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IVESA DYNAMICAL MODEL FOR THE MESON SPECTRUM

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ABSTRACT

A model for the meson spectrum is constructed under the following dynamical assumptions :

- (A) Dominance of processes with low thresholds;
- (B) Dominance of one-particle-exchange forces;
- (C) SU_3 symmetry;
- (D) Dominance of quasi-two-particle states in production processes.

A bootstrap system is sought which satisfies the following conditions :

- i) "Consistency" in that all exchanged particles are bound as poles in the appropriate direct channels;
- ii) "Closure" in that no further low-lying states are bound other than those whose exchange contributes to the forces;
- iii) "Completeness" in that all "low" energy processes involving any of the particles in the spectrum are taken into account.

Starting from the 0^- octet (π, K, η) and the 1^- octet (ρ, K^*, ϕ), these consistency considerations lead to the meson supermultiplets listed in Table 1. Interesting regularities in the predicted spectrum are noted. In particular one sees that apart from "Regge" recurrences, the spectrum happens to coincide with that contained in the $(6^*, 6) - (6, 6^*)$ representation of the recently proposed $U_6 \times U_6$ supersymmetry.

A tentative assignment is attempted of known meson states into the scheme. Apart from the $K^+ - \pi^+ - \pi^0$ enhancement at 1270 MeV, all reasonably well-established resonances to-date fit comfortably into the spectrum. Further empirical regularities are noted in the assignment, which include one in the Bronzan-Low quantum number A.

Wherever feasible, the dynamical parameters (masses, coupling constants) are computed by requiring numerical "bootstrap" consistency using a method proposed by Zachariasen and Zemach. These values are listed in Table 2, where it is seen that qualitative agreement is obtained for all parameters so far calculated.

I. INTRODUCTION

In recent years a large number of highly unstable, strongly interacting particles have been discovered ¹⁾ and it appears likely that the present trend will continue. The number of these states has in fact increased to such an extent, and the system has become so intricate, that we are faced with a situation not very different from that prior to the introduction of the periodic system for atoms and nuclei. In view of this and of the absence of a reliable theory of strong interactions, it seems desirable to construct a model capable of describing the gross features of the system, which ideally would develop into a sort of mesonic and baryonic shell model. The problem of strong interaction may be an intrinsically difficult one. Even when a system is at low energy where only a few real particles exist, these are still coupled to a large number of virtual particles. Thus the simplest scattering problem is in fact a many-body problem in disguise. It may well be that when a system of "field" equations is found which describes correctly elementary particle physics, one may still need some sort of working model for carrying out actual calculations for strong interactions.

In this paper we shall report the beginning of an attempt to construct such a model for the meson system based on the following three dynamical assumptions :

- A) dominance of low-threshold processes at low energy ;
- B) dominance of one-particle exchange graphs ;
- C) SU_3 symmetry.

Our starting point is thus similar to that of the investigations by Cutkosky ²⁾ and Martin and Wali ³⁾ on the baryonic system. The meson system is however particularly amenable to this sort of treatment. Its simpler kinematics allows us to investigate much more thoroughly the question of self-consistency. Moreover, the wider spacing of thresholds and the weaker dependence on cut-offs give us somewhat more confidence in assumptions A) and B). In any case, the solution of the meson problem is a necessary step towards the full understanding of the baryons.

Assumption A), being the starting point in almost all existing "theories" of strong interactions ⁴⁾, need not be discussed here. Assumption B) is current in bootstrap calculations ⁴⁾ and has also been applied with some degree of success to various other problems such as nucleon electromagnetic form factors ⁵⁾, the Goldberger-Treiman relation ⁶⁾, the peripheral models of meson production ⁷⁾, and low-energy nucleon-nucleon scattering ⁸⁾. Although no accurate quantitative success can be claimed in any of these cases, the predictions seem to be in qualitative agreement with experiment. In the present context hypothesis B) means the following. We maintain firstly that the forces mainly responsible for the binding of "particles" are of long or medium range, and are thus dominated by the exchange of low mass states. The exchange of higher masses gives rise only to core interactions which we claim, contribute but little to the binding of meson states in the intermediate energy range. Secondly, we consider that the exchange of multimeson states with a low total energy is itself dominated by the resonances which may exist between the exchanged mesons. This means, in dispersive language, that we consider the discontinuity across the left-hand cut to be dominated by the nearby poles on the crossed-channel unphysical sheet.

We are yet unclear whether the assumption of SU_3 symmetry C) is in fact necessary in such a dynamical model ^{9),10)}. However, the simplification achieved thereby allows us to take into account a large amount of the important inelastic effects, which would otherwise be quite impossible to handle. The assumption is therefore kept throughout the present paper.

Assumptions A) and B) and the requirement of crossing symmetry lead naturally to the idea of particle "bootstrap" ⁴⁾. According to crossing symmetry, any "particle" which occurs as a bound state or resonance pole in the direct channel must also appear as an exchanged particle if the crossed channel represents the same physical process.

Now by B) we have assumed that the binding of particles is effected by forces due to particle exchange. It follows that in any scattering process, the "composite" particles which occur, and their dynamical parameters (masses, coupling constants) must be such that the forces due to their exchange be just sufficient to bind the same particles again as poles in the direct channel. The poles must occur at energies equal to the particles' masses, and have residues equal (apart from kinematical factors) to the coupling constants. This bootstrap self-consistency condition, if taken seriously, is thus a very stringent one. We shall in what follows investigate in some detail its consequences on the low energy meson system.

Although the idea of particle bootstrap is physically very attractive, no formulation of the actual procedure has yet been found which is free from ambiguity. One of the main difficulties encountered is the old problem of divergence. Several crude mechanisms, however, have been suggested which seem to give sensible answers at low energies ¹¹⁾. We shall adopt here for definiteness exclusively the mechanism suggested by Zachariasen and Zemach ¹¹⁾, though we have no particular reason to prefer this mechanism over others apart from that of simplicity.

Z) Zachariasen-Zemach bootstrap mechanism.

It is believed however, that many interesting features of the model are consequences of the general dynamical assumptions A) - C) listed above, and are independent of the method Z). Some of these are obvious, such as the signs of the forces (attractive or repulsive) in the various SU_3 and J^P channels, and the parity doubling of the spectrum discussed in Sections III and IV. Others such as the relative strengths of the coupling constant, etc., are more sensitive to the details of the dynamics, though not necessarily peculiar to the Z) method. We have not attempted in this paper to separate very carefully the more general features from the results peculiar to our calculation. Now, although general statements concerning

the nature of the forces depend merely on the Born terms and the SU_3 crossing matrix elements, definite predictions on the binding or otherwise of "particles" cannot be made without a more precise criterion. We regard the Z) method mainly as just such a criterion. It is noteworthy, however, that all the parameters calculated have so far come out surprisingly close to the expected values, and should not perhaps be summarily discarded.

It is not our purpose here to pursue the ambitious programme of obtaining the existing system of particles as the unique solution to bootstrap equations. We attempt only to investigate closely the consequences of the bootstrap hypothesis starting from a limited amount of well-established experimental facts. We thus take for granted the existence of the following sets of particles, together with the usual assignment of their quantum numbers and the rough order of magnitude of their masses :

- S) i) $0^-(8^+)$; π , K, η ; $m \sim 410$ MeV 12),13) ;
 ii) $1^-(8^-)$; ρ , K^* , ϕ ; $m \sim 870$ MeV 12),13) ;
 iii) baryons, $m \sim 1$ GeV.

It is, however, not obvious that the existence of these particles S) is consistent with the assumptions A), B) and Z) above. In fact we shall have the occasion to test their mutual consistency.

Experimentally, at least nine vector mesons are known, ρ , K^* , ϕ , and ω , where the ϕ and ω mesons are currently interpreted as mixtures of pure SU_3 octet and singlet $I = 0$ states. Since the present model is completely SU_3 invariant, we shall not be able to take account of $\phi - \omega$ mixing. Moreover, the singlet vector meson does not occur as an intermediate state in the low energy problem (see Sections II and III). Its existence will be deduced as a consequence of our assumptions in an extension to higher energies, (Section IV).

We now proceed to investigate the meson system within the framework we have constructed. We consider first of all the scattering of two pseudoscalar mesons belonging to the set $0^-(8^+)$, this being the process with the lowest threshold, [assumption A)]. The existence of the $1^-(8^-)$ particles and assumption B) then imply that the interactions between pseudoscalar mesons are dominated by the exchange of these vector resonances. Moreover, for consistency, these forces must be such as to bind the same vector resonances again as "composite particles" in the direct channel. This problem has been investigated in detail in Ref. ¹⁴⁾. It was found that the forces in the $1^-(8^-)$ direct channel are in fact attractive and that the parameters of the vector resonances may be adjusted so as to bind the particles again as poles, Z). However, the forces in the $0^+(1^+)$ and $0^+(8^+)$ channels were also attractive and were strong enough to bind poles in these s wave states. For consistency, therefore, these should be considered as scalar resonances and be exchanged also, giving rise to additional forces. These new forces were taken into account and the parameters of the particles readjusted until over-all self-consistency was obtained. The system was then found to be "closed" as far as the scattering of $0^-(8^+)$ mesons are concerned, in the sense that no further particles are required for consistency apart from the $0^+(8^+)$, $0^+(1^+)$ and $1^-(8^-)$ mesons already considered and their "Regge" recurrences at higher energies. (The results of Ref. ¹⁴⁾ are briefly summarized in Section II for quick reference.)

We have yet, however, to require self-consistency in another direction. The $0^+(8^+)$ octet of Section II was found to have a mass of the same order of magnitude as the original pseudoscalar octet $0^-(8^+)$. If these scalar mesons are considered as genuine particles then the processes involving them as external particles have similar thresholds to that of $0^-(8^+) - 0^-(8^+)$ scattering, and cannot by A) be ignored. We are thus led to consider the enlarged problem represented symbolically by Fig. 2, Section III. Again, we seek a solution to the problem self-

consistent and "closed" in the sense described above. A tentative solution has been suggested previously ¹⁵⁾, which though qualitatively correct, was incomplete, being dependent on further approximations based on experiment. In Section III, a complete solution to the problem is presented which involves no further assumptions other than those which have already been discussed. It is found that in addition to the meson supermultiplets discovered in Ref. ¹⁴⁾, a further singlet $0^-(1^+)$ and an octet $1^+(8^-)$ are required for consistency. But the system is then completely self-consistent and "closed" in the sense defined.

We next attempt to extend the model to higher energies under the same assumptions A), B), C), and Z). Although the solution described above. (Section III), is now logically complete, an extension to higher energy is a necessity if the model is to be physically acceptable. The energy region of experimental interest at present is around 1.0 to 1.5 GeV where a large number of meson resonances have been discovered recently ¹⁾. At such energies it is difficult to believe that multimeson effects, ignored so far, can still be negligible. Moreover, most of the newly discovered resonances ¹⁾ (e.g., the B , A_1 , A_2 , C , X_0 resonances) are found to decay predominantly into multimeson states.

We shall restrict ourselves to energies below the threshold for baryon-pair production, i.e., ~ 2 GeV. Thus the only inelastic effect we have to consider is multiple-meson production. However, the technical difficulty involved in dealing with multiparticle states is very great. In order to study these inelastic effects in a manner in keeping with the simplicity of the present model, we shall make a further dynamical assumption, namely :

D) quasi-two particle assumption.

We claim that in a system of several particles among which strongly resonating states exist, their interaction is dominated by processes in which the particles emerge in two resonating groups. This assumption,

though crude and difficult to justify theoretically, is a physically attractive one. Recent experiments on multimeson production at the energy range under discussion have shown that a large proportion ($\sim 50\%$) of the reaction does indeed go via quasi-two particle intermediate states ¹⁶⁾. This figure may well be an underestimate since some of the resonances, hence also some quasi-two-particle channels, may not have been properly recognized. A case in mind here is the Rutherford Laboratory ¹⁷⁾ experiment ($\pi^- + p \rightarrow K^+ \pi^- \Lambda$ or $K^+ \pi^- \Sigma^0$) in which the $K\pi$ enhancement at 1420 MeV was discovered. The recognition of this resonance brings the number of quasi-two particle events up to $\sim 80\%$ of the total in these channels.

By assumption D) therefore, we claim that the multimeson processes at 1.0 to 1.5 GeV are dominated by the production of a spin 1 meson together with a spin 0 meson. The production of two spin 1 mesons, or of spin 2 mesons will bring us close to the baryon-antibaryon threshold, which we therefore neglect. Assumption B) then allows us to analyze the forces in the various $J^P(X^C)$ channels by simply computing all the OPE graphs involving the particles discussed above, (Section III). The computation is straightforward though complicated and will be reported in Section IV. One would like of course to obtain numerical self-consistency here also in the same manner as in the low energy problem (Sections II, III). This, however, we have not been able to do because of sheer complication, but have merely analyzed the forces in much the same way as Martin and Wali ³⁾ did for the baryon problem. This analysis is not free from ambiguity, being dependent on a cut-off, and on further hypotheses based either on experiment or on theoretical proposals requiring further confirmation. These last include the concept of "universality" for vector meson coupling, as proposed by Sakurai ¹⁸⁾, and the A) selection rule recently suggested by Bronzan and Low ¹⁹⁾. With these restrictions we then find that of the 36 channels possible with $J \leq 2$, only two new ones show conspicuous attractive forces, namely $1^-(1^-)$ and $1^+(1^-)$. Thus besides those particles which we have already discussed above, the present extension of the model yields only two further spin 1 resonances in the energy region under consideration.

The meson spectrum predicted by this model is summarized in Table 1 of Section V. We have attempted a tentative assignment of the known resonances into our scheme ²⁰⁾, and found that apart from the $I_Z = 3/2$ enhancement at 1270 MeV ²¹⁾ reported by a CERN group, all reasonably well-established resonances fit comfortably into our scheme. The resulting list shows some remarkable regularities. Possible relations of our model to other works on mesonic resonances are also discussed, especially as regard the empirical quantum number A, and the $U_3 \times U_3$ symmetry suggested by Gell-Mann, and Salam and Ward ²²⁾. We point out in particular that the spectrum predicted by our model, apart from Regge recurrences, coincides with that contained in the $(6^*, 6) - (6, 6^*)$ representation of the $U_6 \times U_6$ symmetry scheme proposed recently by Feynman, Gell-Mann and Zweig ²³⁾.

Our model also allows us to calculate some of the dynamical parameters, Z). Taking the mean mass of the pseudoscalar octet $0^-(8^+)$ as 410 MeV, we obtained the values listed in Table 2 of Section V. Although the crudeness of the model makes dubious any quantitative significance of the calculated parameters, it is still gratifying that the calculation yields values never very far from those estimated from experiment.

II. SCATTERING OF PSEUDOSCALAR MESONS

In this Section we summarize very briefly the result of Ref. 14) with slight changes of notation.

According to assumption A), the interaction of mesons at low energies is dominated by processes with low thresholds. We thus consider first the mutual scattering of two pseudoscalar mesons belonging to the supermultiplet $0^-(8^+)$, this being the process with the lowest threshold, S). Assumption B) then implies that the interaction forces are dominated by the exchange of the vector resonances $1^-(8^-)$. These exchange forces were calculated and were found to be attractive in the direct $1^-(8^-)$ channel, so that by adjusting the mass and the coupling constant, the same vector mesons could be bound again in the direct channel as poles. The first self-consistency requirement is thus satisfied. However, the forces calculated by Z), using the self-consistent parameters, also show attraction in the channels $0^+(8^+)$ and $0^+(1^+)$ and were in fact strong enough to bind new poles. It is necessary therefore for consistency to consider also the forces due to the exchange of these scalar resonances. As pointed out in Ref. 14), the exchange of the scalar singlet $0^+(1^+)$ may, for our purpose here, be neglected. The remaining graphs with $1^-(8^-)$ and $0^+(8^+)$ exchange, (Fig. 1a,b) were then computed and decomposed into the various spin states. This gives for the vector exchange graph, (Fig. 1a) ²⁴⁾, the function

$$4\pi F^J(s) = \frac{2}{s-4\mu^2} (m_1^2 + 2s - 4\mu^2) Q_J\left(\frac{2m_1^2 + s - 4\mu^2}{s - 4\mu^2}\right) - \delta_{J,0} \quad (\text{II.1a})$$

and for the scalar exchange graph, (Fig. 1b), the function

$$4\pi G^J(s) = \frac{4\mu^2}{s-4\mu^2} Q_J\left(\frac{2m_0^2 + s - 4\mu^2}{s - \mu^2}\right), \quad (\text{II.1b})$$

where $m_1(m_0)$ is the mass of the exchanged vector (scalar) mesons, s the square of the c.m. energy, and μ the degenerate mass of the external 0^- octet, taken here as the scale, (410 MeV). Q_J is the Legendre function of the second kind of order J .

Fig. 1a

Fig. 1b

The mechanism Z) then yields for the partial amplitudes $T^{JP(XC)}$ the following expressions

$$T^{JP(XC)} = - N^{JP(XC)} / D^{JP(XC)}, \quad (\text{II.2})$$

$$N^{JP(XC)} = -\lambda^{(XC)} \eta_1^2 F^J(s) - \kappa^{(XC)} \eta_0^2 G^J(s), \quad (\text{II.3})$$

$$D^{JP(XC)} = 1 - \lambda^{(XC)} \eta_1^2 \alpha^J(s) - \kappa^{(XC)} \eta_0^2 \beta^J(s), \quad (\text{II.4})$$

where η_1 and η_0 are the coupling constants of respectively (ps, ps, V) and (ps, ps, s), and α^J and β^J are dispersion integrals subtracted once at $s = s_1$,

$$\alpha^J(s) = \frac{s-s_1}{\pi} \int_{4\mu^2}^{\infty} ds' \sqrt{\frac{s'-4\mu^2}{s'}} \frac{F^J(s')}{(s'-s_1)(s'-s-i\varepsilon)}, \quad (\text{II.5a})$$

$$\beta^J(s) = \frac{s-s_1}{\pi} \int_{4\mu^2}^{\infty} ds' \sqrt{\frac{s'-4\mu^2}{s'}} \frac{G^J(s')}{(s'-s_1)(s'-s-i\varepsilon)}. \quad (\text{II.5b})$$

The constants $\lambda^{(XC)}$ and $\kappa^{(XC)}$ are SU_3 crossing matrix elements (Racah coefficients) for, respectively, vector and scalar octet exchange in the direct SU_3 channel (X^C), which in our convention take the following values ²⁵⁾

$$\lambda^{(1+)} = 1, \lambda^{(8+)} = 1/2, \lambda^{(8-)} = 1/2, \lambda^{(10)} = \lambda^{(\overline{10})} = 0, \lambda^{(27+)} = -1/3; \quad (II.6a)$$

$$\kappa^{(1+)} = 1, \kappa^{(8+)} = -3/10, \kappa^{(8-)} = 1/2, \kappa^{(10)} = \kappa^{(\overline{10})} = -2/5, \kappa^{(27+)} = 1/5. \quad (II.6b)$$

A positive crossing matrix element signifies attraction, and a negative one a repulsion in that particular channel. Notice that because of the Pauli principle only states with $J^P = (\text{even})^+$ exist in the SU_3 channels (X^C) = (1^+) , (8^+) and (27^+) ; and only $J^P = (\text{odd})^-$ exist in (X^C) = (8^-) , (10) , $(\overline{10})$.

The bootstrap condition then requires that the exchanged mesons be bound again as poles in the direct channel, so that the functions $D^{1^-(8^-)}$ and $D^{0^+(8^+)}$ should have zeros respectively at $s = m_1^2$ and $s = m_0^2$. Hence,

$$1 = g_1^2 \lambda^{(8-)} \alpha'(m_1^2) + g_0^2 \kappa^{(8-)} \beta'(m_1^2), \quad (II.7a)$$

$$1 = g_1^2 \lambda^{(8+)} \alpha^0(m_0^2) + g_0^2 \kappa^{(8+)} \beta^0(m_0^2). \quad (II.7b)$$

Moreover, the residue of $T^{1^-(8^-)}$ and $T^{0^+(8^+)}$ at these poles should be equal, apart from kinematic factors, to the coupling constants g_1^2 and g_0^2 , namely

$$g_1^2 = \frac{12\pi}{m_1^2 - 4\mu^2} \frac{[g_1^2 \lambda^{(8-)} F^1(m_1^2) + g_0^2 \kappa^{(8-)} G^1(m_1^2)]}{[g_1^2 \lambda^{(8-)} \alpha^{1'}(m_1^2) + g_0^2 \kappa^{(8-)} \beta^{1'}(m_1^2)]}, \quad (\text{II.8a})$$

$$g_0^2 = 4\pi \cdot \frac{[g_1^2 \lambda^{(8+)} F^0(m_0^2) + g_0^2 \kappa^{(8+)} G^0(m_0^2)]}{[g_1^2 \lambda^{(8+)} \alpha^{0'}(m_0^2) + g_0^2 \kappa^{(8+)} \beta^{0'}(m_0^2)]}. \quad (\text{II.8b})$$

The conditions (II.7), (II.8) were to be solved for the four parameters g_1^2 , g_0^2 , m_1^2 , m_0^2 . This was performed numerically and a solution was found with

$$\begin{aligned} m_1 &= 2.33\mu, & m_0 &= 1.2\mu, \\ g_1^2/4\pi &= 5.56, & g_0^2/g_1^2 &= 0.043. \end{aligned} \quad (\text{II.9})$$

The calculated mass of the vector meson, $m_1 = 2.33\mu$, corresponds to a mean mass of the $1^-(8^-)$ octet of ~ 960 MeV, as compared to the experimental value of 870 MeV. The value of $g_1^2/4\pi$ estimated from $\rho \rightarrow 2\pi$ decay is ~ 1.8 , so that the calculated value is a factor of 3 too large. A discussion on the prediction of a scalar octet will be postponed until Section V. We note here only the smallness of g_0^2 as compared to g_1^2 , and the approximate equality of m_0 to μ .

These self-consistent dynamical parameters (II.9) were then used to study the forces in the other channels, $J^P(X^C)$. The signs of the crossing matrix elements (II.6) and the small value of g_0^2/g_1^2 , (II.9) implied that the total forces due to both graphs (Fig. 1a,b) were repulsive

in the 10 , $\overline{10}$, 27^+ , and could bind no resonance poles in these channels. Our system is thus "closed" in the sense that no further "particles" are required for internal consistency, other than the supermultiplets $0^-(8^+)$, $1^-(8^-)$, $0^+(1^+)$, $0^+(8^+)$, and their possible "Regge" recurrences.

Throughout this paper, as also in Ref. ¹⁴⁾, we always ignore the forces due to the exchange of higher Regge recurrences ($J \geq 2$). The contention is that, because of the centrifugal barrier, these will have higher masses. Their exchange will therefore contribute mainly to short-range core interactions which we systematically ignore. (See also Section V.b.) One may, however, use the calculated parameters, (II.9), to estimate the masses of possible higher Regge recurrences in the "bound" SU_3 channels by solving the mass equations

$$1 = \eta_1^2 \lambda^{(xc)} \alpha^J(m_J^{x2}) + \eta_0^2 \kappa^{(xc)} \beta^J(m_J^{x2}). \quad (\text{II.10})$$

In Ref. ¹⁴⁾, the equation (II.10) has been solved for the lowest lying Regge recurrences, namely $2^+(1^+)$ and $2^+(8^+)$, and one obtained, for $\mu = 410$ MeV, the rough values $m_2^1 \sim 1$ GeV ; $m_2^8 \sim 1.4$ GeV.

III. "COMPLETE" LOW ENERGY PROBLEM

The solution in Section II to the bootstrap problem is self-consistent and "closed" as far as the scattering of pseudoscalar mesons is concerned. However, if its predictions are accepted, it is necessary to satisfy consistency in yet another direction. The O^+ octet predicted in Section II has a mass of the same order of magnitude as the O^- octet, (II.9). Scattering processes between the O^+ mesons, and between the O^+ and O^- mesons, thus have thresholds not very different from that of processes involving the O^- mesons alone. For consistency with assumption A) therefore, one should consider all these processes together.

Ideally one should consider in the bootstrap problem also the processes involving as external particle the O^+ singlet predicted in Section II since this singlet was found to have a mass lower even than the O^+ octet. We have, however, avoided bootstrapping these processes partly for the sake of simplicity, but mostly because we are yet unclear as to the actual physical significance of the singlet "meson". (See Section V.a.) For a discussion on the possible effects due to processes involving this singlet, the reader is referred to the end of this Section. In the actual bootstrap calculation we consider only processes involving the octet mesons as external particles.

One is then led to the problem represented diagrammatically ²⁴⁾

i) by Fig. 2 for states with spatial parity $P = (-1)^J$,

Fig. 2

ii) by Fig. 3 for states with spatial parity $P = (-1)^{J+1}$.

Fig. 3

Again one seeks a solution to this problem self-consistent and "closed" in the sense explained before.

In a previous investigation ¹⁵⁾ it was found that one requires for self-consistency at least the following supermultiplets ¹²⁾

$$0^+(1^+), 0^-(1^+), 0^+(8^+), 0^-(8^+), 1^-(8^-), 1^+(8^-) \quad (\text{III.1})$$

and a tentative solution was suggested with only these mesons. The solution given there was however incomplete since the forces due to certain diagrams, (0^+ and 1^+ exchange), had been neglected. We shall present here a "complete" solution to the problem with the supermultiplets (III.1) which involves no further assumptions other than those we have already discussed.

According to assumption B) we shall insert as input forces all one-particle exchange graphs exchanging the particles (III.1). Again as in Section II the exchange of the singlet mesons may be neglected. We then have as input diagrams the graphs ²⁴⁾

i) (Fig. 4) for $P = (-1)^J$

Fig. 4

Fig. 4 (cont.)

ii) (Fig. 5) for $P = (-1)^{J+1}$.

Fig. 5

These exchange graphs then give rise to forces in the various $J^P(X^C)$ states in the direct channel, the forces depending on the values of the coupling constants and masses of the exchanged particles.

The bootstrap condition then requires that the forces be such as to bind the same particles again as poles in the direct channel at energies equal to the particles' masses and with residues proportional to the corresponding coupling constants. Thus for bootstrapping the vector octet $1^-(8^-)$ at mass m_V^2 , we may write the conditions symbolically in the form of Fig. 6 ²⁴⁾

$$\det. \begin{pmatrix} \text{Diagram 1} & \text{Diagram 2} \\ \text{Diagram 3} & \text{Diagram 4} \end{pmatrix}^{1^-(8^-)} = 0, \quad S = m_V^2$$

The diagram shows a 2x2 matrix of Feynman diagrams. Each diagram is a vertex labeled 'D' with four external lines. The top-left and bottom-right diagrams have solid internal lines, while the top-right and bottom-left diagrams have wavy internal lines. The matrix is labeled with $1^-(8^-)$ at the top and $S = m_V^2$ at the bottom right.

Fig. 6

$$\text{Res.} \left(\begin{array}{cc} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \right)_{S=m_V^2}^{1^-(8^-)} = \left(\begin{array}{cc} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \right)$$

Fig. 6 (cont.)

and for bootstrapping the pseudovector octet at mass m_{pV}^2 in the form of Fig. 7

$$\left(\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \right)_{S=m_{pV}^2}^{1^+(8^-)} = 0,$$

$$\text{Res.} \left(\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \right)_{S=m_{pV}^2}^{1^+(8^-)} = \left(\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \right)$$

Fig. 7

where D and T are respectively the denominator function and the scattering amplitude as calculated from the input N function using the Zachariasen-Zemach procedure Z). The conditions for bootstrapping the $0^+(8^+)$ and $0^-(8^+)$ are similar. These bootstrap equations represent altogether 10 conditions on the parameters of the problem.

The adjustable parameters of the problem consist of five coupling constants : $g_{ps,ps}^V$, $g_{s,s}^V$, $g_{ps,s}^{pV}$, $g_{ps,ps}^s$, $g_{s,s}^s$; and six masses : the external mesons' masses μ_s , μ_{ps} , and the internal mesons' masses m_s , m_{ps} , m_V , m_{pV} . Since, however, we have no energy scale, only the ratios of the masses are meaningful. We have thus 10 independent parameters, just sufficient to be determined by the 10 bootstrap conditions described above.

We have, however, in this problem two further conditions to satisfy in addition to the usual bootstrap conditions. Namely we should require that the internal spin 0 mesons be identical to the external ones, thus :

$$\mu_s = m_s, \quad \mu_{ps} = m_{ps}. \quad (\text{III.2})$$

The system then becomes overdetermined and the existence of a solution is not trivial.

It is straightforward to write down explicitly the general form of the bootstrap equations (Figs. 6, 7) and the similar ones for bootstrapping the spin 0 mesons. They will, however, not be necessary for the following discussion since we shall not attempt to find all solutions to the problem. We shall show only that a simple solution to the equations exists which satisfies the following ansatz :

$$\begin{aligned} \mu_s = \mu_{ps} = \mu; \quad m_s = m_{ps} = m_0; \quad m_V = m_{pV} = m_1; \\ g_{ps,ps}^s = g_{s,s}^s = g_0; \quad g_{ps,ps}^V = g_{s,s}^V = g_{ps,s}^{pV} = g_1, \end{aligned} \quad (\text{III.3})$$

and that the solution satisfies the Eqs. (III.2) to as good an approximation as can reasonably be expected from such a crude model. For this, it is sufficient to write down the bootstrap equations under the restriction (III.3).

We note first that because of the charge conjugation properties of the supermultiplets listed in (III.1), the couplings (ps, ps, s) and (s, s, s) are the totally symmetric $8 \times 8 \rightarrow 8$ coupling in SU_3 , while the couplings (ps, ps, V) , (ps, s, pV) , and (s, s, V) are totally antisymmetric²⁶⁾. Moreover, under the conditions (III.3) the graphs (Fig. 4a,c,e) are kinematically identical to one another and to

graph (Fig. 5a), while their crossed graphs are identical to (Fig. 5b). Similarly, under conditions (III.3), the graphs (Fig. 4b,d,f) and the corresponding crossed graphs are identical to (Fig. 5c) and (Fig. 5d), respectively. It can be seen then that on projecting out the contribution to a specific SU_3 channel (X^C), and spin-parity state J^P , one has for the N function the very simple form

i) for $P = (-1)^J$

$$N^{JP(X^C)} = - \left[\lambda^{(X^C)} \eta_1^2 F^J(s) + \kappa^{(X^C)} \eta_0^2 G^J(s) \right] \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \quad (\text{III.4})$$

ii) for $P = (-1)^{J+1}$

$$N^{JP(X^C)} = - 2 \left[\lambda^{(X^C)} \eta_1^2 F^J(s) + \kappa^{(X^C)} \eta_0^2 G^J(s) \right]. \quad (\text{III.5})$$

The constants $\lambda^{(X^C)}$, $\kappa^{(X^C)}$ are SU_3 crossing matrix elements, and the functions F^J , G^J the partial wave projections of OPE graphs, as defined in Section II. There is a factor 2 different between the $P = (-1)^{J+1}$ and $P = (-1)^J$ amplitudes because of the Pauli principle.

Similarly the D function for the channel $J^P(X^C)$ is of the form

i) for $P = (-1)^J$

$$D^{JP(X^C)} = I - \left[\lambda^{(X^C)} \eta_1^2 \alpha^J(s) + \kappa^{(X^C)} \eta_0^2 \beta^J(s) \right] \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; \quad (\text{III.6})$$

ii) for $P = (-1)^{J+1}$

$$D^{JP(X^C)} = 1 - 2 \left[\lambda^{(X^C)} \eta_1^2 \alpha^J(s) + \kappa^{(X^C)} \eta_0^2 \beta^J(s) \right]. \quad (\text{III.7})$$

Note that the scattering amplitude $T^{JP(XC)}$ defined as

$$T^{JP(XC)} = -N^{JP(XC)} [D^{JP(XC)}]^{-1} \quad (\text{III.8})$$

is automatically symmetric, i.e., time-reversal invariant, since N and D commute.

We next specialize to the channels $J^P(X^C) = 0^-(8^+)$, $0^+(8^+)$, $1^-(8^-)$, $1^+(8^-)$ for the bootstrap equations of these particles. For the mass of the 1^- mesons, m_1 , we have

$$\det \left[I - \left\{ \lambda^{(8-)} \eta_1^2 \alpha'(m_1^2) + \kappa^{(8-)} \eta_0^2 \beta^1(m_1^2) \right\} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] = 0,$$

or simply

$$1 - 2 \left\{ \lambda^{(8-)} \eta_1^2 \alpha'(m_1^2) + \kappa^{(8-)} \eta_0^2 \beta^1(m_1^2) \right\} = 0, \quad (\text{III.9a})$$

while for the mass of the 0^+ mesons, we have similarly,

$$1 - 2 \left\{ \lambda^{(8+)} \eta_1^2 \alpha^0(m_0^2) + \kappa^{(8+)} \eta_0^2 \beta^0(m_0^2) \right\} = 0. \quad (\text{III.9b})$$

The coupling constant η_1^2 , according to the procedure Z), is given by an equation of the form

$$\eta_1^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \approx -N^{1-(8-)} \bar{D}^{1-(8-)} / \frac{d}{ds} [\det D^{1-(8-)}]$$

at $s = m_1^2$, where

$$\bar{D} / \det D = D^{-1}.$$

Explicitly,

$$\bar{D}^{1-(8-)} = \begin{pmatrix} 1-A & A \\ A & 1-A \end{pmatrix}$$

with $A = \lambda^{(8-)} g_1^2 \alpha^1(m_1^2) + \kappa^{(8-)} g_0^2 \beta^1(m_1^2)$. This gives for g_1^2 the equation

$$g_1^2 = \frac{1}{2} \cdot \frac{12\pi}{m_1^2 - 4\mu^2} \cdot \frac{[g_1^2 \lambda^{(8-)} F^1(m_1^2) + g_0^2 \kappa^{(8-)} G^1(m_1^2)]}{[g_1^2 \lambda^{(8-)} \alpha^{1'}(m_1^2) + g_0^2 \kappa^{(8-)} \beta^{1'}(m_1^2)]}. \quad (\text{III.10a})$$

Similarly for the coupling constants g_0^2 , we have

$$g_0^2 = 2\pi \frac{[g_1^2 \lambda^{(8+)} F^0(m_0^2) + g_0^2 \kappa^{(8+)} G^0(m_0^2)]}{[g_1^2 \lambda^{(8+)} \alpha^{0'}(m_0^2) + g_0^2 \kappa^{(8+)} \beta^{0'}(m_0^2)]}. \quad (\text{III.10b})$$

From the bootstrap condition for the 1^+ and 0^- mesons in the $P = (-1)^{J+1}$ amplitudes we also obtain equation for the masses m_1^2 , m_0^2 and for the coupling constants g_1^2 , g_0^2 . These equations, however, are readily seen to be identical to the equations above, (III.9) and (III.10), thus verifying the consistency of the restrictions (III.3) with the bootstrap equations.

We are then left with four conditions for the four unknowns, m_1^2 , m_0^2 , g_1^2 and g_0^2 in terms of the external meson mass. These equations, (III.9) and (III.10), however, differ from Eqs. (II.7) and

(II.8) only by factors of $\frac{1}{2}$. It can readily be seen then that one obtains a solution for the new equations simply by putting : $(m_{1,0}^2)_{\text{new}} = (m_{1,0}^2)_{\text{old}}$ and $(\lambda_{1,0}^2)_{\text{new}} = \frac{1}{2}(\lambda_{1,0}^2)_{\text{old}}$. Thus for the new equations,

$$\begin{aligned} m_1 &= 2.33 \mu, & m_0 &= 1.2 \mu \\ \lambda_1^2/4\pi &= 2.73, & \lambda_0^2/\lambda_1^2 &= 0.043. \end{aligned} \quad (\text{III.11})$$

It has already been pointed out in Section II that the value of m_1 obtained there was in crude agreement with experiment, while the value of $\lambda_1^2/4\pi$ was a factor of 3 too large. The present reduction of λ_1^2 by a factor 2 through the inclusion of inelastic effects of scalar mesons is thus in the right direction and brings the calculated value to within 1.5 times the value estimated from ρ decay.

We have then obtained a solution to the usual bootstrap equations (III.9), (III.10). However, one requires in addition (III.2), namely $m_0 = \mu$. The numerical solution (III.11) yields $m_0 = 1.2\mu$, which is as good an agreement as one can expect from such a crude calculation.

We next examine the forces in the other SU_3 channels in order to ascertain that the system is now "closed". Namely we wish to show that the forces calculated with the bootstrap parameters are such that no further particles are bound besides those listed (III.1) and their Regge recurrences. Poles in the channels $J^P(X^C)$ are given by the equations

$$i) \quad P = (-1)^J$$

$$\det \left[I - \left\{ \lambda^{(X^C)} \lambda_1^2 \alpha^J(m_{J^X}^2) + K^{(X^C)} \lambda_0^2 \beta^J(m_{J^X}^2) \right\} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] = 0$$

or simply

$$1 - 2 \left\{ \lambda^{(X^C)} \eta_1^2 \alpha^J(m_J^{X^C}) + \kappa^{(X^C)} \eta_0^2 \beta^J(m_J^{X^C}) \right\} = 0. \quad (\text{III.12})$$

- ii) for $P = (-1)^{J+1}$, the same Eq. (III.12). This means in this solution particles of any $J(X^C)$ must occur in parity-doublets, or not at all.

Moreover, the present equations differ from Eq. (II.10) only by a numerical factor 2, which, however, is compensated by the reduction in the values of $\eta_{1,0}^2$. All conclusions drawn in Section II can thus be taken over directly for any $J(X^C)$ and interpreted here as holding for both parities. We therefore expect no binding in $X^C = 10, \overline{10}, 27^+$ channels for either parity at "low" energies where this model applies. Whereas in channels $X^C = 8^-$; $J^P = 1^\pm, 3^\pm, \dots$, and $X^C = 1^+, 8^+$; $J^P = 0^\pm, 2^\pm, \dots$, one expects attractive forces and "particles" to occur. The channels $0^\pm(8^+)$ and $1^\pm(8^-)$ are the channels we have bootstrapped. The channels $0^\pm(1^+)$ are the mysterious low-lying singlets we have so far deliberately ignored. Apart from these, our system is then "closed" in the sense explained before.

The present solution to the low-energy meson bootstrap problem is "closed" in yet another sense; namely, all particles appearing in the problem occur as intermediate states in some scattering processes and bootstrap one another. Thus whereas the pseudoscalar octet $0^-(8^+)$ occurs in the problem of Section II only as external particles and were not bootstrapped, here they occur also as bound states of the (ps, s) system.

Again, as in Section II, one may use Eq. (III.12) to estimate the masses of higher Regge recurrences in the "bound" channels. This yields in particular for the spin 2 "recurrences" the approximate values $m(2^\pm(1^+)) \sim 1 \text{ GeV}$, $m(2^\pm(8^+)) \sim 1.5 \text{ GeV}$.

We turn next to the question of the low-lying spin 0 singlets which represents a major difficulty in the present model. The $0^{\pm}(1^+)$ channels show strong attraction, stronger even than in the $0^{\pm}$ octets, thus yielding states of very low mass. It has been suggested that these states are connected with the Pomeranchuk ghost²⁷⁾ or with the possible antibound state π ²⁸⁾ in the $\pi - \pi$ system. Because of their unclear physical significance, we have avoided taking them into account in our actual calculations. However, their effects can be crudely estimated in case they exist as genuine particles, and we claim that they will not alter our qualitative predictions.

The forces due to 0^+ singlet exchange in octet-octet scattering has already been discussed in Ref. 14). Because of the relative smallness of the crossing matrix elements and the coupling constant, and the fact that the forces are attractive in all channels, we have argued there that our predictions will not be qualitatively altered. If the singlets are genuine "particles", they can also occur as external particles. The scattering of singlet from singlet contributes only to the singlet $0^{\pm}$ channel, and the force is attractive. Octet-singlet scattering has contributions to $0^{\pm}(8^+)$ and $1^{\pm}(8^+)$ states; the crossing matrix elements are such that the forces in the $0^{\pm}(8^+)$ channels are attractive and in the $1^{\pm}(8^+)$ repulsive. We claim therefore that the effect of the singlet mesons is just to bind the $0^{\pm}(1^+)$ and $0^{\pm}(8^+)$ supermultiplets tighter without altering qualitatively our predictions.

Finally, we remark that we have found a solution to the bootstrap equations without claiming that it is the only solution. However, arguments have been given¹⁵⁾ why we believe that the present solution is most likely the simplest non-trivial solution involving the least number of particles.

IV. EXTENSION TO HIGHER ENERGY

a) General

Although the solution to the bootstrap problem at low energy given in Section III is now logically complete, an extension of the model to higher energy is necessary if it is to be physically meaningful. This we attempt to do by means of the quasi-two-particle assumption D) as explained in the Introduction. We take this assumption only at a low phenomenological level and shall ignore the many theoretical problems it poses ²⁹⁾. The quasi-particles are taken as real and stable, thus giving rise to some computational difficulties connected with the Peierls singularities ³⁰⁾, which we have only artificially avoided, (IV.f).

We have considered up to now only the mutual scattering of $0^\pm(8^+)$ mesons. From the spectrum predicted in Section III, the next lowest threshold is connected with $0^\pm(8^+) - 1^\pm(8^-)$ scattering. Neglecting the $0^\pm$ singlets as in Section III, we are then led to an S matrix represented symbolically as in Fig. 8 ³¹⁾

$$S = \begin{pmatrix} \text{diagram 1} & \text{diagram 2} \\ \text{diagram 3} & \text{diagram 4} \end{pmatrix}$$

Fig. 8

Following our previous procedure, we should supply as input forces all possible one-particle exchange graphs with the spin of the exchanged particle $J \leq 2$. Ideally then we should again seek a solution to the bootstrap problem, self-consistent and "closed" in the sense explained before. The system of equations so obtained,

however, is enormously complicated; the independent trilinear couplings possible between the various mesons themselves number more than 20. A detailed solution of such a system is clearly impractical even if feasible with our very crude tools. It happens, however, that the system can be considerably simplified by using its symmetry properties. One is then able to analyze the forces in the various channels in fair detail, and with the help of some experimental information, make definite predictions concerning the meson spectrum in the intermediate energy range.

We note first that our present bootstrap system, in complete analogy to that of Section III, admits a solution with degenerate parity doublets. Explicitly, if in the input diagrams we impose the conditions (III.3) and analogous relations between other coupling constants (IV.6), it will be seen that the input forces in each $J(X^C)$ channel are independent of the parity P . This means that the bootstrap equations, and hence the "output" parameters, are also P independent, thus verifying the consistency of this parity-doubling ansatz. Again, as in Section III, we believe that this solution is the most natural one though not necessarily unique. By restricting our consideration to this case, the complexity of the problem is drastically reduced.

Next we note that in the decomposition into SU_3 channels, we have, in the scattering of $0^{\pm}(8^+)$ with $0^{\pm}(8^+)$

$$8^+ \times 8^+ = 1^+ + 8^+ + 8^- + 10 + \overline{10} + 27^+, \quad (\text{IV.1})$$

and in the scattering of $0^{\pm}(8^+)$ with $1^{\pm}(8^-)$:

$$8^+ \times 8^- = 1^- + 8^- + 8^+ + 10 + \overline{10} + 27^-, \quad (\text{IV.2})$$

(see Appendix A). Moreover, even J states in $0^{\pm}(8^+) \times 0^{\pm}(8^+)$ can

occur only in 1^+ , 8^+ , and 27^+ , while odd J states only in 8^- , 10 , and $\overline{10}$. By charge conjugation invariance therefore, of the various $J^P(X^C)$ states contained in (IV.1) and (IV.2), only the following can be coupled to both channels in Fig. 8

$$\begin{aligned} &0^\pm(8^+), 1^\pm(8^-), 2^\pm(8^+), \dots \\ &1^\pm(10), 1^\pm(\overline{10}), \dots \end{aligned} \tag{IV.3}$$

The others can only be coupled to either one or the other. Thus coupled to $0^\pm(8^+) \times 0^\pm(8^+)$, we have

$$0^\pm(1^+), 0^\pm(27^+), 2^\pm(1^+), 2^\pm(27^+), \dots ; \tag{IV.4}$$

and coupled to $0^\pm(8^+) \times 1^\pm(8^-)$, we have

$$\begin{aligned} &0^\pm(1^-), 0^\pm(8^-), 0^\pm(10), 0^\pm(\overline{10}), 0^\pm(27^-), \\ &1^\pm(1^-), 1^\pm(8^+), 1^\pm(27^-), \\ &2^\pm(1^-), 2^\pm(8^-), 2^\pm(10), 2^\pm(\overline{10}), 2^\pm(27^-), \dots \end{aligned} \tag{IV.5}$$

For the (IV.4) and (IV.5) channels, therefore, the S matrix of Fig. 8 reduces to only one diagonal block, and the forces can be analyzed in each block separately. This analysis for the states (IV.4) has already been performed in Section III.

For the remaining states (IV.3), we note a recent proposition of Bronzan and Low¹⁹⁾. These authors have suggested for mesons an empirical quantum number A to explain the suppression of certain modes of decay otherwise allowed by known selection rules. A parity was supposed to be violated by strong interactions to the extent of

a few percent in the rates. In this scheme, the pseudoscalar mesons π, K, η , namely our $0^-(8^+)$, are assigned $A = -$, while the vector mesons ρ, K^*, ϕ [our $1^-(8^-)$] are assigned $A = +$. (See also Section V.) This means that the initial and final states of the off-diagonal elements of the S matrix in Fig. 8 have opposite A parities. Approximate A conservation then implies that the coupling between the two channels even for the states (IV.3) has to be weak. Unfortunately, however, the situation is complicated by the current concept of $\phi - \omega$ mixing, which is in contradiction to the assignment of different A parities to the two particles¹⁹⁾. We shall, therefore, in what follows, keep open the possibility of a violation of the Bronzan-Low selection rule.

In view of the considerations given above we propose to analyze the forces in the various $J^P(X^C)$ channels as follows. We first study separately the forces in the diagonal blocks of Fig. 8, then consider the effect of a "weak", "A violating" coupling between the channels for the states (IV.3), (see IV.g)³²⁾.

b) Trilinear couplings between meson supermultiplets

Throughout this paper we shall encounter only one-particle exchange graphs so that only trilinear couplings will occur. Lorentz invariance and parity conservation admit the following types of coupling between the $0^\pm(8^+)$ and $1^\pm(8^-)$ supermultiplets

$$\begin{aligned}
 (ps, ps, V) &= (s, s, V) = (ps, s, pV) \\
 (V, V, V) &= (V, pV, pV) \\
 (V, V, pV) &= (pV, pV, pV) \\
 (ps, ps, s) &= (s, s, s) \\
 (ps, V, V) &= (ps, pV, pV) = (s, V, pV) \\
 (s, V, V) &= (s, pV, pV) = (ps, \bar{V}, pV).
 \end{aligned}
 \tag{IV.6}$$

These are put equal in groups as shown, because of the parity-doubling ansatz, (IV.a).

The forms of these couplings are further restricted by SU_3 and charge conjugation symmetries. For (ps, ps, V) there is only one independent coupling term, namely,

(ps, ps, V) :

$$\mathcal{L}_I = \sum_{abc} i g_1 F_{abc} (p_a - p_b)_\mu \tilde{\varphi}_a \tilde{\varphi}_b V_c^\mu,$$

(IV.7)

where, without loss of generality, we have employed the real meson field $\tilde{\varphi}_a$ and V_c^μ . The indices $a, b, c = 1, \dots, 8$ are SU_3 indices for the adjoint representation, while the constants F_{abc} are Clebsch-Gordan coefficients in the totally antisymmetric coupling $8 \times 8 \rightarrow 8$. The corresponding interaction Lagrangians of (s, s, V) and (ps, s, pV) are simply obtained by replacing the pseudoscalar fields $\tilde{\varphi}_a$ by the scalar fields φ_a , and the vector fields V_c^μ by the pseudovector fields $\tilde{V}_c^\mu$.

For (V, V, V) we have two independent terms in the interaction Lagrangian,

(V, V, V) :

$$\begin{aligned} \mathcal{L}_I = & \frac{1}{3} \sum_{abc} i g_3 F_{abc} (p_b^\nu V_{a\nu})(p_c^\lambda V_{b\lambda})(p_a^\mu V_{c\mu}) \\ & + \sum_{abc} (-i) g_4 F_{abc} (p_a - p_b)^\mu V_a^\nu V_{b\nu} V_{c\mu}, \end{aligned}$$

(IV.8)

while for the coupling (pV, pV, pV) , we have again only one

(pV, pV, pV) :

$$\mathcal{L}_I = \sum_{abc} i g_5 F_{abc} \varepsilon_{\mu\nu\rho\sigma} p_a^\mu p_b^\nu \tilde{V}_b^\rho \tilde{V}_c^\sigma (p_a^\lambda \tilde{V}_{a\lambda}).$$

(IV.9)

The remaining couplings (pV, pV, V) and (V, V, pV) are similar.

We note that all the couplings listed above are F couplings in SU_3 , as is necessary because of charge conjugation, (see Appendix A). Moreover, they are symmetric under the interchange of two similar particles, as is required by the Pauli principle.

Similarly, the remaining couplings in (IV.6) are necessarily D couplings, thus,

(ps, ps, s) :

$$\mathcal{L}_I = \sum_{abc} \mathcal{Y}_0 D_{abc} \tilde{\varphi}_a \tilde{\varphi}_b \varphi_c, \quad (IV.10)$$

(ps, V, V) :

$$\mathcal{L}_I = \sum_{abc} i \mathcal{Y}_2 D_{abc} \varepsilon_{\mu\nu\rho\sigma} (p_a^\mu V_a^\nu p_b^\rho V_b^\sigma) \tilde{\varphi}_c, \quad (IV.11)$$

(ps, V, pV) :

$$\begin{aligned} \mathcal{L}_I = & \sum_{abc} \mathcal{Y}_6 D_{abc} (V_a^\mu \tilde{V}_{b\mu}) \tilde{\varphi}_c \\ & + \sum_{abc} (-\mathcal{Y}_7) D_{abc} (p_a^\mu \tilde{V}_{b\mu}) (p_b^\nu V_{a\nu}) \tilde{\varphi}_c. \end{aligned} \quad (IV.12)$$

In a complete bootstrap model, all the coupling constants $\mathcal{Y}_0 - \mathcal{Y}_7$, are supposed to be determined by the requirement of self-consistency. We have not attempted to do this in the present paper but shall try instead to obtain information concerning these parameters either directly from experiment or otherwise.

The constant $\mathcal{Y}_1$ may be estimated from the decays $\rho \rightarrow 2\pi$ and $K^* \rightarrow K + \pi$, and yields a value of approximately $\mathcal{Y}_1^2/4\pi \sim 1.8$. The bootstrap calculation of Section III yields a value of $\mathcal{Y}_1^2/4\pi \sim 2.7$.

The constant $\mathcal{Y}_4$, being the "vector" coupling parameter between three vector mesons, is not at present estimable directly from experiment. It is, however, related to $\mathcal{Y}_1$ by the concept of "universality" as proposed by Sakurai¹⁸⁾ (extended to SU_3) on

the basis of the Yang-Mills theory ³³⁾. The difficulty of the non-zero meson mass in this proposal has been in part by-passed by its derivation via pole-dominant dispersion relations, as suggested, e.g., by Gell-Mann ⁶⁾, in analogy to the Goldberger-Treiman relation ⁶⁾. Since we are here also dealing with a pole-dominant model, there is good reason to expect that "universality" will remain valid ; namely in our notation $g_1 \approx g_4$. (We note in passing that "universality" is consistent with the parity-doubling ansatz.)

Concerning the "tensor" couplings of 3 vector (pseudovector) mesons, i.e., g_3 and g_5 , no experimental information is available. We shall try to estimate indirectly their rough orders of magnitude within the framework of our model using some experimental data.

The remaining couplings (IV.10) - (IV.12) are all D type and all "A violating", and thus, according to (IV.a) may be strongly suppressed.

If the suggestion $J^P = 0^+$ for the $K(725)$ be accepted ³⁴⁾, then because of its narrow width and the fact that it is then an s wave resonance, its coupling to $K\pi$, namely (ps, ps, s) in (IV.10), should be weak ¹⁹⁾. [The bootstrapped value obtained for this parameter in Section III was indeed very small, ~ 0.04 of (ps, ps, V).]

The coupling (V, V, ps), (IV.11), occurs in the decay $\phi \rightarrow \rho + \pi$. If we accept that ϕ is a pure octet meson, as is required by consistency with A conservation, then the absence of established $\phi \rightarrow 3\pi$ decay implies that the coupling (V, V, ps) is suppressed.

Nothing of course is known about the still highly speculative coupling terms (IV.12). However, if our present assignment of the B resonance ¹⁾ to the $1^+(8^-)$ octet be accepted (Table 1), this coupling should be related to the decay $B \rightarrow \phi + \pi$. No example of this decay mode has been reported, though it may just be due to the lack of phase space and of statistics.

It appears then that present experimental data are not in contradiction with the Bronzan-Low suppression of the D type couplings (IV.10) - (IV.12), though the issue needs clarification as concerns $\phi - \omega$ mixing, (IV.a). Considering ϕ and ω as mixed SU_3 states, the coupling constant U_2 (IV.11) may be estimated from the decay $\omega \rightarrow \rho + \pi \rightarrow 3\pi$ ³⁵⁾, using current values of the mixing angle. It shall be seen that the contributions of U_2 terms to our forces are in most cases negligible and do not alter our main conclusions, (IV.g).

c) The input diagrams

In accordance with assumption B) we insert as input forces all OPE graphs which are allowed by the A conserving couplings (IV.7) - (IV.9).

The lower diagonal block of (Fig. 8) represents the submatrix of (Fig. 9) ²⁴⁾ ;

Fig. 9

The only new OPE graphs to be computed are those listed in (Fig. 10) ²⁴⁾.

Fig. 10

These graphs may be expressed as functions of the square of the total c.m. energy s , the scattering angle θ , and the SU_3 indices of the incoming and outgoing particles. Because of the parity-doubling ansatz (III.3), (IV.6), it is readily seen that many of the diagrams have the same functional dependence on these variables; thus in this sense,

$$\begin{aligned} P(i) &= P(ii) = P(iii) ; P'(i) = P'(ii) = P'(iii) ; \\ Q(i) &= Q(ii) = Q(iii) ; Q'(i) = Q'(ii) = Q'(iii) ; \\ R(i) &= R(ii) = R(iii) = R(iv). \end{aligned} \quad (IV.13)$$

Moreover, apart from the over-all intrinsic parities of the scattering particles, the following diagrams are also kinematically identical :

$$P = P' ; Q = Q'. \quad (IV.14)$$

We exhibit functional dependence explicitly as follows :

$$\begin{aligned} (P) \quad & \sum_s F_{acs} F_{bds} \eta_1 \eta_3 \Phi_{(P1)}(s, \cos \theta) \\ & + \sum_s F_{acs} F_{bds} \eta_1 \eta_4 \Phi_{(P2)}(s, \cos \theta); \\ (Q) \quad & \sum_s F_{asd} F_{csb} \eta_1^2 \Phi_{(Q)}(s, \cos \theta); \\ (R) \quad & \sum_s F_{acs} F_{bds} \eta_1 \eta_5 \Phi_{(R)}(s, \cos \theta). \end{aligned} \quad (IV.15)$$

where $a(b)$ and $c(d)$ denote respectively the incoming and outgoing spin $0(1)$ mesons.

Because of the further degrees of freedom corresponding to the spins of the external vector mesons, the functions Φ are still 3×3 matrices. We express

$$\Phi = \Phi_{\lambda\mu} ; \quad \lambda, \mu = -1, 0, 1 ; \quad (\text{IV.16})$$

where λ and μ are the helicities of respectively the incoming and outgoing (pseudo-)vector mesons. The calculation of $\overline{\Phi}_{(P,Q,R)}$ in terms of s and $\cos\theta$ is straightforward and the results are given in the Appendix B.

Because of (IV.13), it is convenient to write the input diagrams, (Fig. 9), in the form of a direct product :

$$N = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} (P)+(Q) & (R) \\ (R) & (P')+(Q') \end{pmatrix}. \quad (\text{IV.17})$$

The first matrix on the right-hand side represents the parity doubling, its rows and columns referring to two-particle states of the same intrinsic parity, i.e., either to $|ps, V\rangle$ and $|s, pV\rangle$, or to $|ps, pV\rangle$ and $|s, V\rangle$. The rows and columns of the second matrix, however, refer to two-particle states of opposite intrinsic parities, i.e., either to $|ps, V\rangle$ and $|ps, pV\rangle$, or to $|s, V\rangle$ and $|s, pV\rangle$. This is the same notation as that employed in (III.4).

a) The N function in the channel $J^P(X^C)$

We start by diagonalizing the first matrix in the product (IV.17),

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{IV.18})$$

thus in effect replacing the matrix by a numerical factor 2.

Next we define helicity amplitudes for fixed J as follows ³⁶⁾

$$M_{\lambda\mu}^J = 2\pi \int_0^\pi d\theta \sin\theta (\theta 0\mu | M | 00\lambda) d_{\lambda\mu}^J(\theta) \quad (\text{IV.19})$$

where $d_{\lambda\mu}^J$ are rotation matrices as defined by Jacob and Wick ³⁶⁾. States with definite helicities are in general not eigenstates of parity. Introduce then the following combinations as basis :

$$\begin{aligned} \psi_1 &= \frac{1}{\sqrt{2}} \{ |\lambda = +1\rangle + |\lambda = -1\rangle \} ; \quad P \psi_1 = (-1)^{J+1} (-1)^\zeta \psi_1 ; \\ \psi_2 &= \frac{1}{\sqrt{2}} \{ |\lambda = +1\rangle - |\lambda = -1\rangle \} ; \quad P \psi_2 = (-1)^J (-1)^\zeta \psi_2 ; \\ \psi_3 &= |\lambda = 0\rangle ; \quad P \psi_3 = (-1)^{J+1} (-1)^\zeta \psi_3 ; \end{aligned} \quad (\text{IV.20})$$

ζ being the total intrinsic parity of the mesons. These states will then diagonalize the amplitude in both J and P .

The diagonalization of (IV.17) with respect to SU_3 representations is also straightforward, involving crossing matrix elements which can be obtained for example, from Ref. ¹⁴⁾ (Section V) by trivial re-ordering of indices.

The resulting N functions for the channel $J(X^C)$ are the same for both parities apart from a trivial ordering of the matrix elements, and can be written as follows

$$N^{JP(X^C)} = 2 \times \begin{pmatrix} (P)_-^{J(X^C)} + (Q)_-^{J(X^C)} & (R)_+^{J(X^C)} & \sqrt{2} (R)_+^{J(X^C)} \\ (R)_-^{J(X^C)} & [(P)_+^{J(X^C)} + (Q)_+^{J(X^C)}] & \sqrt{2} [(P)_{1,0}^{J(X^C)} + (Q)_{1,0}^{J(X^C)}] \\ \sqrt{2} (R')_{0,1}^{J(X^C)} & \sqrt{2} [(P)_{1,0}^{J(X^C)} + (Q)_{1,0}^{J(X^C)}] & [(P)_{0,0}^{J(X^C)} + (Q)_{0,0}^{J(X^C)}] \end{pmatrix} \quad (\text{IV.21})$$

For $J = 0$, this formula reduces for obvious kinematical reasons

$$N^{OP(XC)} = 2 \times [(P)_{0,0}^{\sigma} + (Q)_{0,0}^{\sigma}] \quad (\text{IV.22})$$

In (IV.21) and (IV.22)

$$(P)^{J(XC)} = \lambda^{(XC)} \eta_1 \eta_3 \Phi_{(P1)}^J(s) + \lambda^{(XC)} \eta_1 \eta_4 \Phi_{(P2)}^J(s),$$

$$(Q)^{J(XC)} = \lambda^{(XC)} \eta_1^2 \Phi_{(Q)}^J(s),$$

$$(R)^{J(XC)} = \lambda^{(XC)} \eta_1 \eta_5 \Phi_{(R)}^J(s),$$
(IV.23)

and

$$(M)_{\pm}^J = (M)_{1,1}^J \pm (M)_{1,-1}^J \quad (\text{IV.24})$$

for $(M) = (P), (Q)$, or (R) . The functions $\Phi^J(s)$ are obtained by (IV.19) from the functions $\Phi(s, \cos\theta)$ given in Appendix B, while the crossing matrix elements $\lambda^{(XC)}$ in our convention take the following values

$$\lambda^{(1-)} = 1, \lambda^{(8-)} = \frac{1}{2}, \lambda^{(8+)} = \frac{1}{2}, \lambda^{(10)} = \lambda^{(\overline{10})} = 0, \lambda^{(27-)} = -\frac{1}{3};$$

$$\lambda'^{(1-)} = -1, \lambda'^{(8-)} = -\frac{1}{2}, \lambda'^{(8+)} = \frac{1}{2}, \lambda'^{(10)} = \lambda'^{(\overline{10})} = 0, \lambda'^{(27-)} = \frac{1}{3}.$$
(IV.25)

e) The D function in the channel $J^P(X^C)$

In our normalization convention, (Appendix B), unitarity gives

$$D^{J(X^C)}(s) = 1 + \frac{1}{32\pi^3} \int_{s_0}^{\infty} \frac{k'}{\sqrt{s'}} \frac{N^{J(X^C)}(s') ds'}{s' - s} \quad (\text{IV.26})$$

Following Zachariasen and Zemach ¹¹⁾, we subtract the integral once to normalize D on the left-hand cut. The once-subtracted integral is still divergent for most graphs and an additional cut-off is necessary. This makes any prediction on absolute magnitudes dubious. However, since the same cut-off and subtraction parameters are used for all graphs and all channels, we believe that a comparison of the relative strengths of the forces is still meaningful. For this reason, an exponential cut-off is preferred to a sharp one, since it allows us to compare the forces in the various channels over a wider energy range, while giving the dominant weight to the low-energy (long-range) region, in accordance with assumption B).

The formula (IV.26) is then modified to read

$$D^{J(X^C)}(s) = 1 + \frac{s - s_1}{32\pi^3} \int_{s_0}^{\infty} \frac{k'}{\sqrt{s'}} \frac{N^{J(X^C)}(s') e^{-s'/\bar{s}} ds'}{(s' - s)(s' - s_1)} \quad (\text{IV.27})$$

where the second term may be obtained by replacing in (IV.21) - (IV.23) the symbols $\Phi_{(M)}^J$ for $(M) = (P), (Q),$ or (R) by the corresponding $\alpha_{(M)}^J$, where

$$\alpha^J(s) = \frac{s - s_1}{32\pi^3} \int_{(m+\mu)^2}^{\infty} ds' \frac{k'}{\sqrt{s'}} \frac{\Phi^J(s') e^{-s'/\bar{s}}}{(s' - s)(s' - s_1)} \quad (\text{IV.28})$$

For $s > (m+\mu)^2$, we take for $\alpha_{(M)}^J$ the principal value of the integral.

A practical difficulty occurs in the evaluation of the D integral for graph (Q) where the left-hand cut overlaps with the right giving rise to singularities in the physical region, (the Peierls singularity³⁰⁾). The difficulty, however, is artificial, having been introduced, by the approximation D), in treating the unstable spin 1 mesons as stable. For the actual 3 particle amplitudes, to which graphs like (P), (Q) and (R) are meant merely as approximations, the Peierls singularities are separated from the physical region by another cut³⁰⁾. We simply avoid this difficulty by displacing the exchanged meson mass. The smallness of the displacement required, and the knowledge of the weak dependence on the exchanged mass of the D integral³⁷⁾, makes it unlikely that substantial alterations will be made to the crude picture for which the model is intended.

The integrals α^J for $J = 0, 1, 2$ were evaluated numerically on a computer, and the results for $J = 0, 1$ are presented in Appendix C, (Fig. 11). The values shown were calculated with $m = 1$ (mass scale) and $\mu = 0.45$, in accordance with (II.9). Following Zachariasen and Zemach, the subtraction point s_1 is chosen near the beginning of the left-hand cut for spin 1 meson exchange, $s_1 = 0.7$ ³⁸⁾. The cut-off is arbitrarily fixed at $\bar{s} = 7$, which on our scale corresponds to ~ 2.5 nucleon masses. For graph (Q), as explained above, a higher mass μ_e is inserted for the exchanged meson, so that $m < \mu + \mu_e$. For the graphs shown, (Fig. 11), $\mu_e = 0.65$.

We note that $\text{Re } \alpha_{(P2)}^1$ for the helicity amplitudes $(0,0)$, $(1,0)$ and $(+)$, (Fig. 11b,c,e), have a cusp-like behaviour at threshold. These helicity combinations for $J = 1$ correspond to a 1^+ state in a $0^- - 1^-$ system. They thus contain an s wave component which is known to have such a behaviour.

f) Analysis of the forces

In our simplified dynamics, the criterion for the occurrence of "particles" is just the vanishing of the determinant of the approximate D function. This depends on the signs and magnitudes of the integrals (IV.27) which may thus conveniently be termed the "forces". The total force is attractive (repulsive) in channel $J^P(X^C)$ if $\det D^{JP(X^C)} < 1$ (> 1).

We note first that in all the graphs (Fig. 11), the integrals $\alpha_{(P2)}^J$ are easily the largest, with $\alpha_{(Q)}^J$, much smaller, second. Except for $J = 0$, the values of $\alpha_{(P1)}^J$ are in general quite insignificant. The terms $\alpha_{(R)}^J$ in (IV.27) which are proportional to U_5 , occur only for $J > 0$ and only in the off-diagonal elements of the D matrix. They thus affect the total forces in these channels indirectly. Hence, unless the coupling constant U_3 happens to be much larger than U_1 and U_4 , the forces (IV.27) in most channels are dominated by the graphs (P2) and (Q). We shall show that a large value for U_3 is in fact unlikely, at least within the framework of our model. Then since U_1 is well known from P decay, and U_4 related to it by "universality", one is able to make reasonable estimates of the various forces without detailed knowledge of U_3 or U_5 .

For the $J = 0$ states the forces are particularly easy to analyze. Consider first only (P1) and (Q). Reading the crossing matrix elements λ, λ' from (IV.25), we note that the states $0^\pm(8^+)$ are enhanced since here both (P1) and (Q) are attractive, whereas in all other (X^C) channels the forces oppose each other and tend to cancel. However, in the singlet $(X^C = 1^-)$, because of the largeness of the crossing matrix element, the total force is still attractive enough to bind a low-lying resonance, unless it is cancelled by the (P1) term. Since no $0^\pm$ states are known experimentally with

abnormal charge conjugation ($C = -$) up to energies of ~ 1.5 GeV, we shall use this information to determine $\mathcal{U}_3$ in such a way as to cancel the attraction in the ($X^C = 1^-$) channel. The force (P1) in this state is attractive or repulsive according as $\mathcal{U}_3 > 0$ or < 0 . However, $\mathcal{U}_3$ cannot be too negative otherwise the attraction due to (P1) will bind $0^\pm$ resonances in the channel $X^C = 27^-$, which at low energies have not been observed³⁹). These limits for $\mathcal{U}_3$, though crude, are already sufficient for our purpose since the contribution of (P1) are insignificant for $J > 0$, provided $\mathcal{U}_3$ is of the same order as $\mathcal{U}_1$ and $\mathcal{U}_4$. In Fig. 12a, we have plotted the D function for $0^\pm(X^C)$ with $\mathcal{U}_1 = \mathcal{U}_4$, $\mathcal{U}_1^2/4\pi = 1.8$, and $\mathcal{U}_3$ chosen just sufficiently negative to cancel the attraction in the ($X^C = 1^-$) channel up to threshold, namely $\mathcal{U}_1 \mathcal{U}_3/4\pi = 1.05$. It is noticed then that the forces in the $0^\pm$ channels almost all cancel except for the $0^\pm(8^+)$ which shows the enhanced attraction already noted.

For the $J = 1$ states, the forces, being represented by a determinant, are more complicated to analyze. However, their qualitative features can be seen by studying merely the diagonal elements. The graphs (Fig. 11b-e) are entirely dominated by the integrals $\alpha_{(P2)}^J$ which alone already give a fair indication of the nature of the forces. The crossing matrix elements λ (IV.25), show that the (P2) force is attractive in $X^C = 1^-, 8^-, 8^+$, and that it is "twice" as attractive in (1^-) as in (8^-) and (8^+); the same force in (27^-) and (10), ($\overline{10}$) is repulsive or zero. The determinant of $D^{1^\pm(X^C)}$, however, depends on the unknown parameter $\mathcal{U}_5$ in the off-diagonal element, graph (R). The det D's are independent of the sign of $\mathcal{U}_5$, and their nature (attractive or repulsive) and relative sizes are insensitive to its magnitude within reasonable limits. However, variations of $\mathcal{U}_5$ do alter their energy dependence. As seen in (Fig. 11), the integrals $\alpha^J(s)$ all have a near

linear behaviour at low energy. Extreme values of $\mathcal{U}_5$ give rise to a "quadratic" behaviour and double zeros of $\det D$ which means in our scheme more than one particle with the same quantum number. We have chosen in Fig. 12b, $\mathcal{U}_5$ in such a way that $\det D_{1^\pm(1^-)}$ is approximately linear up to threshold, thus avoiding this possibility. The actual value employed is $\mathcal{U}_1, \mathcal{U}_5/4\pi = 5.2$. We note (Fig. 12b) an enormous attraction in $1^\pm(1^-)$ and less conspicuous attractions in $1^\pm(8^\pm)$. Forces in other channels are not significant.

As already noted in the previous Section, the "forces" in the $J = 1$ states have cusps at threshold. This represents the lack of a centrifugal barrier in the s wave component which disallows the existence of resonances above threshold without additional repulsive forces. In our context, therefore, $\det D$ has no meaning above threshold for $J = 1$ and is not plotted. In fact even below threshold, the cusp induces such a rapid rise in the integral α^J near threshold that it is doubtful whether they have any meaning there when interpreted as forces.

In Fig. 12c, we have plotted $\det D$ for the $J = 2$ states, using values of $\mathcal{U}_3$ and $\mathcal{U}_5$ estimated above. Again, one notices an enhancement of the $2^\pm(8^+)$ channel, the forces in all other channels being insignificant at energies we are considering.

To sum up, our analysis gives :

- i) a very conspicuous attraction in the $1^\pm(1^-)$;
- ii) weaker attractions in the $0^\pm(8^+)$, $1^\pm(8^-)$, $2^\pm(8^+)$, and $1^\pm(8^+)$ states ;
- iii) forces in all other channels occurring being insignificant at energies under consideration.

Whether the attractive forces are strong enough to "bind" resonances depends on the absolute values of the integrals $\alpha^J(s)$ and hence

on the cut-off. For our present value corresponding to a cut-off of 2.5 nucleon masses, which seems reasonable, the only channel likely to "bind" a low-lying resonance is the $1^{\pm}(1^{-})$ singlet. The channels $0^{\pm}(8^{+})$, $1^{\pm}(8^{-})$ and $2^{\pm}(8^{+})$ are all coupled to $0^{\pm} - 0^{\pm}$ scattering, namely to the upper diagonal block of Fig. 8. They were already found there to give resonances, Section III. Besides $1^{\pm}(1^{-})$, therefore, the only candidate for binding possible new resonances here is $1^{\pm}(8^{+})$. However, because of the cusp behaviour of the D function at threshold and the uncertain significance of the cut-off, we hesitate to make a prediction in this channel.

g) Possible corrections to the forces

We shall discuss here qualitatively two possible corrections to the forces analyzed in (IV.f), namely i) the possible violation of the A selection rule, and ii) the effects of the $1^{\pm}(1^{-})$ singlets suggested in (IV.f).

A violating couplings (IV.10) - (IV.12) can contribute to the forces in our model in two ways. Firstly, they may contribute to the OPE forces in (Fig. 9). Thus from the η_2 coupling we may have the graph (W), (Fig. 13), and other graphs related to it by parity-doubling.

Fig. 13

$$(W); \sum_s D_{ads} D_{cbs} \eta_2^2 \Phi_{(W)}(s, \cos \theta), \quad (IV.29)$$

where $\Phi_{(W)}(s, \cos \theta)$ is given in Appendix B. Its contribution to the D function

$$I_{(W)}^{J(xc)}(s) = K'^{(xc)} \eta_2^2 \alpha_{(W)}^J(s) \quad (IV.30)$$

are to be added to those from (P) and (Q) in (IV.27) and (IV.28) with the following crossing matrix elements

$$\begin{aligned} K'^{(1-)} &= 1, \quad K'^{(8-)} = -3/10, \quad K'^{(8+)} = -1/2, \\ K'^{(10)} &= K'^{(\overline{10})} = 2/5, \quad K'^{(27-)} = 1/5. \end{aligned} \quad (IV.31)$$

The integrals $\alpha_{(W)}^J$ were plotted in Fig. 11.

Assuming $\theta \neq 3\pi$ and a mixing angle of 30° , an estimation from $\omega \rightarrow \rho + \pi \rightarrow 3\pi$ decay³⁵⁾ yields a value $\eta_2^2/4\pi \approx 4.6$. The contributions of (W) with this value of η_2^2 were already included in Fig. 12. Because of the smallness of $\alpha_{(W)}^J$ in all channels, the difference due to their inclusion is unimportant.

Our assignment of $K(725)$ to the 0^+ octet implies that (IV.10) is negligibly small, (see IV.b). For the other coupling (IV.12) present information is insufficient to give an estimate of its effects.

The second way the A violating terms can contribute is via the off-diagonal elements of Fig. 8. As pointed out in (IV.a), only the states (IV.3) are affected. Unfortunately we cannot calculate quantitatively the effects due to this coupling of the channels since

they involve the unknown terms (IV.12). However, since $0^\pm(8^+)$, $1^\pm(8^-)$ and $2^\pm(8^+)$ were already bound in $0^\pm - 0^\pm$ scattering, Section III, and have attractive forces also in $0^\pm - 1^\pm$, (IV.f), a coupling between the channels will only help to bind these states further. For the states $1^\pm(10)$ and $(\overline{10})$, the forces in either channel are in any case insignificant.

We turn next to the forces due to the $1^\pm(1^-)$ singlets. The only trilinear couplings involving the supermultiplets $1^\pm(1^-)$, $0^\pm(8^+)$ and $1^\pm(8^-)$ allowed by SU_3 , Lorentz and charge conjugation invariance are the following, with their corresponding parity-doubling terms :

(ps, V, V) :

$$\mathcal{L}_I = \sum_{ab} i \mathcal{K}_1 D_{ab} \varepsilon_{\mu\nu\rho\sigma} (p_a^\mu V_a^\nu p^\rho V^\sigma) \tilde{\varphi}_b \quad (\text{IV.32})$$

(ps, V, pV) :

$$\begin{aligned} \mathcal{L}_I = & \sum_{ab} \mathcal{K}_2 D_{ab} (V_a^\mu \tilde{V}_\mu) \tilde{\varphi}_b \\ & + \sum_{ab} (-\mathcal{K}_3) D_{ab} (p_a^\mu \tilde{V}_\mu)(p^\nu V_{\nu}) \tilde{\varphi}_b. \end{aligned} \quad (\text{IV.33})$$

$V^\mu, \tilde{V}^\mu$ (without index a) represent the singlets, and D_{ab} are Clebsch-Gordan coefficients in the coupling $8 \times 8 \rightarrow 1$. These (A conserving) couplings are kinematically the same as the couplings (IV.11) and (IV.12). The coupling constant $\mathcal{K}_1$ may be estimated from $\omega \rightarrow \rho + \pi \rightarrow 3\pi$ decay, and according as whether we have mixing or not, this gives $\mathcal{K}_1^2/4\pi \approx 9.2$ or 13.8 ³⁵⁾. Further, if our tentative assignments of B and H in Table 1, Section V, be accepted, the constants $\mathcal{K}_2$ and $\mathcal{K}_3$ may be estimated from the decays $B \rightarrow \omega + \pi$ and $H \rightarrow \rho + \pi$. We shall consider here, however, only (IV.32).

The forces due to the singlets $1^{\pm}(1^{-})$ may enter in two ways. Firstly, the singlet mesons may be exchanged in Fig. 9 giving rise to a diagram similar to (W), (Fig. 13). Indeed, if the masses of the singlets and of the $1^{\pm}(8^{-})$ octets are equal, the contributions of the singlet exchange graph (W1) differ from (W) only in the crossing matrix elements. Thus

$$I_{(W1)}^{J(xc)} = \sigma'_{(xc)} \mathcal{H}_1^2 \alpha_{(w)}^J(s) \quad (\text{IV.34})$$

where

$$\begin{aligned} \sigma'_{(1-)} &= 1/8, \quad \sigma'_{(8-)} = 1/8, \quad \sigma'_{(8+)} = -1/8, \\ \sigma'_{(10)} &= \sigma'_{(\overline{10})} = -1/8, \quad \sigma'_{(27-)} = 1/8. \end{aligned} \quad (\text{IV.35})$$

Now although $\mathcal{H}_1^2$ in (IV.34) is 2 to 3 times larger than $\mathcal{H}_2^2$ in (IV.30), the crossing matrix elements are only $\sim 1/4$ as large. Hence the effects of (W1) are only of the same order as that of (W), which we have already found to be unimportant.

The second way in which the $1^{\pm}$ singlets can effect the forces is via their scattering with the $0^{\pm}(8^{+})$ octets. Because of our ignorance of $\mathcal{H}_2$ and $\mathcal{H}_3$ the only diagram which at present can be analyzed is (W2) in Fig. 14.

Fig. 14

Again, for equal masses of $1^-(1^-)$ and $1^-(8^-)$, this graph is kinematically identical to (W) apart from the crossing matrix element. Its only contribution is to the $X^C = 8^-$ state, giving

$$I_{(W2)}(s) = \mathcal{K}_1^2 \alpha_{(W)}^J(s) \quad (\text{IV.36})$$

Reading $\alpha_{(W)}^J$ from Fig. 11 we find that $I_{(W2)}$ for $J = 0$ is repulsive and small. For $J = 1$, it is stronger and attractive; while for $J = 2$, (not shown), it is again negligible. The net effect is thus some extra attraction in the $1^+(8^-)$ channel, which was already found to bind resonances in other scattering channels, (Section III).

It appears then that all the corrections which one can calculate at the present state of knowledge will not materially affect the analysis of the previous Section.

V. RESULTS AND DISCUSSION

a) The predicted spectrum and tentative assignments of existing resonances

The meson supermultiplets predicted by the present model are listed together in Table 1. The resulting spectrum shows some surprising regularities ; beside the parity doubling already noted, we have a singlet and an octet of mesons with the same charge conjugation properties in each J^P state, giving it the appearance of a rotation spectrum. It also entails the feature of "dynamical octet enhancement" envisaged by various authors in that only the singlet and octet channels in each J^P state are strongly bound.

Many of the particles listed here have also been suggested by other authors on various grounds. It is particularly interesting to compare our spectrum with the representations of some recently proposed symmetry schemes. In the approximate $U_3 \times U_3$ symmetry suggested by Gell-Mann and others ²²⁾, mesons are required to belong to super-supermultiplets containing particles of both parities. The pseudoscalar octet has been assigned by Gell-Mann ²²⁾ to the $(3^*, 3) - (3, 3^*)$ representation in order to satisfy the Goldberger-Treiman relation in the symmetric limit. This requires exactly the 18 spin 0 mesons with the charge conjugation properties listed in Table 1. The 2^+ , 2^- resonances suggested by our model, being Regge recurrences of the 0^+ , 0^- mesons, presumably have the same assignment in the $U_3 \times U_3$ scheme. Similarly the 18 spin 1 mesons listed may be accommodated in a $(3^*, 3) - (3, 3^*)$ super-supermultiplet. Gell-Mann however, has chosen the more natural assignment, $(1, 8) - (8, 1)$ for the spin 1 mesons ²²⁾, namely the same as that for the currents. The pseudovector octet of his assignment has then the opposite charge conjugation parity to that suggested here.

In an extension of the symmetry scheme to $U_6 \times U_6^{23)}$, the mesons may belong either to the representation of the currents, i.e., $(1,35) - (35,1)$, or to the representation $(6^*,6) - (6,6^*)$. The $(6^*,6) - (6,6^*)$ supermultiplet contains nine spin 0 and nine spin 1 mesons for each parity, whose SU_3 contents and charge conjugation properties are exactly those of the mesons listed in our Table 1.

Although the spectrum predicted by our model seems thus in some way related to the symmetry schemes $U_3 \times U_3$ and $U_6 \times U_6$, it is by no means claimed that the two approaches are equivalent. In fact, the dynamics in our model, which govern the masses of the spin 0 and spin 1 mesons, and the SU_3 octets and singlets, are quite different, so that these supermultiplets here are not, and have no reason to be degenerate. Nor are the coupling constants between the supermultiplets related in a way required by the supersymmetry schemes.

In a previous paper ²⁰⁾ we have attempted to fit the known mesonic resonances into our scheme, (Table 1). In this connection it should be noted that this model differs from pure symmetry schemes in one important aspect. Namely, since definite dynamical assumptions have been made it allows us not only to predict the existence of certain supermultiplets but also to exclude the existence of others. Although our dynamics are not free from ambiguity, the existence or otherwise of a meson supermultiplet is not arbitrary. The contention of the model in its present form is thus that only those supermultiplets listed in Table 1 should exist at "low" energy ⁴⁰⁾.

Supermultiplet	Assignments	A	Supermultiplet	Assignments	A
$0^-(1^+)$	$X_0 ?$	+	$0^+(1^+)$	$\sigma ?$	+
$0^-(8^+)$	π, K, η	-	$0^+(8^+)$	$\kappa(725),$	-
$1^-(1^-)$	ω	-	$1^+(1^-)$	H	-
$1^-(8^-)$	ρ, K^*, ϕ	+	$1^+(8^-)$	B,	+
$2^-(1^+)$	$X_0 ?$	+	$2^+(1^+)$	f_0	+
$2^-(8^+)$	A_1, C, E	-	$2^+(8^+)$	$A_2,$	-

Table 1

Reasons for the assignment in Table 1 have already been presented ²⁰⁾. Since then a further $K - \pi$ enhancement (1420 MeV) has been reported for which the quantum numbers $I = \frac{1}{2}$, $J^P = 1^-, 2^+$ slightly favouring 2^+ have been suggested ⁴¹⁾. This would fit well into the $2^+(8^+)$ supermultiplet. It may be seen that all reasonably well-established mesonic enhancements reported to date fit comfortably into the scheme of Table 1 except for the $K^+ - \pi^+ - \pi^0$ at 1270 MeV ²¹⁾. We note also the existence of enhancements near threshold in both the $\pi - \pi$ (σ) ²⁸⁾ and $K - \bar{K}$ systems in the $J = 0$, $I = 0$ state. We shall return later to a discussion of these elusive objects.

The present assignment of existing resonances into the scheme has some further interesting regularities :

- i) the singlets have lower masses than the octets for the same J^P ;
- ii) for the same $J(X^C)$, the particles with $P = +$ have higher masses than those $P = -$;
- iii) within an octet where all members are known, we have $m(I = 1) < m(I = \frac{1}{2}) < m(I = 0)$. Where only one or two members are known in a supermultiplet they are $I = 1, I = \frac{1}{2}$ in that order, so that the unknown members may be assumed to have higher masses, consistent with the mass inequalities above. An exception is the $0^+(8^+)$ octet.

The regularity i) was in most cases imposed by the choice of assignment to agree with the dynamics of the model, which requires that the singlets be bound tighter than the octets.

The parity doublets play a symmetric role so far in our dynamical scheme and may thus always be obtained with equal masses as a solution to the bootstrap equations. In fact this symmetry is seen to be much more general and is likely to persist in any reasonable self-consistent model involving mesons alone. The symmetry however, will be broken

when baryons are taken into account. Parity-doublet splitting in our model may thus be considered as an effect of virtual baryon-pair production. In this connection the reader is reminded of the parallel situation in the $U_3 \times U_3$ scheme ²²⁾. The degeneracy of parity-doublets persists there only so long as the baryon mass term is neglected. Taking it into account, the 0^+ octet for example, acquires a higher mass than the 0^- . It is conceivable that in our model also a similar splitting may occur, thus explaining the observed regularity ii). A more detailed analysis of this point is, however, wanting.

As for SU_3 symmetry itself, no really convincing mechanism for spontaneous break-down in a self-consistent model has yet been invented. Semi-phenomenological calculations, however, have been done ⁴²⁾, whereby the splitting of the 1^- octet was obtained as a consequence of that of the 0^- by consistency requirements. These results, together with the symmetric role of parity doublets in our model and the fact that the $2^\pm$ resonances are just "Regge" recurrences of the spin 0 mesons, then afford a partial explanation of the mass regularity iii) observed.

We turn next to the empty spaces in Table 1. In the supermultiplets $1^+(8^-)$ and $2^+(8^+)$, only the $I = 1$ members have been assigned to existing resonances. No enhancements, [except possibly $K - \pi$ (1420)] ⁴¹⁾, which may correspond to the other isospin members have yet been reported. If we accept the regularity iii) as genuine, however, this may be due to their having masses above the B and A_2 resonances, say $\gtrsim 1.5$ GeV.

The other empty spaces are in the 0^+ supermultiplets. The absence of well-established low-lying scalar mesons is a major difficulty not only in our model but also in such symmetry schemes as $U_3 \times U_3$ ²²⁾ or $U_6 \times U_6$ ²³⁾. A possible explanation of this, however, exists within the present dynamical framework. In the degenerate calculations of Section III, the 0^+ mesons appear as poles

below the $0^- - 0^-$ threshold, i.e., as genuine bound states, which is of course experimentally unacceptable. Presumably, therefore, when symmetry-breaking interactions are switched on, these poles will move above some $0^- - 0^-$ thresholds and become unstable. If there is no additional long-range repulsion, then the absence of a centrifugal barrier in the s state implies that these poles cannot exist as resonances with non-zero lifetime. They will appear as antibound state poles on the second sheet below the physical threshold, corresponding experimentally to large scattering lengths.

Now in the present model the forces mainly responsible for binding the 0^+ "particles" are those due to the exchange of the spin 1 octets, with a range of roughly $1/m_1$. The exchange forces of the spin 0 octets themselves are weaker because of the smaller coupling constant, but have a longer range, namely $\sim 1/m_0$. The SU_3 crossing matrix elements are such that the latter forces are repulsive in the $0^+(8^+)$ channel, but are again attractive in the $0^+(1^+)$. Since all forces in the $0^+(1^+)$ channel with a range $\gtrsim 1/m_1$ are attractive, there is no possibility of a true resonance. Any pole in this channel must be either a bound or antibound state. In the latter case, it could well correspond to the elusive σ (ABC) near the $\pi - \pi$ threshold¹⁸⁾. In the $0^+(8^+)$ channel, however, the weak long-range repulsion due to spin 0 octet exchange may act in the same way as a centrifugal barrier. Thus, depending on the strength and range of this repulsion, the poles may appear either as true resonances, or as threshold enhancements like the σ . It is conceivable that the $K(725)$ be an example of the first case and the $I = 0$, $K_0\bar{K}_0$ threshold enhancement an example of the second. Another possible candidate for the $I = 0$ member of the 0^+ octet is the ϵ_0 enhancement in $\pi^0\pi^0$ and $\pi^+\pi^-$ at ~ 700 MeV, for the existence of which there is growing evidence⁴³⁾. These assignments then leave only the 0^+ $I = 1$ multiplet, for the apparent absence of which we have no explanation⁴⁴⁾.

Finally we note the apparent regularity of the spectrum in Table 1 as regards the empirical quantum number A ⁴⁵⁾.

b) The dynamical assumptions

In this Section we discuss briefly our dynamical assumptions A), B), C), and Z), in connection with our results obtained so far, and with other related topics.

In Section I, we have asserted A) that it is a fair approximation at low energies to consider only processes with "low" thresholds. In actual calculations we have attempted to consider all processes with thresholds $\lesssim 1.5$ GeV. At such energies there is good reason to believe that absorption into other channels plays an essential role. Peripheral model calculations ⁷⁾ of meson production at medium energies (e.g., $K^+p \rightarrow K^{*+}p$ at 4 GeV/c) show that lower partial waves, $J \lesssim 4$, have to be strongly absorbed in order to reproduce the sharp forward peaking. These results at energies corresponding to a c.m. momentum $k \approx 1$ GeV, give us a picture of the nucleon as a grey absorbing disc with a radius of $\sim 1/2 m_\pi$. An analysis of the asymptotic diffraction peak ⁴⁶⁾ gives a similar radius of absorption for the nucleon. In our model the scattering of $0^\pm(8^+)$ mesons at 1.5 GeV corresponds to a c.m. momentum of $k \approx 0.6$ GeV. If one takes the same absorption radius for the meson as for the nucleon, i.e., $1/2 m_\pi$, large absorption effects are then expected for all $l/k \lesssim 1/2 m_\pi$, namely for $l \leq 2$. This is indeed well illustrated by the following sequence of calculated values for the coupling constant g_1 :

$$\begin{aligned} g_1^2/4\pi &= 8.1 \text{ for } \rho \text{ bootstrap in } \pi - \pi \text{ scattering,} \\ &= 5.4 \text{ for } 1^-(8^-) \text{ bootstrap in the scattering of } 0^-(8^+) \\ &\quad \text{octets, Section II,} \\ &= 2.7 \text{ for } 1^\pm(8^-) \text{ bootstrap in the scattering of } 0^\pm(8^+) \\ &\quad \text{octets, Section III.} \end{aligned}$$

The experimental value estimated from ρ decay is $g_1^2/4\pi \sim 1.8$. It is noted that each time more channels are included (more absorption), the calculated value of $g_1^2/4\pi$ is improved. Similar effects of inelastic channels in bootstrap problems have already been noticed by other authors ⁴⁷⁾.

Throughout our paper we have assumed B) that the binding of "particles" is effected mainly by long and medium range forces. In our calculations, we have attempted to include all OPE diagrams with exchanged mass $\lesssim 1$ GeV. Our conclusions therefore, are expected to hold only so long as $\ell/k \gg 1 \text{ GeV}^{-1}$. For the s wave state, because of the absence of a centrifugal potential, the wave function reaches into the core region so that our conclusions here must be taken particularly cautiously. Thus the scalar meson mass and coupling constant were found to be more sensitive to the subtraction point ¹⁴⁾, as expected. Our spin 1 mesons are "bound" in the region 1 GeV, corresponding in $0^+(8^+)$ octets scattering to $k \lesssim 0.5$, while our spin 2 mesons have mass ~ 1.5 GeV corresponding to $k \lesssim 0.75$ GeV, and are thus both well within the limit, $\ell/k \gg 1 \text{ GeV}^{-1}$.

We have in this scheme introduced SU_3 as a group external to dynamics and have deliberately avoided the question whether such higher symmetries have a dynamical origin ⁹⁾. Without a deeper understanding of their real significance in strong interactions, the assumption C) and "parity doubling" in our scheme should be regarded as dynamical approximations which, though crude, allow us to take into account a large number of inelastic channels, (Fig. 8 was, before reduction, a 512×512 matrix !) whose effects, as pointed out above, are essential.

The weakest link in our scheme is undoubtedly the assumption Z) which has so far little or no justification. This should, however, be taken in two stages : the criterion for binding "resonances" by looking for zeros of the approximate D function depends on little more than the crossing matrix elements and the signs and relative magnitudes of the various Born terms, whereas the calculation of the coupling constants and masses of particles by (II.2) depends on the explicit form of the approximate T matrix and is thus much harder to believe. It is interesting to note, however, that the quantities so far computed all turn out to have reasonable orders of magnitudes, as can be seen in Table 2.

	Calculated Values	Values from experiment
μ	scale	410 MeV, R.M.S. of (π, K, η)
m_1	960 MeV	R.M.S. of $(\rho, K^*, \phi) = 870$ MeV, B = 1220 MeV
m_0	500 MeV	$K = 725$ MeV
m_2^0	1.0 GeV	$f_0 = 1.25$ GeV
m_2^8	1.4 GeV	$A_2 = 1.32$ GeV
$\lambda_1^2/4\pi$	2.7	1.8 from ρ decay, $\rho \rightarrow 2\pi$
$\lambda_0^2/4\pi$	$0.04 \times \lambda_1^2/4\pi$	small, from $K(725)$ decay, $K \rightarrow K + \pi$

T a b l e 2

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A P P E N D I X A

CHARGE CONJUGATION OF MESON SUPERMULTIPLETS

Throughout this Appendix we follow the notation of de Swart ⁴⁸⁾ for the representations and Clebsch-Gordan coefficient in SU_3 .

Charge conjugation carries a particle state into the antiparticle state. In SU_3 , the antiparticle belongs to the conjugate representation and has opposite I_z and Y . Thus if the particle is represented by ψ_ν^μ , where μ denotes the SU_3 representation and $\nu = I_z, Y$, the antiparticle is represented by $\psi_{-\nu}^{\mu*} (-1)^{I_z + Y/2}$ in de Swart's phase convention. Under charge conjugation C ,

$$C \psi_\nu^\mu = c (-1)^{I_z + Y/2} \psi_{-\nu}^{\mu*}$$

where $c = \pm 1$. We shall call c the charge conjugation parity of the meson supermultiplet. For mesons, the exponent $I_z + Y/2$ is just the charge, so that c for the supermultiplet is just the normal charge conjugation parity of its neutral members, if such exist.

Consider an SU_3 invariant interaction between three meson supermultiplets, say $\phi_{\nu_1}^{\mu_1}$, $\psi_{\nu_2}^{\mu_2}$ and $\chi_{\nu_3}^{\mu_3}$. Following de Swart's notation for the Clebsch-Gordan coefficients, we may write the interaction Lagrangian as

$$\begin{aligned} \mathcal{L}_I = \sum_{\nu_1 \nu_2 \nu_3} \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} \phi_{\nu_1}^{\mu_1} \psi_{\nu_2}^{\mu_2} \begin{pmatrix} \mu_3 & \mu_3^* & 1 \\ \nu_3 & -\nu_3 & 0 \end{pmatrix} \chi_{-\nu_3}^{\mu_3^*} \\ + \text{h.c.} \end{aligned}$$

In strong interactions, charge conjugation is believed to leave the system invariant: thus $C \mathcal{L}_I = \mathcal{L}_I$. Now,

$$C \mathcal{L}_I = \sum_{\nu_1, \nu_2, \nu_3} \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \gamma \\ \nu_1 & \nu_2 & \nu_3 & \end{pmatrix} c_1 \varphi_{-\nu_1}^{\mu_1*} c_2 \psi_{-\nu_2}^{\mu_2*} \cdot \\ \cdot \begin{pmatrix} \mu_3 & \mu_3^* & 1 \\ \nu_3 & -\nu_3 & 0 \end{pmatrix} c_3 \chi_{\nu_3}^{\mu_3},$$

where we have used the condition that charge is conserved, namely

$\nu_1 + \nu_2 - \nu_3 = 0$. In particular if the representations μ_i are all self-conjugate, i.e., $\mu_i = \mu_i^*$, we may rewrite $C \mathcal{L}_I$ as

$$C \mathcal{L}_I = \sum_{\nu_1, \nu_2, \nu_3} c_1 c_2 c_3 \begin{pmatrix} \mu_1^* & \mu_2^* & \mu_3^* & \gamma \\ -\nu_1 & -\nu_2 & -\nu_3 & \end{pmatrix} \varphi_{\nu_1}^{\mu_1} \psi_{\nu_2}^{\mu_2} \cdot \\ \cdot \begin{pmatrix} \mu_3^* & \mu_3 & 1 \\ -\nu_3 & \nu_3 & 0 \end{pmatrix} \chi_{-\nu_3}^{\mu_3*}.$$

From the symmetry properties of the Clebsch-Gordan coefficients, we have

$$\begin{pmatrix} \mu_1^* & \mu_2^* & \mu_3^* & \gamma \\ -\nu_1 & -\nu_2 & -\nu_3 & \end{pmatrix} = \xi_3(\mu_1, \mu_2, \mu_3, \gamma) \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \gamma \\ \nu_1 & \nu_2 & \nu_3 & \end{pmatrix}$$

where $\xi_3(\mu_1, \mu_2, \mu_3) = \pm 1$ depending on μ_i and γ . We thus have

$$C \mathcal{L}_I = c_1 c_2 c_3 \xi_3(\mu_1, \mu_2, \mu_3, \gamma) \xi_3(\mu_3, \mu_3^*, 1) \mathcal{L}_I$$

For $\mathcal{L}_I$ to be invariant under C , the over-all phase factor must be +1. ξ_3 have been tabulated by de Swart for the most common representations of SU_3 . Thus given c_1 and c_2 , and the type of coupling, c_3 is uniquely determined by C invariance so long as μ_i are self-conjugate representations.

In the present paper, we meet frequently the case when $\mu_1 = \mu_2 = 8$.
 For the mutual scattering of the spin 0 octets, $c_1 = c_2 = +1$, thus

$$8^+ \times 8^+ = 1^+ + 8_D^+ + 8_F^- + 10 + \overline{10} + 27^+.$$

For the scattering of the spin 0 octets with the spin 1 octets, we
 have $c_1 = +1$, $c_2 = -1$, thus

$$8^+ \times 8^- = 1^- + 8_D^- + 8_F^+ + 10 + \overline{10} + 27^-.$$

A P P E N D I X B

INPUT DIAGRAMS ⁴⁹⁾

We present here the results of our computation of the one-particle-exchange input diagrams listed in (IV.14). The helicity amplitudes $\bar{\Phi}_{\lambda\mu}$ are represented as functions of the energy and the scattering angle, both in the c.m. system of the direct channel.

For four-vectors we use the following conventions :

$$p = (p_0, i\mathbf{p}),$$

$$p^2 = p_0^2 - |\mathbf{p}|^2,$$

$$\epsilon_{0123} = 1; \quad \epsilon_{\mu\nu\rho\sigma} \quad \text{totally antisymmetric.}$$

For graphs in the lower diagonal block of (IV.2), namely (P), (Q) and (R), we define :

$$\underline{k} = \text{momentum of incoming } 1^\pm \text{ meson,}$$

$$\underline{k}' = \text{momentum of outgoing } 1^\pm \text{ meson,}$$

$$\underline{k} \cdot \underline{k}' = k^2 \cos \theta,$$

$$\omega' = \sqrt{k^2 + \mu^2}$$

$$E = \sqrt{k^2 + m^2}$$

where m and μ are respectively the degenerate masses of the spin 1 and spin 0 octets. The polarization vectors ϵ_μ of the spin 1 mesons are normalized as follows :

$$\epsilon_\mu \epsilon_\mu = -1.$$

and satisfy the condition :

$$p_\mu \epsilon_\mu = 0$$

for meson momentum p_μ . Choosing a right-handed co-ordinate system with the z axis along $\underline{k}$ and the x axis in the scattering plane: namely

$$\underline{k} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}; \quad \underline{k}' = \begin{pmatrix} k \sin \theta \\ 0 \\ k \cos \theta \end{pmatrix};$$

we have for the incoming $1^\pm$ meson,

$$\underline{\varepsilon} = (\varepsilon_0, i \underline{\varepsilon})$$

$$\underline{\varepsilon}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}, \quad \underline{\varepsilon}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \quad \varepsilon_0 = \frac{E}{m} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\varepsilon_0 = (\underline{k} \cdot \underline{\varepsilon}) / E;$$

and for the outgoing $1^\pm$ meson,

$$\underline{\varepsilon}' = (\varepsilon'_0, i \underline{\varepsilon}'),$$

$$\underline{\varepsilon}'_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos \theta \\ -i \\ \sin \theta \end{pmatrix}, \quad \underline{\varepsilon}'_- = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \theta \\ -i \\ -\sin \theta \end{pmatrix}, \quad \varepsilon'_0 = \frac{E}{m} \begin{pmatrix} \sin \theta \\ 0 \\ \cos \theta \end{pmatrix},$$

$$\varepsilon'_0 = (\underline{k}' \cdot \underline{\varepsilon}') / E.$$

In this notation we obtained for the graphs :

(P1) :

$$\Phi_{\lambda\mu} = \Psi_{\lambda\mu} / [2k^2(1 - \cos \theta) + m^2],$$

where

$$\begin{aligned}\Psi_{0,0} &= (4k^2) \frac{E^2}{m^2} [(2\omega'E + k^2) + k^2 \cos \theta] (1 - \cos \theta)^2, \\ \Psi_{1,1} &= (-2k^2) [(2\omega'E + k^2) + k^2 \cos \theta] (1 - \cos^2 \theta), \\ \bar{\Psi}_{1,-1} &= (2k^2) [(2\omega'E + k^2) + k^2 \cos^2 \theta] (1 - \cos^2 \theta), \\ \bar{\Psi}_{1,0} &= (2\sqrt{2} k^2) \frac{E}{m} \sin \theta \times \\ &\quad \times [(2\omega'E + k^2) - 2\omega'E \cos \theta - k^2 \cos^2 \theta];\end{aligned}$$

(P2) :

$$\Phi_{\lambda\mu} = \Psi_{\lambda\mu} / [2k^2(1 - \cos \theta) + m^2],$$

where

$$\begin{aligned}\Psi_{0,0} &= 4 \left[\frac{2k^2}{m^2} (2\omega'E + E^2 + m^2) + \right. \\ &\quad \left. + \frac{1}{m^2} (2k^2 m^2 - 4\omega'E k^2 + 4\omega' m^2 E) \cos \theta - \right. \\ &\quad \left. - \frac{2k^2 E^2}{m^2} \cos^2 \theta \right], \\ \Psi_{1,1} &= 4 [(2\omega'E + 3k^2) + (2k^2 + 2\omega'E) \cos \theta - k^2 \cos^2 \theta], \\ \bar{\Psi}_{1,-1} &= 4 [(2\omega'E - k^2) - 2\omega'E \cos \theta + k^2 \cos^2 \theta], \\ \bar{\Psi}_{1,0} &= (-2\sqrt{2}) \frac{E}{m} \sin \theta \left[\left(k^2 + \frac{2\omega' m^2}{E} \right) - k^2 \cos \theta \right];\end{aligned}$$

(Q) :

$$\Phi_{\lambda\mu} = \Psi_{\lambda\mu} / [(E - \omega')^2 - 2k^2(1 + \cos\theta) - \mu'^2],$$

where μ' is the mass of the exchanged spin 0 meson, and

$$\Psi_{0,0} = (-16k^2) \frac{E^2}{m^2} \left(\frac{\omega'}{E} + \cos\theta \right)^2,$$

$$\Psi_{1,1} = (8k^2)(1 - \cos^2\theta),$$

$$\Psi_{1,-1} = -\Psi_{1,1},$$

$$\Psi_{1,0} = 8\sqrt{2} \frac{E}{m} k^2 \sin\theta \left(\frac{\omega'}{E} + \cos\theta \right);$$

(R) :

$$\Phi_{\lambda\mu} = \Psi_{\lambda\mu} / [2k^2(1 - \cos\theta) + m^2],$$

where

$$\Psi_{0,0} = 0,$$

$$\Psi_{1,1} = 0,$$

$$\Psi_{1,-1} = (4ik) \left[(3k^2E + 2k^2\omega' + 2\omega'E^2) - 2\omega'E^2\cos\theta - (3k^2E + 2k^2\omega')\cos^2\theta \right],$$

$$\begin{aligned} \Psi_{1,0} = & -2\sqrt{2}i \frac{k}{m} \sin\theta \\ & \times \left[\left\{ -2k^2E(E + \omega') + 2m^2\omega'E + m^2k^2 \right\} \right. \\ & \left. + \left\{ 2k^2E(E + \omega') + k^2m^2 \right\} \cos\theta \right]; \end{aligned}$$

(W) :

$$\Phi_{\lambda\mu} = \Psi_{\lambda\mu} / [-2E\omega' - 2k^2 \cos\theta + \mu^2],$$

where

$$\Psi_{0,0} = 4k^2 m^2 \sin^2\theta,$$

$$\Psi_{1,1} = -2k^2 [m^2 \cos^2\theta + s \cos\theta + (s - m^2)],$$

$$\Psi_{1,-1} = -2k^2 [-(2k^2 + m^2) \cos^2\theta + (E - \omega')^2 \cos\theta + (2\omega'E - \mu^2)],$$

$$\Psi_{1,0} = 2\sqrt{2} m k^2 \sin\theta (\omega' + E \cos\theta).$$

All other helicity amplitudes are then given by parity and time reversal.

A P P E N D I X C

FIGURE CAPTIONS

- Figure 11 The integrals d^J (Section IV.e) are plotted as functions of s (the square of the c.m. energy) in units of $m_1 = 1$. Dotted curves are to be multiplied by (-1) .
- Figure 12 The determinants $\det D^{J(XC)}$ are plotted as functions of s in units $m_1 = 1$. The small correction terms from graph (V) (Section IV.g) have already been included. The total force in channel $J(XC)$ is attractive (repulsive) for $\det D^{J(XC)} < 1$ (> 1).

FIG. 11a

FIG. 11b

FIG.11c

FIG.11d

$J = 1, (0, 1)$

FIG.11e

FIG.11f

FIG.12a

FIG. 12b

FIG. 12 c

FOOTNOTES AND REFERENCES

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- 39) This lower limit on η_3 may have to be modified in view of recent reports of possible enhancements in the $K^+ - \pi^+ - \pi^0$ $^{21),1)}$ and $K^+ - K^+$ $^{1)}$ systems. Such modifications need not, however, alter critically our analysis for $J > 0$.
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- 45) See Ref. $^{20)}$. Note, however, the ambiguous assignment here of A for X_0 resonance. Also the assignment of $K - \pi$ (1420) $^{41)}$ (presumably $A = +$) to the $2^+ (8^+)$ octet would violate this regularity.
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- 47) J.R. Fulco, G.L. Shaw and D.Y. Wong, Phys.Rev. 137, B1242 (1965).
- 48) J.J. de Swart, Revs.Modern Phys. 35, 916 (1963).
- 49) Part of the calculation in this Appendix has been performed in conjunction with Dr. K. Dietz in connection with another work, Nuovo Cimento 34, 250 (1964).