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ANALYTIC AND ASYMPTOTIC PROPERTIES OF SOLUBLE N/D MODELS

A. P. Contogouris

CERN - Geneva

A B S T R A C T

A class of relativistic models, defined in terms of the N/D equations for asymptotically varying left-hand discontinuities and constant inelasticity, is considered. This is shown to present an alternative, analogous in certain respects to, but simpler than, the class of models based on the ladder approximation of the Bethe-Salpeter equation. In terms of an N/D amplitude with an asymptotically vanishing left-hand discontinuity an explicit strong coupling calculation of the Pomeranchuk trajectory in agreement with the Regge pole analysis of the diffraction scattering experimental data is given; the corresponding residue function is also determined. An N/D amplitude with an asymptotically constant left-hand discontinuity is shown to be associated with a non-shrinking diffraction peak. Certain analogies to the solutions for non-relativistic scattering by regular or singular energy independent potentials are also discussed.

A framework in which various properties conjectured for elementary particle interactions can be studied in detail is usually provided by potential scattering ; typical examples are the analytic properties of the amplitude in the momentum and angular momentum variables. It is believed, however, that a parallel development and study of relativistic models is certainly desirable and sometimes even necessary.

A class of relativistic models that occupies a prominent position in the literature is the Bethe-Salpeter (B-S) equation in the ladder approximation. The study of the analytic properties in the coupling g and complex angular momentum ℓ of the corresponding amplitudes has been used to extract information about the nature of certain renormalizable and non-renormalizable theories and about the asymptotic behaviour in the crossed channel ¹⁾⁻⁶⁾. However, as the B-S equation is an off-the-mass-shell approach, the resulting integral equations are rather complicated and, excepting the weak coupling limit, the explicitly calculated results very much limited in number.

As an alternative to the B-S ladder formalism the present paper considers a class of relativistic models defined by the N/D equations in the approximation of elastic unitarity (or of constant inelasticity) and corresponding to left-hand discontinuities of varying asymptotic behaviour. By studying analyticity in g and ℓ a number of analogies between appropriate N/D and B-S solutions is established. On the other hand, due to the relatively simple integral equations which arise in the N/D approach, the superiority of the N/D models in providing explicit results for strong couplings is made, we think, clear.

In Section 2 an N/D model with an asymptotically vanishing left-hand discontinuity (scalar-scalar interaction) is considered ⁷⁾. Its analytic properties in g and ℓ are shown to be analogous to the ladder approximation of the B-S equation in the ϕ^3 interaction ^{1), 2)}. In non-relativistic scattering by local energy independent potentials similar analytic properties in g and ℓ characterize the amplitude defined in terms of the Schrödinger equation for a potential which is regular at the origin ($\sim r^{-\beta}$, $\beta < 2$).

In Section 3 it is shown that the relativistic model of Section 2 allows an explicit calculation of a Regge trajectory, at least in the strip $-\frac{1}{2} < \text{Re } l < \frac{3}{2}$. By adjusting the parameters of the model the calculated shape is in good agreement with the form of the Pomeranchuk Regge trajectory, as it is determined by certain Regge pole analysis of the high-energy experimental data. Note that this is a strong coupling calculation; to the author's knowledge, explicit results on Regge poles from relativistic models have been obtained only in the weak coupling limit ^{2), 8)} (with the exception of the forward direction ⁹⁾). Moreover it is shown that the N/D method allows the detailed calculations of the residue function of the moving pole, as well.

Section 4 considers an N/D example with an asymptotically constant left-hand discontinuity (scalar-vector interaction). The general solution of the resulting "marginally" singular integral equations and their analytic properties in g for physical ℓ have been derived and studied elsewhere ^{10), 11)}. Here the explicit solution for a simplified N/D amplitude is given for complex ℓ and the existence of a fixed (energy independent) branch point is demonstrated in general (strong coupling and non-zero total energy). This model, which can be associated with a non-shrinking diffraction peak in the crossed channel, is analogous in many respects to the ladder approximation of the B-S equation for a ϕ^4 interaction or for the scattering of two scalar particles by exchange of vector mesons with propagator $g_{\mu\nu}(q^2 - m^2)^{-1}$.

In non-relativistic scattering similar analytic properties in g and ℓ characterize the Schrödinger solutions of a "marginally" singular potential (i.e., behaving at the origin like $1/r^2$).

Finally, Section 5 discusses briefly the analytic properties in g of the solutions of certain N/D equations corresponding to left-hand discontinuities which asymptotically increase as powers of the energy¹²⁾. Here the analogous (with respect to analyticity in g) B-S examples exhibit certain features typical of non-renormalizable theories^{4),5),6)}; and the analogous non-relativistic models correspond to strongly singular potentials ($\sim r^{-\beta}$, $\beta > 2$). However, the resulting solutions of the B-S models are known to suffer from uncertainties associated with the allowability of the Wick rotation. The corresponding N/D approach is, of course, free of this objection; but it leads to denominator functions which, in general, contain an infinity of zeros¹²⁾.

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2. THE SCALAR-SCALAR INTERACTION

Consider the partial wave amplitude $A_\ell(\nu)$ for the elastic scattering of two scalar particles of mass μ ; ν is related to the square s of the total centre-of-mass energy by $s = 4(\nu + \mu^2)$. The left-hand discontinuity ($\equiv$ the "force" of the interaction) is defined from the exchange of a single scalar meson of mass m (scalar-scalar interaction); this gives the Born term:

$$B_\ell(\nu) = \frac{g}{\nu} Q_\ell\left(1 + \frac{m^2}{2\nu}\right) \quad (2.1)$$

where g is the square of the coupling and Q_ℓ the Legendre function of the second kind. For $\ell = 0, 1, 2, \dots$, $B_\ell(\nu)$ has a non-vanishing discontinuity only along $-\infty < \nu < -m^2/4$; for complex ℓ , the discontinuity along $-m^2/4 < \nu < 0$ is non-zero, too¹³⁾. However, in the function

$$\beta_\ell(\nu) = \nu^{-\ell} B_\ell(\nu) \quad (2.2)$$

the last piece is easily seen to cancel.

Similar properties characterize the amplitude $A_\ell(\nu)$, which, near the physical threshold ($\nu = 0$), behaves like $\nu^{\ell/2}$; and for complex ℓ it develops an additional kinematic branch point¹⁴⁾. This is factored out by making the N/D decomposition as follows:

$$\nu^{-\ell} A_\ell(\nu) = N_\ell(\nu) / D_\ell(\nu) \quad (2.3)$$

Now, in the approximation of elastic unitarity $N_\ell(\nu)$ and $D_\ell(\nu)$ satisfy the following system of coupled integral equations

$$N_\ell(\nu) = \frac{1}{\pi} \int_{-\infty}^{-m^2/4} \frac{d\nu'}{\nu' - \nu} \Delta \beta_\ell(\nu') D_\ell(\nu') \quad (2.4)$$

$$D_\ell(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^\infty \frac{d\nu'}{\nu' - \nu} p(\nu') \nu'^\ell \frac{N_\ell(\nu')}{\nu' - \nu_0} \quad (2.5)$$

In (2.4) $\Delta\beta_\ell(\nu)$ is the discontinuity along the left-hand cut ; from (2.2) and (2.1) it follows that

$$\Delta\beta_\ell(\nu) = \frac{\pi}{2} g \nu^{-(\ell+1)} P_\ell \left(1 + \frac{m^2}{2\nu}\right) \quad (2.6)$$

In (2.5) $p(\nu) \equiv (\nu/(\nu + \mu^2))^{\frac{1}{2}}$ is the usual phase-space factor. For the numerator function a non-subtracted dispersion relation has been assumed.

Replacing (2.4) and (2.6) in (2.5) and taking $\nu_0 = 0$ one ends up with the following integral equation

$$D_\ell(\nu) = 1 - \frac{g}{2} \nu \int_{-\infty}^{-m^2/4} d\nu' H_\ell(\nu, \nu') \nu'^{-(\ell+1)} P_\ell \left(1 + \frac{m^2}{2\nu'}\right) D_\ell(\nu') \quad (2.7)$$

where

$$H_\ell(\nu, \nu') = \frac{1}{\pi} \int_0^\infty \frac{d\bar{z} \bar{z}^{\ell-1}}{(\bar{z} - \nu)(\bar{z} - \nu')} p(\bar{z}) \quad (2.8)$$

Equation (2.7) will be our model equation.

It will be shown that for

$$-\frac{1}{2} + \varepsilon < \text{Re} \ell \leq \frac{3}{2} \quad (2.9)$$

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where ε is arbitrarily small and positive, Eq. (2.7) is of Fredholm type. For this let $\gamma = -\omega$ and

$$q(\omega) \equiv \left[\omega^{-\ell} P_{\ell} \left(1 - \frac{m^2}{2\omega} \right) \right]^{1/2}$$

The definitions

$$f(\omega) \equiv \omega^{-1} q(\omega) D_{\ell}(-\omega) \quad f_0(\omega) \equiv \omega^{-1} q(\omega)$$

and

$$K(\omega, \omega') \equiv \frac{1}{2} q(\omega) H_{\ell}(-\omega, -\omega') q(\omega')$$

lead to the following equivalent equation with symmetric kernel

$$f(\omega) = f_0(\omega) + g \int_{m^2/4}^{\infty} d\omega' K(\omega, \omega') f(\omega') \quad (2.10)$$

At first, it is easy to establish that, for $\text{Re} \ell > -1$, $f_0(\omega)$ is square integrable. Next, to show the square integrability of the kernel note that

$$\bar{z} + \omega \geq \left| \bar{z}^{\frac{\ell}{2} + \frac{1}{4}} - \frac{\varepsilon}{2} \omega^{-\frac{\ell}{2} + \frac{3}{4} + \frac{\varepsilon}{2}} \right|$$

for $\bar{z}, \omega \geq 0$ and $-\frac{1}{2} + \varepsilon < \text{Re} \ell < \frac{3}{2} + \varepsilon$ where ε is positive (and small). Then :

$$\int_{m^2/4}^{\infty} d\omega \int_{m^2/4}^{\infty} d\omega' |K(\omega, \omega')|^2 < \left(\int_{m^2/4}^{\infty} d\omega \omega^{-\frac{3}{2} - \varepsilon} \left| P_{\ell} \left(1 - \frac{m^2}{2\omega} \right) \right| \right)^2 \int_{m^2/4}^{\infty} d\bar{z} \bar{z}^{-\frac{3}{2} + \varepsilon}$$

which is finite for $0 < \varepsilon < \frac{1}{2}$ q.e.d. In particular, for $\nu, \nu' < 0$ $H_{\ell}(\nu, \nu')$ is finite provided that $-\frac{1}{2} < \text{Re} \ell < 2$.

The Fredholm character of (2.10) [or (2.7)] implies immediately that $f(\omega)$ [or $D_\ell(\nu)$] is a meromorphic function of the coupling constant g . The same is true for the partial wave amplitude $A_\ell(\nu)$; its poles in the complex g plane being defined from the zeros of $D_\ell(\nu)$ considered as a function of g .

In terms of a theorem on operators $K(\ell)$ depending analytically on a parameter ℓ ¹⁵⁾ it can be shown that, at least for ℓ in the strip (2.9), $A_\ell(\nu)$ is meromorphic in ℓ , as well. According to this theorem, if $K(\ell)$ is a Fredholm operator for ℓ in a connected domain $\mathcal{D}$ and if $|K(\ell_0)| < 1$ at least at one point $\ell_0 \in \mathcal{D}$, then $[1+K(\ell)]^{-1}$ is operator-meromorphic in $\mathcal{D}$. For appropriate values of the parameters g and m the operator defined by the kernel of (2.9) satisfies the provisions of this theorem; thus $f(\omega)$ is meromorphic in ℓ . This implies the meromorphy of $A_\ell(\nu)$; its poles in the complex ℓ plane are defined from the zeros $\ell_i = \ell_i(\nu)$ of the denominator function $D_\ell(\nu)$.

Note that, in view of the Fredholm character of (2.10) for ℓ in (2.9), the absence of branch points of $f(\omega)$ should be expected; for, otherwise $f(\omega)$ would be multivalued and this would contradict the well-known uniqueness theorems of Fredholm integral equations.

Within the strip (2.9) $D_\ell(\nu)$ has only a finite number of zeros. This could be argued as follows: consider for simplicity the approximation of non-relativistic phase space $\rho(\nu) = \nu^{\frac{1}{2}}$ (which is also used in the next Section) and assume that for $\ell = i|y| \rightarrow i\infty$ along the strip (2.9) $D_\ell(\nu)$ does not increase exponentially. Now

$$H_\ell(-\omega, -\omega') = \frac{1}{\cos \pi \ell} \frac{\omega^{\ell-\frac{1}{2}} - \omega'^{\ell-\frac{1}{2}}}{\omega - \omega'}$$

Using the asymptotic form of $P_{iy}(\cos \theta)$ for $|y| \rightarrow \infty$ ¹³⁾ one has

$$\left| \frac{P_{iy}(\cos \theta)}{\cosh \pi y} \right| \sim |y \sin \theta|^{-\frac{1}{2}} e^{-(\pi - |\theta|)|y|}$$

Hence, for $\ell = i|y| \rightarrow i\infty$ Eq. (2.7) gives

$$D_\ell(\nu) \rightarrow 1$$

(in agreement with the initial assumption). This implies that the poles of $D_\ell^{-1}(\nu)$ [i.e., the zeros of $D_\ell(\nu)$] cannot accumulate at infinity. As they can form no finite accumulation point their total number is finite.

Note that the proof of meromorphy of $A_\ell(\nu)$ as a function of complex g or complex ℓ can be extended to ℓ outside the strip (2.9).

The basic feature of the present model (scalar-scalar interaction) is that the discontinuity along the left-hand cut

$$\Delta B_\ell(\nu) = \frac{\pi}{2} g \nu^{-1} P_\ell \left(1 + \frac{m^2}{2\nu}\right)$$

vanishes sufficiently rapidly as $\nu \rightarrow -\infty$. This combined with elastic unitarity leads to the Fredholm nature of the model Eq. (2.7). As a result, the only singularities of $A_\ell(\nu)$ as a function of g or ℓ are poles (at least for $\text{Re} \ell > -\frac{1}{2}$). These are properties in common with the partial wave amplitude for non-relativistic scattering by a potential which is regular at the origin (e.g., a Yukawian).

The analytic structure of $A_\ell(\nu)$ in ℓ and g is also similar to that of the B-S amplitude for scattering of scalar mesons in the ladder approximation of exchange of scalar mesons (ϕ^3 theory)^{1),2)}. This type of theory is usually termed super-renormalizable. However, unlike the B-S case, the N/D model leads to simple and tractable integral equations. This will become clear in the next Section.

3. DIFFRACTION SCATTERING IN THE SCALAR-SCALAR INTERACTION

In the model of the previous Section the poles of the partial wave amplitude are defined from the zeros of the solution $D_\ell(\nu)$ of the integral Eq. (2.7). The kernel of this equation is non-separable. In general, a non-separable continuous Fredholm kernel can be approximated by a separable one to any degree of accuracy; and then standard methods can be used to solve the resulting equation¹⁶⁾.

For simplicity in the calculations take $\rho(\nu) = \nu^{\frac{1}{2}}$; clearly, this approximation cannot affect the basic conclusions. Now, at $\ell = 1$ and with the substitutions $\nu = -\omega$, $\nu' = -\omega'$ Eq. (2.8) takes the form

$$H_1(-\omega, -\omega') = (\omega^{\frac{1}{2}} + \omega'^{\frac{1}{2}})^{-1} \quad (3.1)$$

To approximate this by a separable form notice that, given a kernel $\lambda K(x, x')$, there is, in general, a resolvent $R(x, x'; \lambda)$ such that

$$K(x, x') = \lim_{\lambda \rightarrow 0} R(x, x'; \lambda) \quad (3.2)$$

provided that the limit exists. It has been proved¹⁰⁾ that the general resolvent of the kernel $\lambda \pi^{-1}(x+x')^{-1}$ is

$$R(x, x'; \lambda) = \frac{1}{2\pi i x'} \int_{\sigma-i\infty}^{\sigma+i\infty} ds \left(\frac{x'}{x}\right)^s (\sin \pi s - \lambda)^{-1} \quad (3.3)$$

with $0 < \sigma < 1$. The displacement of zeros of $\sin \pi s - \lambda$ for real values of λ is shown in Fig. 1. Clearly, for $\lambda = 0$ one can take $\sigma = \frac{1}{2}$; then it follows that for $\lambda \rightarrow 0+$ the resolvent (3.3) tends to $\pi^{-1}(x+x')^{-1}$. Hence, after a change of variable $s = \frac{1}{2} + it$:

$$\begin{aligned}
 H_1(-w, -w') &= \frac{(ww')^{-\frac{1}{2}}}{2} \int_{-\infty}^{\infty} dt \frac{\exp(\frac{i}{2} t \log \frac{w'}{w})}{\cosh \pi t} = \\
 &= \frac{(ww')^{-\frac{1}{2}}}{2} \sum_{m=0}^{\infty} (-1)^m \frac{|E_{2m}|}{(2m)! 4^{2m}} \left(\log \frac{w'}{w}\right)^{2m}
 \end{aligned} \tag{3.4}$$

where E_{2m} are Euler numbers ¹³⁾. Defining

$$A_m = |E_{2m}| / (2m)! 4^{2m}$$

one finds : $A_1 = 1$, $A_2 = 3.12 \times 10^{-2}$, $A_3 = 8.12 \times 10^{-4}$, $A_4 = 2.07 \times 10^{-5}$, $A_5 = 5.3 \times 10^{-7}$. This suggests that a good approximation may be obtained by keeping only a few of the terms of the series.

Consider the approximation of keeping in (3.4) only the first term ; the result will subsequently be tested in the region of interest. In this approximation Eq. (2.7) is easily solved, with the result

$$\hat{D}_1(-w) = 1 - \frac{2}{45} \left(\frac{2}{m}\right)^{5/2} \left(1 - \frac{2}{3m}\right)^{-1} w^{3/4} \tag{3.5}$$

Consider now the crossed channel for which $s = 4(\nu + \mu^2)$ is the momentum transfer variable. The requirement that for forward scattering ($s = 0$) a pole of the partial wave amplitude $A_\ell(\nu)$ passes through $\ell = 1$ (Pomeranchuk pole) gives the condition $\hat{D}_1(-\mu^2) = 0$.

The same procedure can be repeated at $\ell = 0$. Here Eq. (2.8) takes the form

$$H_0(-w, -w') = (ww')^{-\frac{1}{2}} H_1(-w, -w')$$

where $H_1(-\omega, -\omega')$ is given by (3.1). With the same approximations Eq. (2.7) gives

$$\hat{D}_0(-\omega) = 1 - \frac{g}{3} \left(\frac{2}{m} \right)^{3/2} \left(1 + \frac{g}{m} \right)^{-1} \omega^{1/4} \quad (3.6)$$

The condition $\hat{D}_0(-\omega) = 0$ defines the intersection of the (Pomeranchuk) trajectory with the $\ell = 0$ axis. Eliminating g between $\hat{D}_1(-\mu^2) = 0$ and $\hat{D}_0(-\omega) = 0$ and setting $\mu/m = r$ one has

$$\omega = \omega_0 = 4m^2 \left(1 + \frac{\sqrt{2}}{15} r^{3/2} \right)^4 \quad (3.7)$$

In a similar way one can determine the intersection of the calculated trajectory with the line $\ell = \frac{1}{2}$. Here

$$H_{\frac{1}{2}}(-\omega, -\omega') = \frac{1}{\pi} \frac{\log(\omega'/\omega)}{\omega' - \omega}$$

This can be approximated by a separable form by observing that the general resolvent of the kernel $(\lambda/\pi^2)(\log \frac{x'}{x})/(x' - x)$ was found to be ¹⁰⁾

$$R(x, x'; \lambda) = \frac{1}{2\pi i x'} \int_{\sigma-i\infty}^{\sigma+i\infty} ds \left(\frac{x'}{x} \right)^s (\sin^2 \pi s - \lambda)^{-1} \quad (3.8)$$

with $0 < \sigma < 1$. The displacement of the zeros of $\sin^2(\pi s) - \lambda$ with real λ is shown in Fig. 2. For $\lambda = 0$ one can take $\sigma = \frac{1}{2}$ and, with the change $s = \frac{1}{2} + it$, get:

$$\begin{aligned} \frac{\log \frac{x'}{x}}{x' - x} &= \frac{\pi}{2} (xx')^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt \frac{\exp(it \log \frac{x'}{x})}{\cosh^2 \pi t} = \\ &= (xx')^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(\log \frac{x'}{x})^{2m}}{(2m)!} (2^{1-2m} - 1) B_{2m} \end{aligned} \quad (3.9)$$

B_{2m} are the Bernoulli numbers ; one can show that with increasing m , $B_{2m}/(2m)!$ decreases fast ¹³⁾. Again, this suggests that for a practical calculation only the first few terms of (3.9) be kept. Keeping only the first term and using the approximation $P_{\frac{1}{2}}(z) \simeq 1 - \frac{1}{2} \log 2 + \log(z+1)$ ($-1 < z \leq 1$) one finds

$$\hat{D}_{\frac{1}{2}}(-\omega) = 1 - \frac{g}{\pi m^2} \left(1 + \frac{2 \log 2}{\pi} \frac{g}{m}\right)^{-1} \omega^{1/2} \quad (3.10)$$

Eliminating g between $\hat{D}_1(-\mu^2) = 0$ and $\hat{D}_{\frac{1}{2}}(-\omega) = 0$:

$$\omega = \omega_{\frac{1}{2}} = 4m^2 \left(\log 2 + \frac{\pi}{6} + \frac{\sqrt{2}\pi}{45} r^{3/2} \right)^2 \quad (3.11)$$

For a given value of the ratio r Eqs. (3.7) and (3.11) along with $\omega = \omega_1 = \mu^2$ (for $\ell = 1$) define the trajectory of a Regge pole which for $s = 0$ passes from $\ell = 1$. Figure 3 gives the trajectories corresponding to $r = 1, 3$ and 4 ; m is identified with the mass of the pion.

To test these numerical results we shall confront the so-defined moving pole with the leading Regge trajectory of certain analysis of diffraction data. Thus, we accept that elastic scattering at energies 10-30 GeV and momentum transfers $0 \lesssim |t| \lesssim 1$ (GeV/c)² can be described by the exchange of a (relatively) small number of Regge poles. Now, to determine the trajectory of the leading pole we can proceed in two slightly different ways :

- (a) We can use directly the corresponding proton-proton experimental data ¹⁷⁾, because for this process the total contribution of the non-leading trajectories at small $|t|$ can be made to cancel, roughly. Then, for $3 < r < 4$ the present model is in very good agreement (Fig. 3) ¹⁸⁾.
- (b) We may accept the analysis of Phillips and Rarita ¹⁹⁾ ; the corresponding leading trajectory (denoted by P-R) is plotted in Figure 3, as well. Here for $r = 4$ our model is in very satisfactory agreement, too.

The approximation of keeping in (3.4) and (3.9) only the first term can be tested by introducing $\hat{D}_{\ell}(\nu)$ in the integral of the right-

hand side of (2.7) and carrying the integration with the exact $H_\ell(\nu, \nu')$; the result is to be compared to $\hat{D}_\ell(-\omega) - 1$ [given by (3.3), (3.6) and (3.10)]. At $\ell = 1$ and $s = 0$ ($\nu = -\omega_1 = -\mu^2$) the error is 15 %. At $\ell = 0$ and $s = -50 \text{ m}^2$ (corresponding $r = 3$) the error is 20 %.

The same procedure can also be used to calculate the numerator (or residue) function $N_\ell(\nu)$. At first, by substituting (2.5) in (2.4) one has the following integral equation :

$$N_\ell(\nu) = \beta_\ell(\nu) + \frac{1}{\pi} \int_0^\infty d\nu' \frac{\nu \beta_\ell(\nu) - \nu' \beta_\ell(\nu')}{\nu - \nu'} \rho(\nu') \nu'^{\ell-1} N_\ell(\nu') \quad (3.12)$$

where

$$\beta_\ell(\nu) = g \nu^{-(\ell+1)} Q_\ell \left(1 + \frac{m^2}{2\nu}\right)$$

At $\ell = 1$, with the substitutions

$$x \equiv \frac{4\nu}{m^2} + 1 \quad N_1(\nu) \equiv f(x)$$

(3.12) is replaced by

$$f(x) = \beta_1(m^2 \frac{x-1}{4}) + \frac{gm}{4\pi} \int_1^\infty dx' \left[(2x'-1) \frac{\log \frac{x'}{x}}{x'-x} + S(x, x') \right] (x'-1)^{\frac{1}{2}} f(x')$$

$$S(x, x') = \frac{2 - \log x}{x-1} - 2 \frac{x+x'-2}{(x-1)^2(x'-1)} \log x$$

The part $S(x, x')$ of the kernel is separable. The non-separable part ($\sim (\log \frac{x'}{x}) / (x'-x)$) can be treated again by the expansion (3.9). The calculation of $N_\ell(\nu)$ at $\ell=0$ leads again to a non-separable kernel of the form $\log \frac{x'}{x} / (x'-x)$ and thus presents no additional difficulty.

An improved treatment, including a better approximation for $D_\ell(\nu)$ and $N_\ell(\nu)$ and more points in the interval $0 < \ell < 1$, is certainly possible. However, for the purpose of the present work, this is completely unnecessary.

As has been mentioned, a different relativistic example which leads to the same qualitative conclusions is the B-S equation in the ladder approximation of the ϕ^3 theory ²⁾ - or the multiperipheral model ²⁰⁾. The basic physical feature of these is that diffraction scattering appears as the shadow of inelastic contributions containing a large number of (rather weakly correlated) secondaries; this seems indeed to be the basic feature of diffraction scattering ²¹⁾. The fact that our model gives equivalent results is, perhaps, not surprising. In both cases the "force" is defined by the same Born term: one-particle exchange in the scalar-scalar interaction. Then, our N/D approach could, roughly, be viewed as the completion of a "ladder" by an iteration in which unitarity and analyticity are satisfied in every step (unitary-dispersive iteration) ²²⁾.

Thus, explicit calculation of a Regge trajectory and of the associated residue function $N_\ell(\nu)$ has been possible through an N/D model, even for strong coupling. As mentioned in the Introduction, equivalent calculation through B-S models does not exist, except in the weak coupling limit ($g \rightarrow 0$). The price paid is that through the N/D approach the simple multiperipheral picture becomes, to some extent, obscure.

The present method of calculation can be extended to other Regge trajectories and/or types of interaction, provided that they lead to Fredholm integral equations.

4. THE SCALAR VECTOR INTERACTION

Consider again the elastic scattering of two scalar mesons of mass μ , but define the left-hand discontinuity from the exchange of a single vector meson of mass m ; this gives the Born term

$$B_\ell(\nu) = g \frac{8\nu + 4\mu^2 + m^2}{8\nu} Q_\ell \left(1 + \frac{m^2}{2\nu}\right) \quad (4.1)$$

Again, it is easily seen that the function $\beta_\ell(\nu) = \nu^{-\ell} B_\ell(\nu)$ has, for any ℓ , a non-vanishing discontinuity along $-\infty < \nu < -\nu_L = -m^2/4$ only.

Here, after an N/D decomposition like (2.3), it will be assumed that both $N_\ell(\nu)$ and $D_\ell(\nu)$ satisfy once subtracted dispersion relations :

$$N_\ell(\nu) = N_\ell(\nu_0) + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{-\nu_L} \frac{d\nu'}{\nu' - \nu} \frac{\Delta\beta_\ell(\nu') D_\ell(\nu')}{\nu' - \nu_0} \quad (4.2)$$

$$D_\ell(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^\infty \frac{d\nu'}{\nu' - \nu} \frac{\rho(\nu') \nu'^\ell N_\ell(\nu')}{\nu' - \nu_0} ; \quad (4.3)$$

$\Delta\beta_\ell(\nu)$, the discontinuity along the left-hand cut, is

$$\Delta\beta_\ell(\nu) = \pi g \frac{8\nu + 4\mu^2 + m^2}{16\nu^{\ell+1}} P_\ell \left(1 + \frac{m^2}{2\nu}\right) \quad (4.4)$$

With the definitions

$$D_\ell(\nu)/(\nu - \nu_0) \equiv f(\nu) \quad (\nu - \nu_0)^{-1} \equiv f_0(\nu) \quad (4.5)$$

and with the substitution of (4.4) into (4.5) one finds the following model integral equation :

$$D_\ell(\nu) = 1 - (\nu - \nu_0) N_\ell(\nu_0) H_\ell(\nu, \nu_0) + \frac{g(\nu - \nu_0)}{16} \int_{-\infty}^{-\nu_L} d\nu' H_\ell(\nu, \nu') \frac{8\nu' + 4\nu'^2 + m^2}{\nu'^{\ell+1}} P_\ell\left(1 + \frac{m^2}{2\nu'}\right) \frac{D_\ell(\nu')}{\nu' - \nu_0} \quad (4.6)$$

where

$$H_\ell(\nu, \nu') = \frac{1}{\pi} \int_0^\infty \frac{d\tilde{\nu} \tilde{\nu}^\ell}{(\tilde{\nu} - \nu)(\tilde{\nu} - \nu')} \quad (4.7)$$

Unlike (2.7), here the model equation is not Fredholm for any ℓ ; this is due to the behaviour of the kernel for $\nu, \nu' \rightarrow \infty$ ("marginally" singular). Yet, an explicit solution can be obtained by following the methods developed in Ref. ¹⁰⁾. As discussed there, it is sufficient to obtain the resolvent of (4.6) for the asymptotic part of its kernel; then, the determination of the complete $D_\ell(\nu)$ reduces to solving a Fredholm integral equation.

As in Ref. ¹⁰⁾, the asymptotic part of the kernel is defined by taking the limits $\rho(\nu) \rightarrow 1$ and $(8\nu + 4\nu^2 + m^2) P_\ell(1 + \frac{m^2}{2\nu}) \rightarrow 8\nu$ (for $\nu \rightarrow \infty$); also, the limit of integration in (4.6) will be taken $\nu_L = 0$. It has been proved in detail ¹⁰⁾ that these substitutions do not alter the analytic structure of the resolvent with respect to the coupling constant g ; the same is plausible with respect to ℓ . Then, the substitutions $\nu = -\omega$, $\nu_0 = -\omega_0$ and

$$\psi(\omega) = -\frac{D_\ell(-\omega)}{\omega - \omega_0} \quad \psi_0(\omega) = -\frac{1}{\omega - \omega_0} - N_\ell(-\omega_0) H_\ell(-\omega, -\omega_0) \quad (4.8)$$

reduce (3.6) to the form

$$\psi(w) = \psi_0(w) - \frac{g}{2\sin\pi l} \int_0^\infty dw' \frac{w'^l - w^l}{w' - w} \frac{\psi(w')}{w'^l} \quad (4.9)$$

This integral equation can be inverted by Mellin transform. The general resolvent is

$$R(w, w'; g; l) = \frac{1}{i\pi w} \int_{\sigma-i\infty}^{\sigma+i\infty} ds \left(\frac{w'}{w}\right)^s \left[\sin\pi \left(2s + l - \frac{1}{2}\right) + g\pi + \cos\pi l \right]^{-1} \quad (4.10)$$

The limits of the integration path lie in $0 < \sigma < 1 - \operatorname{Re} l$; hence the Mellin transform of (4.9) exists only for $\operatorname{Re} l < 1$.

It has been shown ¹⁰⁾ that the resolvent (4.10) has, in general, fixed branch points at

$$g\pi + \cos\pi l = 1, -1, \infty \quad (4.11)$$

In fact, a resolvent with only one finite branch point, at $g\pi + \cos\pi l = 1$ is

$$R(w, w'; g; l) = \frac{(ww')^{-1/2}}{[(g\pi + \cos\pi l)^2 - 1]^{1/2}} \left(\frac{w'}{w}\right)^{-\frac{l}{2}} \frac{w' + w}{w' - w} \times \sin\left(\frac{q_0}{2} \log \frac{w'}{w}\right) \quad (4.12)$$

where

$$\cosh\pi q_0 = -g\pi - \cos\pi l$$

The solution of (4.9) corresponding to the resolvent (4.10) can be determined explicitly. First, corresponding to the inhomogeneous term

$$\psi_0^{(1)}(w) = -\frac{1}{w - w_0},$$

the solution $\psi^{(1)}(\omega)$ of (4.9) is computed by using well-known Mellin transforms²³⁾. The result is

$$\psi^{(1)}(\omega) = \frac{1}{\omega - \omega_0} \left\{ -1 + \frac{q\pi}{4i} \left[(q\pi + \cos\pi\ell)^2 - 1 \right]^{-1/2} S(\ell, i \frac{q_0}{2}) \right\} \quad (4.13)$$

where

$$S(\ell, \xi) = \sin 2\pi\xi \left(a(\ell+1) + a(\ell-1) \right) + b(\ell+1, \xi) - b(\ell+1, -\xi) \\ + b(\ell-1, \xi) - b(\ell-1, -\xi)$$

$$a(p) \equiv \left[\sin\pi\left(\frac{p}{2} + \xi\right) \sin\pi\left(\frac{p}{2} - \xi\right) \right]^{-1} \quad b(p, \xi) \equiv \left(\frac{\omega}{\omega_0}\right)^{\frac{p}{2} + \xi} \cot\pi\left(\frac{p}{2} + \xi\right)$$

Next, corresponding to the inhomogeneous term

$$\psi_0^{(2)}(\omega) = -N_\ell(-\omega_0) H_\ell(-\omega, -\omega_0),$$

the solution $\psi^{(2)}(\omega)$ of (4.9) is determined by noting that the resolvent R of the kernel $\frac{\lambda}{\pi} K$ satisfies the integral equation (in operator form) :

$$R = \frac{1}{\pi} K + \frac{\lambda}{\pi} K \cdot R$$

Then it easily follows

$$\psi^{(2)}(\omega) = \frac{\omega_0^\ell N_\ell(-\omega_0)}{[(q\pi + \cos\pi\ell)^2 - 1]^{1/2}} \left(\frac{\omega}{\omega_0}\right)^{\ell/2} \frac{1}{(\omega\omega_0)^{1/2}} \frac{\omega + \omega_0}{\omega - \omega_0} \sin\left(\frac{q_0}{2} \log \frac{\omega}{\omega_0}\right) \quad (4.14)$$

The solution of (4.9) corresponding to the complete inhomogeneous term $\psi_0(\omega) = \psi_0^{(1)}(\omega) + \psi_0^{(2)}(\omega)$ is

$$\psi(\omega) = \psi^{(1)}(\omega) + \psi^{(2)}(\omega) \quad (4.15)$$

For $\omega_0 > 0$ one verifies through (4.8), (4.15), (4.13) and (4.14) that the denominator function $D_\ell(\nu)$ has a cut $0 \leq \nu < \infty$. The determination of $N_\ell(\nu)$ through (4.2) is straightforward.

For $g > 0$ (attractive force) and in the weak coupling limit ($g \rightarrow 0$) the branch points of (4.12) in the complex ℓ -plane are of the square root type and appear at

$$\ell \simeq \pm \sqrt{\frac{2}{\pi} g} \quad (4.16)$$

These branch points are singularities of $A_\ell(\nu)$, as well.

Similar is the structure of the B-S amplitude for scattering of two scalar mesons in the ladder approximation of the ϕ^4 theory or of exchange of vector mesons with propagator $g_{\mu\nu} (q^2 - m^2)^{-1}$: $A_\ell(s)$ has branch points in ℓ of which the position depends only on the coupling³⁾; and for $g \rightarrow 0$ they are of the type (4.16). Again, the analogy of the results for the scalar-vector interaction is not surprising: the force is defined by the same Born term. However, in the B-S treatment explicit calculations exist only in the weak coupling limit or at zero scattering energy ($s = 0$)³⁾. Again the superiority of the N/D approach is clearly demonstrated: the equivalent N/D model has been treated with no a priori restrictions in g or s .

In non-relativistic scattering the phase shift of a "marginally" singular potential of the type g/r^2 is

$$\delta_\ell = \frac{\pi}{2} \left(\ell + \frac{1}{2} - \sqrt{\left(\ell + \frac{1}{2}\right)^2 - g} \right) \quad (4.17)$$

This introduces fixed square root branch points in the partial wave amplitude considered as a function of complex ℓ ²⁴⁾.

For fixed ℓ , in all the discussed cases (N/D, B-S and non-relativistic model) there exist, in general, solutions expandable in powers of g near $g = 0$ (perturbation solutions)¹⁰⁾; hence, this type of theory can be termed renormalizable (see Bastai et al., Ref. ¹⁾).

Finally, consider the case of a relativistic amplitude $F(t,s)$ which satisfies the Froissart bound²⁵⁾ and at the same time has a partial wave amplitude with the same singularities of $A_\ell(\nu) = N_\ell(\nu)/D_\ell(\nu)$ of this Section (Eqs. (4.2), (4.3) and (4.15)), at least near $\ell = 1$. In the crossed channel, where $s = 4(\nu + \mu^2)$ is the momentum transfer variable and t the square of the centre-of-mass energy, the large t behaviour of $F(t,s)$ is determined by the singularities of $A_\ell(\nu)$ near $\ell = 1$. From (4.11) it follows that for $g = 2/\pi$ $A_\ell(\nu)$ has a fixed branch point at $\ell = 1$. This leads to an amplitude which, for $t \rightarrow \infty$, behaves like $f(s):t$ (apart from $\log t$ factors). Thus, for $g = 2/\pi$ the model of this Section can be used to define a relativistic amplitude with a diffraction peak which at high energy remains constant.

5. THE SCALAR TENSOR INTERACTION

To complete the discussion of analogies between certain N/D and B-S models consider the elastic scattering of two scalar mesons of mass μ and define the left-hand discontinuity from the exchange of a single spin 2 meson of mass m . Use of the propagator $g_{\mu\nu} g_{\rho\sigma} (q^2 - m^2)^{-1}$ for the exchanged particle gives a Born term of the form

$$B_\ell(\nu) = g h(\nu, m, \mu) Q_\ell \left(1 + \frac{m^2}{2\nu}\right); \quad (5.1)$$

the function $h(\nu, m, \mu) \rightarrow \nu$ for $|\nu| \rightarrow \infty$. In this Section only physical values of ℓ will be considered; then the N/D decomposition of the partial wave amplitude can be carried in the usual way:

$$A_\ell(\nu) = N_\ell(\nu) / D_\ell(\nu) \quad (5.2)$$

Assume, as in Section 4, once subtracted dispersion relations and take the subtraction point at $\nu_0 = 0$. Then, for $\ell \geq 1$, $N_\ell(\nu_0) = 0$ and the N/D equations take the form

$$N_\ell(\nu) = \frac{\nu}{\pi} \int_{-\infty}^{-\nu_L} d\nu' \frac{\Delta B_\ell(\nu') D_\ell(\nu')}{\nu'(\nu' - \nu)} \quad (5.3)$$

$$D_\ell(\nu) = 1 - \frac{\nu}{\pi} \int_0^\infty d\nu' \frac{P(\nu') N_\ell(\nu')}{\nu'(\nu' - \nu)}, \quad (5.4)$$

where $\nu_L = m^2/4$ and

$$\Delta B_\ell(\nu) = \frac{\pi g}{2} h(\nu, m, \mu) P_\ell \left(1 + \frac{m^2}{2\nu}\right) \quad (5.5)$$

Replacing (5.3) in (5.4) one finds as model equation

$$D_\ell(\nu) = 1 + \frac{g}{2} \nu \int_{-\infty}^{-\nu} d\nu' H(\nu, \nu') h(\nu', m, \mu) P_\ell(1 + \frac{m^2}{2\nu'}) D_\ell(\nu') \quad (5.6)$$

where

$$H(\nu, \nu') = \frac{1}{\pi} \int_0^\infty \frac{d\tilde{\nu} \rho(\tilde{\nu})}{(\tilde{\nu} - \nu)(\tilde{\nu} - \nu')} \quad (5.7)$$

The basic feature of this model is a left-hand discontinuity which increases as $\nu \rightarrow -\infty$. Of course, (5.6) is a non-Fredholm equation (strongly singular). Yet, finite solutions do exist! Halpern¹²⁾ has been able explicitly to solve (5.6) in the approximation of replacing the kernel by its asymptotic (= "most singular") part; this is defined by taking $m = 0$, $h(\nu) \approx \nu$ and $\rho(\nu) \approx 1$ and leads to the integral equation

$$D_\ell(\nu) = 1 - \frac{g}{2\pi} \nu \int_{-\infty}^0 d\nu' \frac{\log \frac{\nu'}{\nu}}{\nu' - \nu} \nu' D_\ell(\nu') \quad (5.8)$$

Note that the same integral equation (in the "most singular" approximation) holds if, instead of a massless spin 2 boson, a pair of vector bosons with complete propagators [i.e., $(g_{\mu\nu} - q_\mu q_\nu / m^2) / (q^2 - m^2)$] is exchanged in the Born term. Now, a solution of (5.8) is¹²⁾

$$D_\ell(\nu) = \frac{i}{2} \int_{-\infty}^{\infty} dw e^{i w \log(-\nu)} \coth \pi(w + i\epsilon) \exp \left\{ i w \log(2\pi g) - \frac{1}{2} \rho \int_{-\infty}^{\infty} \frac{dx e^{-ixw}}{x \sinh^2 \frac{x}{2}} \right\} \quad (5.9)$$

One of the basic features of (5.9) is the existence of a singularity in complex g , at $g = 0$. This is a permanent feature for all the N/D solutions (homogeneous or not), which correspond to asymptotically increasing left-hand inputs. As a result, the iteration solution

of (5.8) in powers of g is termwise divergent. This is in contrast to the scalar-vector N/D model, where one solution expandable in g at $g = 0$ was shown to exist always¹⁰⁾.

In the ladder approximation of the B-S equation solutions non-expandable in powers of g have been obtained (after a Wick rotation) in the four-fermion interaction⁴⁾, vector boson (W) theory⁵⁾ and interaction of scalar mesons with massless spin 2 bosons⁶⁾. Again the iteration series solution is termwise divergent; a renormalization procedure would require an infinity of renormalization constants. Therefore these theories may be termed non-renormalizable.

Non-analyticity in g at $g = 0$ and non-existence of perturbation expansion is also a typical feature of the solutions of the Schrödinger equation for potentials which at the origin behave like $r^{-\beta}$, $\beta > 2$ ²⁶⁾.

The N/D approach towards an explicit solution with non-renormalizable properties (like (5.9)) is free of certain difficulties of the equivalent B-S approach (connected with the Wick rotation and a possible violation of certain field-theoretic axioms²⁷⁾). There is, however, the following pathological feature¹²⁾: the asymptotic expression of (5.9) for $|\nu| \rightarrow \infty$ contains terms of the form $\sim \exp [\pm ic \log^2(-\nu)]$, where c independent of ν . As a result, $D_\ell(\nu)$ undergoes an infinity of oscillations as $\nu \rightarrow -\infty$ [this should be anticipated in view of the Phragmén-Lindelöf theorem and of the fact that for ν sufficiently large and negative the left-hand discontinuity (5.5) exceeds the unitarity limit (4.16)]. Now, this implies an infinite number of zeros of $D_\ell(\nu)$ accumulating at $\nu = -\infty$ and is a permanent trouble of the N/D models with increasing left-hand discontinuities²⁸⁾.

24.

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- 28) A somewhat similar trouble appears in attractive strongly singular potentials : as they overcome the centrifugal potential at short distances they acquire a discrete spectrum of energy states which extends to minus infinity.

FIGURE CAPTIONS

- Figure 1 The displacement of zeros of $\sin(\pi s) - \lambda$ for real values of λ and the contour of integration for the resolvent (3.3).
- Figure 2 The displacement of zeros of $\sin^2(\pi s) - \lambda$ for real values of λ and the contour of integration for the resolvent (3.8).
- Figure 3 The Pomeranchuk trajectories corresponding to various values of the parameter $r = \mu/m$ compared to the experimental data of Ref. ¹⁷⁾. (P-R) denotes the Pomeranchuk trajectory employed in the analysis of Phillips and Rarita (Ref. ¹⁹⁾).

FIG. 1

FIG. 2

FIG. 3