmetry in the low-energy regions where they are unambiguous and at the same time automatically satisfies (2.3) and thus, the infinite set of moment conditions, the use of unitarity yielding some of the information we lack about the left-hand cut.

We shall thus tackle the problem by the fixed angle method used in Ref. ⁴⁾, modifying it to include explicitly the correct asymptotic behaviour.

3. THE EXPONENTIAL REPRESENTATION OF THE FIXED ANGLE AMPLITUDES

We now look for equations for the fixed-angle amplitudes which incorporate the features discussed so far. These may be summarized as follows :

- 1) left and right-hand cuts in positions specified by the Mandelstam representation ;
- 2) the low energy right-hand cut discontinuity specified by unitarity ;
- 3) the low energy left-hand cut given by crossing ;
- 4) exponential asymptotic behaviour as $s \rightarrow \infty$ of the form

$$a \sim e^{-\sqrt{\frac{s}{s_A}}}$$

where the amplitude will be assumed to be real.

These assumptions will be discussed in more detail in the light of the resulting equations in Section 4.

First we shall briefly review the results of Ref. 4) which gave the linear combinations of fixed angle amplitudes that approximate the partial wave amplitudes in the low energy physical region, thus allowing us to use 2). In general we define these amplitudes, $a_{\ell}^L(s)$, as the solutions of the L linear equations

$$A^I(s, z_i^L) = \sum_{\ell=0}^{L-1} a_{\ell}^L(s) \cdot (2\ell+1) \cdot P_{\ell}(z_i^L) \quad i=1,2,\dots,L \quad (3.1)$$

where $P_L(z_i^L) = 0$. We shall describe our method for the special cases

$$a_1(s) \equiv a_{11}^3(s) = \frac{A^1(s, z_3)}{3z_3} \quad z_3 = \sqrt{\frac{3}{5}} \quad (3.2)$$

$$a_0^I(s) \equiv a_{I0}^2(s) = A^I(s, z_2) \quad z_2 = \sqrt{\frac{1}{3}} \quad (3.3)$$

where the symmetry of the $\pi\pi$ amplitudes has been used to simplify the expressions. These are the simplest approximations to the p and s wave $\pi\pi$ amplitudes in the low energy physical region and contain contamination only from $\ell \geq 5$ and $\ell \geq 4$ respectively. It should be stressed, however, that (3.1), (3.2), and (3.3) are the analytic prescriptions that define the a 's; the resemblance to the partial wave amplitudes is used only to give us a good approximation to $\text{Im}a^{-1}$ on the low energy right-hand cut.

The generalization of the method to the general amplitude, given by (3.1), is straightforward and will be sketched in Section 4.

We shall include the unitary behaviour in the usual way by writing ¹⁾

$$a(s) = \frac{n(s)}{d(s)} \quad (3.4)$$

where $n(s)$ and $d(s)$ have only left and right-hand cuts respectively. The denominator function $d(s)$ will satisfy the equation

$$d(s) = 1 - \frac{s-s_0}{\pi} \int_4^{s_F} \frac{\rho(s') n(s') ds'}{(s'-s)(s'-s_0)} \quad (3.5)$$

$$\rho(s) = \sqrt{\frac{s-4}{s}}$$

where s_I is the energy at which the unitarity relation ceases to be a good approximation because high partial waves become important. If inelasticity effects are serious before this, they can be included in the usual way by replacing the density of states, $\rho(s)$, by $\rho \cdot R(s)$ where

$$R(s) = \frac{\sigma_{\text{tot}}^{\ell}(s)}{\sigma_{\text{el}}^{\ell}(s)} \quad (3.6)$$

In principle the high energy part of the right-hand cut must be included in $n(s)$, as suggested by Chew¹⁰⁾, but because of the asymptotic behaviour on the right we believe that its contribution will be small and for the moment we shall ignore it.

The assumption 4) of exponential asymptotic behaviour cannot be included if we write a dispersion relation for $n(s)$ itself, since we shall always obtain power law behaviour at infinity unless the discontinuity satisfies the infinite set of moment conditions (2.11). Instead we write

$$n(s) = e^{-\phi(s)} \quad (3.7)$$

and write a dispersion relation for $\phi(s)$.

In order to achieve the requirements 1), 3), and 4) this must have the form

$$\frac{\phi(s)}{\sqrt{s_L + s}} = \frac{1}{\sqrt{s_A}} + \frac{1}{\pi} \int_{-\infty}^{s_L} \frac{\sigma(s') ds'}{s' - s} \frac{1}{\sqrt{-s' - s_L}} \quad (3.8)$$

where the discontinuity function $\sigma(s)$ is determined by crossing, at least in the low energy region. It follows that

$$\sigma(s) = \log |n(s)| = \log |a(s)| + \log d(s) \quad (3.9)$$

We therefore require from crossing the whole amplitude $a(s)$ in the spectral region and not just its imaginary part as in the dispersion relation for $n(s)$. Combining the above assumptions, we arrive at the basic set of equations

$$\frac{\phi(s)}{\sqrt{s_L + s}} = \frac{1}{\sqrt{s_A}} + \frac{1}{\pi} \int_{-\infty}^{s_L} \frac{\log |a(s')| + \log d(s')}{s' - s} \frac{ds'}{\sqrt{-s' - s_L}} \quad (3.10a)$$

$$d(s) = 1 - \frac{(s - s_0)}{\pi} \int_4^{s_I} \frac{\rho(s') R(s') e^{-\phi(s')}}{(s' - s)(s' - s_0)} ds' \quad (3.10b)$$

$$a(s) = \frac{e^{-\phi(s)}}{d(s)} \quad (3.10c)$$

which must be solved.

As usual we shall look for dynamical resonances in the low energy region, $4 < s < s_I$, which will appear as zeros of the function $d(s)$.

We know of no method of obtaining a closed solution of these transcendental integral equations, given the function $a(s)$ on the left-hand cut. They can, however, be solved by iterated quadratures, starting with $d(s) = 1$. This is equivalent to the "determinantal" ¹¹⁾ method used in the usual N/D approach to the partial wave amplitudes. Moreover, it is clear that none of the convergence difficulties that enter there arise in this method, at least as long as $a(s)$ does not rise exponentially as $s \rightarrow -\infty$, because the left-hand cut discontinuities are logarithmic functions of a and d .

In practice, it will be necessary to apply a cut-off to the integrals over the left-hand cut because we can obtain $a(s)$ from crossing only in the nearby region but, as the integrals are - in principle - convergent, we may expect that the results will be insensitive to its position. Similarly because $\log d(s)$ only enters the equation for $\phi(s)$ we may hope the iteration will converge fairly rapidly.

4. DISCUSSION OF THE APPROXIMATIONS

We shall now discuss in more detail the main assumptions and approximations that we have made in arriving at equations (3.10), and where possible suggest how they may be improved.

1) The use of the unitarity equation

This is justified for our linear combinations of fixed angle amplitudes only in the low-energy region; the cut-off s_I is determined by the onset of important contributions from high partial waves ($\ell \geq 5$). Again, the situation is especially favourable for this approach in the $\pi\pi$ problem where, because of the very short range of the main forces, a cut-off energy between 2-3 GeV seems reasonable giving $s_I \sim 300$.

In principle the equations should be modified by including in Eq. (3.10a) a term corresponding to the far right cut contribution, namely,

$$-\frac{1}{\pi} \int_{s_I}^{\infty} \frac{\delta a(s') ds'}{s'-s \sqrt{s'+s_L}} \quad \delta a = \frac{\text{Im } A}{\text{Re } A} \quad (4.1)$$

This can be done if a reasonable model for $\delta a(s')$ on the far right can be found. As we shall see it probably goes to zero as $s \rightarrow \infty$.

2) The assumption of real asymptotic phase

It is a central assumption of this whole approach that the amplitude at fixed angle has a definite asymptotic value for its phase and that, more specifically, it is real. While we can offer no proof on this tricky point, the following argument makes it seem to us both consistent and plausible.

The argument is based on unitarity and the experimental asymptotic behaviour in both s and z . Let us first consider the elastic contribution to the imaginary part, assuming that the amplitude is real and behaves as

$$A(s, z) \sim e^{-a\sqrt{t(s, z)}} \sim e^{-a\sqrt{s} \cdot \frac{\sin \theta}{2}} \quad s \rightarrow \infty \quad (4.2)$$

as indicated by experiment ⁶⁾. Then we find, taking only the controlling exponential factors into account, using elastic unitarity

$$\frac{\text{Im} A_{el}(s, z)}{A(s, z)} \sim \int d\Omega_n e^{-a\sqrt{s} \left[\sin \frac{\theta_{fn}}{2} + \sin \frac{\theta_{ni}}{2} - \sin \frac{\theta_{fi}}{2} \right]} \quad (4.3)$$

where i , n and f label the initial, intermediate and final states respectively. But ^{*)}

$$\sin \frac{\theta_{fn}}{2} + \sin \frac{\theta_{ni}}{2} - \sin \frac{\theta_{fi}}{2} \geq 0 \quad (4.4)$$

and the range of integration is finite, so that for finite intermediate angles the contribution to $\text{Im}A/A$ vanishes like $e^{-a'\sqrt{s}}$.

The equality in (4.4) arises when either θ_{fn} or θ_{ni} is zero. We must therefore consider separately the contributions which arise when one of the two scatterings in the unitarity integral is close to the forward direction. Let us assume that an angle Σ around the forward direction is involved and, for the moment, that the form (4.2) for the amplitude holds even in this region. Its contribution is then

^{*)} This behaviour can be illustrated by looking at the case $\theta(fi)/2 = \theta_{fn} = \theta_{ni} = \Sigma$ and using $\frac{\sin \theta}{\theta} \leq 1$.

$$\leq \varepsilon d\varepsilon e^{-a\sqrt{s} \left[\sin \frac{|\varepsilon|}{2} + \sin \frac{\theta - \varepsilon}{2} - \sin \frac{\theta}{2} \right]} \quad (4.5)$$

$$\sim \varepsilon d\varepsilon e^{-a\sqrt{s} \varepsilon (1 - \cos \theta)}$$

It is clear that this too vanishes exponentially unless

$$\varepsilon \lesssim \sqrt{\frac{t_0}{s}} \quad (4.6)$$

The constant t_0 is in practice related to the width of the diffraction peak. Note that experimentally the behaviour (4.2) does hold if (4.6) is not satisfied because we are then outside the diffraction region.

The inner region

$$\theta_{ni} < \varepsilon \sim \sqrt{\frac{t_0}{s}} \quad (4.7)$$

gives a contribution

$$\frac{\text{Im } A}{A} \sim \varepsilon^2 \text{Re } A_{ni}(s, t_0) \sim \frac{t_0}{s} \text{Re } A_{ni}(s, t_0) \quad (4.8)$$

which vanishes as long as $\text{Re } A_{ni} < 0(s)$ as $s \rightarrow \infty$, which we know to be necessary theoretically and true experimentally.

Similar arguments can be applied to the inelastic intermediate state contributions, provided that we assume that the asymptotic parameter s_A is universal, as seems to be indicated by experiment. The two particle intermediate states are analyzed in exactly the same way as for the elastic contribution; as long as their number does not rise too fast asymptotically, (we believe that it tends to a constant), their contribution also vanishes. The multiparticle intermediate states have negligible effect, since they necessarily involve the scattering of all the N particles through a total angle θ_{fi} , which has a probability

$$\lesssim e^{-Na\sqrt{s} \sin \frac{\theta_{fi}}{2}} \quad (4.9)$$

so that

$$\frac{\text{Im } A(N)}{A} \lesssim e^{-(N-1)a\sqrt{s} \sin \frac{\theta_{fi}}{2}} \quad (4.10)$$

which goes to zero exponentially.

It therefore seems that the assumption, that the fixed angle amplitude is asymptotically real, is reasonable. It is comfortably that the same arguments applied to scattering near the forward direction, that is, in the diffraction peak, where

$$A(s, z) \sim e^{bt(s, z)} \sim e^{bs \sin^2 \frac{\theta}{2}} \quad (4.11)$$

leads to

$$\frac{\text{Im } A}{A} \gtrsim \int d\Omega_n \cdot e^{-bs \left[\sin^2 \frac{\theta_{fn}}{2} + \sin^2 \frac{\theta_{ni}}{2} - \sin^2 \frac{\theta_{fi}}{2} \right]} \quad (4.12)$$

Here the exponent is positive over part of the integration region, so that the imaginary part should dominate, as it does.

3) Zeros of $a(s)$ on the right-hand cut

In writing our equations we have so far assumed that there are no zeros of $a(s)$ except on the left-hand cut. However, the formalism can easily be modified to take into account the two sorts of zeros that we expect from physical considerations, namely threshold behaviour

$a_\ell(s) \sim (s-4)^\ell$ near $s = 4$ and possible CDD zeros. In the first case we may choose to factor out such behaviour and write our equations for the function $a_\ell(s)/(s-4)^\ell$. The density of states in Eq. (3.5) is then given by

$$\rho(s) = \frac{(s-4)^{\ell+\frac{1}{2}}}{\sqrt{s}} \quad (4.13)$$

CDD zeros will appear as usual as arbitrary poles in $d(s)$.

4) Extension to arbitrary fixed-angle amplitudes

We conclude this Section by sketching the extension of the method to the general fixed-angle amplitudes

$$a_{\ell\ell}^L(s) = \sum_i c_{\ell i}^L A^I(s, z_i^L) \quad (4.14)$$

which are the solutions of Eq. (3.1). Again we utilize the known asymptotic dependence at given angle and write

$$\frac{a_{\ell\ell}^L(s)}{(s-4)^\ell} = \frac{\sum_i c_{\ell i}^L e^{-\phi_i(s)}}{d(s)} \quad (4.15)$$

Each of the functions ϕ_i satisfies an equation of the form (3.10a), where d is given by Eq. (3.5), modified suitably according to (4.9), as before.

5. SOLUTION FOR A SIMPLE MODEL

We shall conclude this paper with a semi-quantitative discussion of the solution of a simple, but fairly realistic, model problem which indicates one new effect that seems to enter in this approach, leaving the detailed numerical solution of Eqs. (3.10) for the s and p wave pion-pion systems for a subsequent paper.

The origin of these ideas lay in the beautiful and surprisingly simple experimental behaviour at high energies and large angles ⁶⁾. Our model is based on a function used in the analysis of these by Henley and Muzinich ¹²⁾, which has the sort of analytic structure that we require. We choose to assume that the function $n(s)$ has the form

$$n(s) = \frac{C(s-4)}{\sqrt{\frac{s_L+s}{s_A}}} e^{-\sqrt{\frac{s_L+s}{s_A}}} \quad (5.1)$$

Assuming that $d(s)$ does not differ by much from unity on the near left-hand cut, we can fix the parameters s_L and C by requiring that the discontinuity of the nearby part of the left-hand cut of $n(s)$ which has the form

$$\rho_L(s) = \frac{C|s-4| \cos \sqrt{\frac{s_L+s}{s_A}}}{\sqrt{\frac{s_L+s}{s_A}}}$$

roughly fits that expected from ρ exchange in magnitude and position. The experimental asymptotic behaviour ⁶⁾ determines s_A , of course.

As we have discussed earlier, such a low energy left-hand cut does not, by itself, produce a resonance. However, when we substitute the expression (5.1) for $n(s)$ into Eq. (3.5) for $d(s)$ we find that the force is stronger than is needed to give a zero of d at the required position.

While we do not take seriously the detailed predictions of this very crude model, we believe that it indicates that the distant left-hand cut, even though it oscillates, has effects of practical importance which should be taken into account. To summarize, we have an example of an amplitude with a left-hand cut structure roughly consistent with the ρ exchange contribution at low energies and the experimental asymptotic behaviour at high energies which gives a much stronger attractive force than the nearby left-hand cut alone provides. It should be stressed that no special features have been imposed on this far cut apart from the oscillations necessary for the asymptotic behaviour.

The detailed properties of this asymptotic potential remain to be investigated but it seems to contain a possible source of the missing attraction in the $J = 1 \pi\pi$ system.

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