CM-P00057395

LIMITING PROCEDURES FOR SINGULAR POTENTIALS - III

Henri Cornille

CERN--Geneva

ABSTRACT

In previous works, for singular potentials, the validity of the use of the cut-off $\theta(r - \xi)$ has been established. Here we show that, in potential scattering, a sufficient condition for the use of non singular cut-off potentials $V(\lambda, r, \xi)$ is:

- i) $V(\lambda, r, \xi)$ for $r > 0$ and $0 \leq \xi \leq \xi_0$ is continuous in (r, ξ) , (or even weaker conditions), and V is well behaved when $r \rightarrow \infty$.
- ii) $V(\lambda, r, \xi) > 0$ for $0 \leq r \leq r_0$, $0 \leq \xi \leq \xi_0$, (r_0 independent of ξ).

If instead of ii) $V(\lambda, r, \xi)$ is attractive near the origin, the situation is more delicate but it turns out that the study of the roots of the cut-off dependent wave function is the key in order to establish the validity of the procedure. There are two cases following that the number of roots increases or does not increase when the cut-off decreases. In the first case we get that the limit of the cut-off going to zero gives a solution (the regular physical solution or a singular solution) and in the second case the limit does not exist. In previous works the properties of the Fredholm formulation of cut-off dependent solutions have been studied in the case $\lambda V(r) \theta(r - \xi)$. Here we study the possibility to extend these properties to other types of $V(\lambda, r, \xi)$.

ERRATA

LIMITING PROCEDURES FOR SINGULAR POTENTIALS - III

TH. 609, 23 September, 1965

by H. Cornille

-
- Abstract, line 15: Replace "increases or does not increase" by "does not increase or increases".
- Page 7, line 8 : Replace " $(9) \Leftrightarrow (9.a)$ " by " $(9) \Rightarrow (9.a)$ ".
- Page 9, Eq. (8'): Replace " $\lambda V(\lambda, x')$ " by " $V(\lambda, x')$ ".
line 13: Replace " $\Upsilon_q(k, r, \epsilon) = \lambda^q \Upsilon_q(k, r)$ " by " $\Upsilon_q(\lambda, k, r, \epsilon) = \lambda^q \Upsilon_q(k, r, \epsilon)$ ".
- Page 16, line 21: Replace " $|\Upsilon'| > 0$ " by " $\Upsilon' \Upsilon > 0$ ".
- Page 17, line 11: Replace " $\lim_{\epsilon \rightarrow 0} | \quad | = 0$ " by " $\lim_{\epsilon \rightarrow 0} | \quad | = 0$ for all r ^{*}".
*) It is sufficient to prove this relation for $0 < r \leq X_0$ because from (6) and (1) we get that if $\lim_{\epsilon \rightarrow 0} \Upsilon(r, \epsilon) = \Upsilon_R(r)$ is true for $0 < r \leq X_0$, it is also true for any r ."
- Page 20, line 24: The sentence "After an infinite..... sign" should be cancelled.
- Page 22, line 13: Replace " $\delta \leq 1$ " by " $\delta \geq 1$ ".
- Page 23, line 17: Replace "in the case $\delta = 1$ " by "in the case $\delta = 0$ ".
- Page 24, line 1 : In the formula " $r \left[1 + \left(\frac{r}{\epsilon} \right)^3 \right]$ " must be read " $r \left[1 + \left(\frac{r}{\epsilon} \right) \right]^3$ ".
- Page 27, Eq. (15): Replace $\lim_{\epsilon \rightarrow 0} \frac{T(k, \epsilon, 0)}{R(-k, \epsilon, 0)}$ by $r \lim_{\epsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{T(k, r, \epsilon)}{R(-k, r, \epsilon)}$ ".

INTRODUCTION

In previous works ^{1),2)} for singular potentials $\lambda V(r)$, where the most singular part is repulsive, it has been shown that if we introduce a cut-off $\lambda V(r)\theta(r-\epsilon)$ we get the following properties :

- i) the Fredholm formulation of the physical solution in co-ordinate space is such that both the numerator and the denominator diverge in the same way when $\epsilon \rightarrow 0$, the limit of the ratio being the right solution and we can obtain approximate methods by connecting in a certain manner $\epsilon \rightarrow 0$ and the order of Fredholm's determinants going to ∞ ;
- ii) similar properties for the Lippmann-Schwinger equation;
- iii) the prescription of summing up the first leading singularities in the perturbation expansion can be very dangerous when the most singular part is not of the power type.

It is the aim of this paper to show that these properties are not restricted to this particular choice of cut-off potential and we will see explicitly that they are valid for instance in the case $\lambda V(r+\epsilon)$.

In fact, the first stage before the use of our method of connecting two limiting procedures or any other approximate method using cut-off potentials $V(\lambda, r, \epsilon)$ is the proof of the validity of the cut-off procedure. This validity has been assumed by Khuri and Pais ³⁾ for a restricted class of potentials. On the other hand, Calogero ⁴⁾ has given explicit counter examples (not in the case considered by Khuri and Pais) where the cut-off does not work. Here we give also some counter examples, also in the case considered by Khuri and Pais ³⁾, but in fact we are more interested in a positive aspect of the question. We want to give some general features of "honest" working cut-off potentials not singular when $r \rightarrow 0$. We recall that a potential $V(\lambda, r)$ is said singular when $r \rightarrow 0$ if $\int_0^b r |V(\lambda, r)| dr$ is divergent for any fixed b . In this paper we show that sufficient conditions for the use of cut-off potentials are mainly :

2.

- 1) continuity in (r, ϵ) for $r > 0$ [or even weaker conditions, see (A.2)] and good behaviour when $r \rightarrow \infty$ [condition (A.1)]:
- 2) $V(\lambda, r, \epsilon)$ repulsive near the origin in a range independent of ϵ .

We note that these conditions are satisfying physical assumptions because $V(\lambda, r)$ is repulsive near the origin.

A study is also made of the more delicate case where $V(\lambda, r, \epsilon)$ is attractive near the origin. Necessarily V is < 0 in a small range going to zero when $\epsilon \rightarrow 0$. If the potential is sufficiently attractive, then there appear roots of the wave functions going also to zero when $\epsilon \rightarrow 0$. Then it is shown that these roots and stationary points of a function associated to the wave function give the necessary information about the validity of the cut-off procedure. If the potential is not too attractive, then the number of roots remains finite when $\epsilon \rightarrow 0$ and we can follow the oscillations of the wave functions and see when $\epsilon \rightarrow 0$ if we get the physical regular solution or a singular solution. If the number of roots increases when $\epsilon \rightarrow 0$, then the number of oscillations of the wave function increases also and we find that the limit of the cut-off dependent wave function does not exist.

P A R T O N E

SOME CONDITIONS ABOUT THE VALIDITY OF CUT-OFF PROCEDURES IN
POTENTIAL SCATTERING

a) Assumptions about the cut-off potentials

We consider $V(\lambda, r)$, singular potentials when $r \rightarrow 0$, (see the introduction) and where the most singular part is repulsive [there is some range $0 < r \leq \text{finite constant}$ where $V > 0$ and $\lim_{r \rightarrow 0} V(\lambda, r) \rightarrow +\infty$.] We introduce a cut-off $\epsilon \gg 0$ such that $V(\lambda, r, \epsilon)$ is a real potential, non-singular when $r \rightarrow 0$, and coincides with $V(\lambda, r)$ for $\epsilon = 0$. Contrary to the previous cases ^{1), 2)} $\Theta(r - \epsilon)$, we assume that ϵ and r can go independently to zero. In supplement, we assume the following properties, (A.1) - (A.2), for $\epsilon \gg 0$:

$$(A.1) \quad \int_r^\infty r'^2 |V(\lambda, r', \epsilon)| dr' \quad \text{exists.}$$

$$(A.2) \quad \text{for any finite } a \gg 0, \quad b > 0 :$$

$$\int_a^b r |V(\lambda, r, \epsilon)| dr \quad \text{exists, (if } \epsilon = 0, \quad a \neq 0)$$

$$\text{for any finite } a > 0, \quad b > 0 :$$

$$\int_a^b r |V(\lambda, r, \epsilon) - V(\lambda, r)| dr \Rightarrow 0 \quad \epsilon \rightarrow 0$$

For instance, if $V(\lambda, r, \epsilon)$ is continuous in (r, ϵ) for $r > 0$ and $0 \leq \epsilon \leq \epsilon_0$, then we see that (A.2) is satisfied. We consider for simplicity S waves ^{*)}. Since the difficulties of singular potentials

*) There are no difficulties to extend to $\ell \neq 0$.

4.

are not connected with the behaviour for large r , we want that the cut-off introduces no difficulties for solutions defined by asymptotic behaviour. For instance, if we consider the cut-off dependent Jost solutions because of (A.1), we have for $\epsilon \gg 0$:

$$R(\pm k, r, \epsilon) \underset{r \rightarrow \infty}{\simeq} e^{\mp i k r}$$

It is shown in Appendix A that the properties (A.1) and (A.2) give

$$|R(k, r, \epsilon) - R(k, r)| \underset{\epsilon \rightarrow 0}{\rightarrow} 0 \quad (r \neq 0) \quad (1)$$

b) General conditions

We now write the physical (cut-off dependent) integral equation

$$\psi(k, r, \epsilon) = \frac{\sin k r}{k} + \int_0^\infty K(k, r, r', \epsilon) \psi(k, r', \epsilon) dr' \quad (2)$$

$$F(k, \epsilon) = - \int_0^\infty \frac{\sin k r}{k} V(r, \epsilon) \psi(k, r, \epsilon) dr \quad (3)$$

where $K(k, r, r', \epsilon) = -k^{-1} V(r, \epsilon) g(r, r', k)$

$$g(r, r', k) = \sin k r' e^{i k r} \Theta(r-r') + \sin k r e^{i k r'} \Theta(r'-r) \quad (4)$$

First we consider the Born series of (2) where $\Psi_q(\lambda, k, r, \epsilon)$ is the q th term, if we substitute this series in (3) we get the perturbation expansion where we call $a_q(\lambda, k, \epsilon)$ the q th term

$$\Psi(k, r, \epsilon) = \sum_{q=0}^{\infty} \Psi_q(\lambda, k, r, \epsilon) \quad F(k, \epsilon) = \sum_{q=1}^{\infty} a_q(\lambda, k, \epsilon) \quad (5)$$

Secondly, similarly as in Refs. 1) and 2), we investigate the solution of (2) in another manner by a change of the integration path in (2) $[0, r] \rightarrow [0, \infty] + [\infty, r]$. Then for the study of (2) and (3) we get another formalism equivalent to (5) and giving for instance the continuation when $\epsilon \rightarrow 0$ of the series $\sum_q a_q(\lambda, k, \epsilon)$

$$\Psi(k, r, \epsilon) = T(k, r, \epsilon) + F(k, \epsilon) R(-k, r, \epsilon) \quad (6)$$

$$F(k, \epsilon) = \frac{-k^{-1} \int_0^{\infty} T(k, r, \epsilon) \sin kr V(\lambda, r, \epsilon) dr}{1 + \int_0^{\infty} k^{-1} R(-k, r, \epsilon) \sin kr V(\lambda, r, \epsilon) dr} = \frac{-T(k, 0, \epsilon)}{R(-k, 0, \epsilon)} \quad (7)$$

In (6) and (7), T and R are the solutions of the Volterra equation

$$\chi(k, r, \epsilon) = \chi_0(k, r) + \int_r^{\infty} \frac{\sin k(r'-r)}{k} V(\lambda, r', \epsilon) \chi(k, r', \epsilon) dr' \quad (8)$$

where $\chi_0 = k^{-1} \sin kr$ for T and $\chi_0 = e^{\pm ikr}$ for $R(\pm k, r, \epsilon)$.

Then T and R satisfy the properties (1) if (A.1) and (A.2) are assumed for $V(\lambda, r, \epsilon)$.

The conditions of the validity of the use of the cut-off are :

$$\begin{cases} \lim_{\epsilon \rightarrow 0} \Psi(k, r, \epsilon) = \Psi_R(k, r) \\ \lim_{\epsilon \rightarrow 0} F(k, \epsilon) = F(k) = \frac{e^{i\delta} \sin \delta}{k} \end{cases} \quad (9)$$

where $\Psi_R(k, r)$ and $F(k)$ are the corresponding right physical quantities for $V(\lambda, r)$. [$\Psi_R(k, r)$ is a regular solution when $r \rightarrow 0$.] As we will see, these conditions are not independent. In fact the study of the validity of the cut-off has two different stages :

- i) is $\lim_{\epsilon \rightarrow 0} \Psi(k, r, \epsilon)$ a solution of the Schrödinger equation ^{*)} for $V(\lambda, r)$ and also does $\lim_{\epsilon \rightarrow 0} F(k, \epsilon)$ exist ?
- ii) are these limits the right physical quantities ?

First we note that with the conditions (A.1) and (A.2), because (1) is satisfied, from (6) we get that: $\lim_{\epsilon \rightarrow 0} \Psi(k, r, \epsilon)$ is a solution for $V(\lambda, r) \iff \lim_{\epsilon \rightarrow 0} F(k, \epsilon)$ exists, or $\lim_{\epsilon \rightarrow 0} \Psi(k, r, \epsilon)$ exists $\iff \lim_{\epsilon \rightarrow 0} F(k, \epsilon)$ exists.

Secondly, from (6) we see that $\Psi(k, r, \epsilon)$, ($\epsilon \neq 0$), is a linear combination of the two singular ($\underset{r \rightarrow 0}{\sim} \text{const}$) solutions T and R , and from (7) we see that $F(k, \epsilon)$ is such that the linear combination is regular near the origin ($\underset{r \rightarrow 0}{\sim} \text{const } r$).

Now we want that these features are conserved when $\epsilon = 0$. More explicitly, $\lim_{\epsilon \rightarrow 0} \Psi(k, r, \epsilon)$ must be a linear combination of the two singular ($\epsilon = 0$) solutions $T(k, r)$, $R(-k, r)$ such that $\lim_{\epsilon \rightarrow 0} F(k, \epsilon)$ removes the singular part ($\epsilon = 0$) from the linear combination.

*) That this problem is a real problem in general will be emphasized in the following when we shall give examples of cut-off potentials such that $V(\lambda, r, \epsilon = 0) = V(\lambda, r)$ but where $\lim_{\epsilon \rightarrow 0} \Psi(k, r, \epsilon)$ does not exist. There is also a preliminary problem which is to take the properly normalized regular solution when $\epsilon \neq 0$. But we have taken into account this condition by the choice that the regular cut-off solution $\Psi(k, r, \epsilon)$ is the solution of (2).

From (1), (2), (6) and (7) it is easy to see that the necessary and sufficient conditions such that Eqs. (9) are satisfied are either

$$\lim_{\epsilon \rightarrow 0} \lim_{r \rightarrow 0} \psi(k, r, \epsilon) = \lim_{r \rightarrow 0} \lim_{\epsilon \rightarrow 0} \psi(k, r, \epsilon) = 0 \quad (9.a)$$

$$\text{or } *) \lim_{r \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{T(k, r, \epsilon)}{R(-k, r, \epsilon)} = \lim_{\epsilon \rightarrow 0} \lim_{r \rightarrow 0} \frac{T(k, r, \epsilon)}{R(-k, r, \epsilon)} \quad (9.b)$$

For instance, if we assume (9.a) from (6) and (7) we get

$$\lim_{\epsilon \rightarrow 0} F(k, \epsilon) = - \lim_{r \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{T(k, r, \epsilon)}{R(-k, r, \epsilon)} = - \lim_{\epsilon \rightarrow 0} \frac{T(k, 0, \epsilon)}{R(-k, 0, \epsilon)} = F(k)$$

and

$$\lim_{\epsilon \rightarrow 0} \psi(k, r, \epsilon) = T(k, r) - \left(\lim_{x \rightarrow 0} \frac{T(k, x)}{R(-k, x)} \right) R(-k, r) = \psi(k, r)$$

Inversely, (9) $\Leftrightarrow$ (9.a). In the same manner we have (9.b) $\Leftrightarrow$ (9).

c) Approximation of the first leading singularities for $V(\lambda, r, \epsilon) = \lambda V(r, \epsilon)$
 $V(\lambda, r, \epsilon) = \lambda V(r, \epsilon)$

Now we assume that (1) and either (9) or (9.a) or (9.b) are satisfied. As in the two preceding works ^{1), 2)} we consider the case $\lambda V(r)$. We have $\gamma_q(k, r, \epsilon) = \lambda^q \gamma_q(k, r)$ and $a_q(\lambda, \epsilon) = \lambda^q a_q(\epsilon)$. We study the approximation of summing up the first leading singularities $\bar{F}(k, \epsilon) = \sum_q \lambda^q \bar{a}_q(k, \epsilon)$. [In the following we call $\bar{X}$ the first leading singular part of X .]

Because of (1) and (9.b) we have

$$\lim_{\epsilon \rightarrow 0} F(k, \epsilon) = \lim_{\epsilon = r \rightarrow 0} \frac{-T(k, r)}{R(-k, r)}$$

$$\lim_{\epsilon \rightarrow 0} \bar{F}(k, \epsilon) = - \lim_{\epsilon = r \rightarrow 0} \frac{\sum \lambda^q \bar{T}_q(k, r)}{\sum \lambda^q \bar{R}_q(-k, r)}$$

*) It is easy to see that for $\ell \neq 0$ (9.b) is

$$\lim_{\epsilon \rightarrow 0} \lim_{r \rightarrow 0} \frac{T_\ell(k, r, \epsilon)}{R_\ell(-k, r, \epsilon)} = \lim_{r \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{T_\ell(k, r, \epsilon)}{R_\ell(-k, r, \epsilon)}$$

where T_ℓ and R_ℓ are the corresponding $\ell \neq 0$ solutions.

where $T_q(k, r)$ and $R_q(-k, r)$ are for T and R the q th order term of the Volterra equation (8) with $\epsilon = 0$. Then we get the result that anyway we have introduced the cut-off if (9) or (9.a) or (9.b) is satisfied $\epsilon \bar{F}_0(k, \epsilon)$ gives always the same result. On the other hand this result is the same as the one we have previously found ^{1), 2)} for the particular choice $\lambda V(r)\theta(r-\epsilon)$. Thus, for the sum of the first leading singularities we have the same result as in Refs. 1) and 2) in the two cases where the most singular part is of the power type or not. For instance, we recall that if the most singular part is like $\lambda r^{-2n} \exp(\gamma/r^\delta)$ [γ arbitrarily > 0 , δ arbitrarily > 0 , n arbitrarily $> \frac{3}{2}$], $\lim_{\epsilon \rightarrow 0} \bar{F}(k, \epsilon)$ gives always the same result $-\epsilon$ ^{*)}. Then we see that, firstly, this

*) We recall that H.H. Aly, Riazuddin and A.H. Zimmerman [Nuovo Cimento 35, 324 (1964)] have been the first to quote this result for the sum of the first leading singularities in the case $V = (\lambda e^{2/r})/r^4$. In fact, their work is mainly concerned with $V = \lambda (e^{2/r}/r^4 + 1/4\lambda r^4)$. But at the end of their paper (p.328) where they discussed the approximation of the first leading singularities for the whole potential, it is easy to see that among other results [concerning this whole potential which is not of the family $\lambda V(r)$ and consequently not considered here or in Ref. 1)] they have got also a result which is that the sum of the first leading singularities in the simple case $\lambda e^{2/r}/r^4$ is $-\epsilon$. That this result about $V = \lambda e^{2/r}/r^4$ is true for $\lambda V(r)\theta(r-\epsilon)$ was shown in Ref. 1) because it was recovered by the analytical continuation of the first leading singularities when $\epsilon \rightarrow 0$. That this result is also true for any kind of $\lambda V(r, \epsilon)$ satisfying (1), (9), (9.a) and (9.b) is shown here in this paper. But this does not mean that this result is true for any kind of $\lambda V(r, \epsilon)$ satisfying the assumptions of Khuri and Pais ³⁾ because, as it will be proved in the following, there exist pathological potentials of the class of Khuri and Pais where $\lim_{\epsilon \rightarrow 0} Y(k=0, r, \epsilon)$ does not exist. [In these pathological cases when the whole cut-off dependence has no limit when $\epsilon \rightarrow 0$, we are not sure it will be very interesting to test if some partial dependence, like the sum of the first leading singularities, has a meaning or not when $\epsilon \rightarrow 0$; this is why we have not considered these potentials in this section.]

approximation is not good and, secondly, it is not able to reproduce any features of the potential, thus this approximation gives nonsensical answer when $\epsilon \rightarrow 0$.

d) A particular case $V(\lambda, r+\epsilon)$

We want now to show that (9.b) is valid for $V(\lambda, r, \epsilon) = V(\lambda, r+\epsilon)$. In (8) we put $x = r+\epsilon$, $x' = r'+\epsilon$ and call $\chi(k, r, \epsilon) = \chi(k, x-\epsilon, \epsilon) = \Phi(k, x, \epsilon)$: we get

$$\Phi(k, x, \epsilon) = \chi_0(k, x-\epsilon) + \lambda \int_x^\infty \frac{\sin k(x'-x)}{k} V(\lambda, x') \Phi(k, x', \epsilon) dx' \quad (8')$$

First we see that in the kernel of the integral equation (8'), ϵ has disappeared and secondly in the two cases for $\chi_0(k, x, \epsilon)$ we can factorize the ϵ dependence. Finally we get

$$\begin{aligned} T(k, r, \epsilon) &= T(k, r+\epsilon) \cos \epsilon - \frac{\sin \epsilon}{2k} [R(-k, r+\epsilon) + R(k, r+\epsilon)] \\ R(-k, r, \epsilon) &= e^{-ik\epsilon} R(-k, r+\epsilon) \end{aligned} \quad (10)$$

and for $k=0$:

$$\begin{aligned} T(0, r, \epsilon) &= T(0, r+\epsilon) - \epsilon R(0, r+\epsilon) \\ R(0, r, \epsilon) &= R(0, r+\epsilon) \end{aligned}$$

where we recall that $T(k, z)$ and $R(-k, z)$ are the solutions of (8) without cut-off. First we verify that (1) is satisfied in this case and secondly it is easy to see that we can reverse the order of the two limiting procedures in (9.b). Thus, for $V(\lambda, r+\epsilon)$ the use of the cut-off is always valid. For instance, for $k=0$ we have:

$$\Psi(0, r, \epsilon) R(0, \epsilon) = T(0, r + \epsilon) R(0, \epsilon) - R(0, r + \epsilon) T(0, \epsilon)$$

and for the particular case $V = \lambda / r^4$ we recall that ¹⁾

$$T(0, r) = r \operatorname{ch}(\lambda^{1/2}/r) \quad R(0, r) = r \lambda^{-1/2} \operatorname{sh}(\lambda^{1/2}/r)$$

Then

$$\Psi(0, r, \epsilon) = (r + \epsilon) \operatorname{sh}\left(\frac{\lambda^{1/2} r}{\epsilon(r + \epsilon)}\right) \left(\operatorname{sh}\left(\frac{\lambda^{1/2}}{\epsilon}\right)\right)^{-1}$$

and (9), (9.a) and (9.b) are satisfied.

But in the general case $V(\lambda, r, \epsilon)$ we cannot, as for $V(\lambda, r + \epsilon)$, reduce $T(k, r, \epsilon)$ and $R(-k, r, \epsilon)$ to a linear combination of the solution $R(\pm k, z)$ such that it is not so easy to test directly any one of the conditions (9), (9.a) and (9.b).

e) Some counter examples ($k=0$)

1. First we retain as assumptions about the cut-off potentials only the hypothesis that $\lim_{\epsilon \rightarrow 0} V(\lambda, r, \epsilon) = V(\lambda, r)$. For instance, in the case $V = \lambda / (r + f(\epsilon))^4$, where $f(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$, the solution is :

$$\Psi(r, \epsilon) = (r + f(\epsilon)) \left[\frac{e^{-\lambda^{1/2}/(r+f(\epsilon))}}{1 - e^{-2\lambda^{1/2}/f(\epsilon)}} + \frac{e^{\lambda^{1/2}/(r+f(\epsilon))}}{1 - e^{2\lambda^{1/2}/f(\epsilon)}} \right]$$

- i) if $f(\epsilon) = -\epsilon$ we get $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = r \exp(\lambda^{1/2}/r) = \Psi_{\text{sing}}(r)$;
- ii) if $f(\epsilon) = \epsilon \sin \frac{1}{\epsilon}$ then $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon)$ does not exist;
- iii) if ^{*)} $f(\epsilon) = |\epsilon| e^{i\theta}$ ($\theta = \text{const}$) we get for $|\theta| < \frac{\pi}{2}$,
 $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_R(r)$, for $\frac{\pi}{2} < |\theta| < \pi$, $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_{\text{sing}}(r)$; for $\theta = \pm \frac{\pi}{2}$, $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon)$ does not exist.

*) We wish to thank Dr. M.B. Halpern who pointed out to us the interest of such a type of complex cut-off in connection with field theory (M.B. Halpern, to be published).

Of course, these potentials are pathological, for instance in case i) we have only shifted the essential singularity of the Schrödinger solutions from $r=0$ for $V(\lambda, r)$ to $r=\epsilon$ for $V(\lambda, r, \epsilon)$. Then such types of counter examples are not significant.

2. Secondly we still assume for $V(\lambda, r, \epsilon)$ the conditions given in a). We note that when we take $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon)$ we have in general a linear combination of $\Psi_R(r)$, the regular ($\epsilon=0$) solution, and $\Psi_{\text{sing}}(r)$, a singular ($\epsilon=0$) solution

$$\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_R(r) \lim_{\epsilon \rightarrow 0} a(\epsilon) + \Psi_{\text{sing}}(r) \lim_{\epsilon \rightarrow 0} b(\epsilon)$$

If (9) is satisfied $\lim_{\epsilon \rightarrow 0} a(\epsilon) = 1$ and $\lim_{\epsilon \rightarrow 0} b(\epsilon) = 0$. A simple situation is when we get only Ψ_R or Ψ_{sing} . Then it is useful to consider potentials like $\lambda V_1(r, \epsilon) + \lambda^{\frac{1}{2}} V_2(r, \epsilon)$ where Ψ for $\epsilon \neq 0$ depends on $\lambda^{\frac{1}{2}}$. For $k=0$ we always put

$$\Psi(r, \epsilon) = r \exp\left(\lambda^{\frac{1}{2}} \gamma \int_r^\infty \phi(r', \epsilon) dr'\right) \quad \gamma = \pm 1$$

where the conditions for Ψ can be taken into account very easily [normalization when $r \rightarrow \infty$, regularity when $r \rightarrow 0$ if $\phi(r, \epsilon)$ is integrable when $r \rightarrow 0$]. The corresponding $V(\lambda, r, \epsilon)$ is

$$V(\lambda, r, \epsilon) = \lambda \phi^2 - \gamma \lambda^{\frac{1}{2}} \left(\phi' + \frac{2\phi}{r} \right)$$

Then it is not difficult to find ϕ such that we get very simple counter examples and such that the assumptions a) about $V(\lambda, r, \epsilon)$ are satisfied. For instance, for

$$V(r) = \frac{\lambda}{r^{2n}} - \gamma \frac{(2-n)\lambda^{\frac{1}{2}}}{r^{n+1}} \quad n \geq 2 \quad (11)$$

we take :

i)

$$\phi = (r+\epsilon)^{-n}, \quad \psi_\gamma(r, \epsilon) = r \exp \left[\gamma \lambda^{\frac{1}{2}} / (n-1)(r+\epsilon)^{n-1} \right] \quad (12.a)$$

$$V_\gamma = \frac{\lambda}{(r+\epsilon)^{2n}} \frac{-\gamma \lambda^{\frac{1}{2}} [(2-n) + \frac{2\epsilon}{r}]}{(r+\epsilon)^{n+1}} \quad (11.a)$$

ii) *)

$$\phi = r^{-n} e^{-\epsilon/r^\beta}, \quad \psi_\gamma = r \exp \left(\gamma \lambda^{\frac{1}{2}} \int_r^\infty \frac{e^{-\epsilon'/r'^\beta}}{r'^n} dr' \right) \quad (12.b)$$

$$V_\gamma = \frac{\lambda}{r^{2n}} e^{-2\epsilon/r^\beta} \frac{-\gamma \lambda^{\frac{1}{2}}}{r^{n+1}} \left[2-n + \frac{\beta \epsilon}{r^\beta} \right] e^{-\epsilon/r^\beta} \quad (11.b)$$

($r > 0, \beta > 0$)

iii)

$$\phi = \exp \left[-n e^{-\epsilon/r} \log r \right] \quad (11.c)$$

*) Calogero ⁴⁾ has given previously such types of counter examples $[n=2, \gamma=1, \beta=1, \gamma=+1 \text{ in (11.b), (12.b)}]$ of (9) or (9.b). But we disagree with his interpretation that such types of difficulties are connected with the impossibility to invert the two limiting procedures $\epsilon \rightarrow 0$ and $r \rightarrow \infty$. In fact, his interpretation is established from the investigation of this particular case ($n=2, \gamma=1, \beta=1, \gamma=+1$) where the corresponding ψ is $\psi(r, \epsilon) = r \exp \left[\frac{\lambda^{\frac{1}{2}}}{\epsilon} (1 - e^{-\epsilon/r}) \right]$. He claims that for ψ if we take first r large we get $r + \lambda^{\frac{1}{2}}$ whereas if we take first $\epsilon \rightarrow 0$ we get $r - \lambda^{\frac{1}{2}}$. But this result is wrong because in the two cases we get in reality $r + \lambda^{\frac{1}{2}}$. (We wish to thank Dr. F. Calogero for a correspondence on this subject.) Now, if we consider all counter examples $\psi_{\gamma=+1}$ given here, (11.a)-(11.b), we get in all cases that we can always invert $\epsilon \rightarrow 0$ and $r \rightarrow \infty$. Then it does not seem that the possibility (or not) to invert $\lim_{r \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \psi(k, r, \epsilon)$ is really related to the problem of the validity of the cut-off procedure. This can be easily understood because with (A.1) and (A.2) we get (1) and from (6) if $\lim_{\epsilon \rightarrow 0} F(k, \epsilon)$ exists then we can always invert $\epsilon \rightarrow 0$ and $r \rightarrow \infty$ in $\psi(k, r, \epsilon)$.

In all cases we get $\lim_{\epsilon \rightarrow 0} \Psi_{\gamma=-1}(r, \epsilon) = \Psi_R(r)$ and $\lim_{\epsilon \rightarrow 0} \Psi_{\gamma=+1}(r, \epsilon) = \Psi_{\text{sing}}(r)$. Many other ϕ of this type can be found. The fact that we know that $V(\lambda, r+\epsilon)$ gives the right result and the similarity of the construction of these counter examples (12.a)-(12.c), gives us the feeling that the use of $e^{-\epsilon/r^\beta} V(r)$ or $(r+\epsilon)^{-n}$ or $r^{-n} \exp(-\epsilon/r)$ is indifferent to the validity of the cut-off procedure. We want to remark that in (11.a)-(11.c) we have always instead of (9.a)

$$\lim_{\epsilon \rightarrow 0} \lim_{r \rightarrow 0} \Psi(r, \epsilon) \neq \lim_{r \rightarrow 0} \lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) \quad (\gamma = +1)$$

But for such types of difficulties where we cannot invert two limiting procedures, we recall that it can be useful to introduce an arbitrary dependence $\epsilon(r) \xrightarrow{r \rightarrow 0} 0$. Generally we find limiting dependences such that

$$\lim_{r \rightarrow 0} \Psi(r, \epsilon(r)) \Rightarrow \begin{cases} \lim_{r \rightarrow 0} \lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) & \text{if } \epsilon(r) < \epsilon_L(r) \\ \lim_{\epsilon \rightarrow 0} \lim_{r \rightarrow 0} \Psi(r, \epsilon) & \text{if } \epsilon(r) \geq \epsilon_L(r) \end{cases}$$

For instance for (11.a) and $\gamma = +1$ when (9.a) is not satisfied

$$\lim_{r \rightarrow 0} \Psi(r, \epsilon(r)) \Rightarrow \begin{cases} \infty & \text{if } \epsilon(r) < \frac{(\frac{\lambda}{n-1})^{\frac{1}{n-1}} + \epsilon'}{|\log r|^{\frac{1}{n-1}}} = \epsilon_L(r) \\ 0 & \text{if } \epsilon(r) > \epsilon_L(r) \end{cases} \quad \epsilon' > 0$$

whereas for $\gamma = -1$ where we can invert $\epsilon \rightarrow 0, r \rightarrow 0$, we always find $\lim_{r \rightarrow 0} \Psi(r, \epsilon(r)) \rightarrow 0$. We have also similar results for (11.b) and (11.c) in the two cases $\gamma = \mp 1$.

At the end we note that for the given examples ^{*)}, (11.a)-(11.c), we have the following features :

- i) for $\eta = -1$: $V(r, \epsilon) > 0$ and $\eta \phi(r, \epsilon) < 0$ for r small and $\epsilon = 0$ or $\epsilon \neq 0$: $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_R(r)$;
- ii) for $\eta = +1$: $V(r, \epsilon) < 0$ and $\eta \phi(r, \epsilon) > 0$ for r small (ϵ fixed $\neq 0$): $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_{\text{sing}}(r)$.

On the other hand, we have not been able to find counter examples [by other choice of $\phi(r, \epsilon)$] such that $V(r, \epsilon) > 0$ for $r < R_0$ small and $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) \rightarrow \Psi_{\text{sing}}(r)$. (R_0 independent of ϵ).

f) $V(\lambda, r, \epsilon)$ repulsive near the origin ($k=0$)

Now, in addition to the assumptions made in a), [mainly (A.1) and (A.2)], we suppose :

$$(A.3) \quad V(\lambda, r, \epsilon) > 0 \quad \text{for} \quad 0 \leq r < R_0, \quad 0 \leq \epsilon \leq \epsilon_0$$

$$(\underline{R_0 \text{ independent of } \epsilon}) .$$

^{*)} We recall that the counter examples given above, e)-1. where $f(\epsilon) = -\epsilon$, $\epsilon \sin \frac{1}{\epsilon}$, $|\epsilon|e^{i\theta}$ do not satisfy the assumptions made in a) and are not considered here. But we remark that if in (12.a) we take $\phi = (r+f(\epsilon))^{-n}$, in all cases we get that $\lim_{\epsilon \rightarrow 0} \Psi_{\eta=-1}(r, \epsilon) = \Psi_R$ and $\lim_{\eta=+1} \Psi(r, \epsilon) = \Psi_{\text{sing}}$. In these cases the results do not depend on the choice of $f(\epsilon)$ but on the value $\eta = \pm 1$ (counter term to the potential). Although, for instance for $|\epsilon|e^{i\theta}$ a study can be made similarly as in the following sections f), g) and i), we shall not do it because we consider only potentials satisfying a).

We can always write the $\Psi(r, \epsilon)$ solution of (2) as :

$$\Psi(r, \epsilon) = r \exp \int_r^\infty \phi(\lambda, r', \epsilon) dr'$$

$$V(\lambda, r, \epsilon) = \phi^2 - \left(\phi' + 2\frac{\phi}{r} \right) = \left(\phi - \frac{1}{r} \right)^2 \left[1 + \frac{d}{dr} \left(\frac{1}{\phi - \frac{1}{r}} \right) \right] \quad (15)$$

We want to study $\phi(\lambda, r, \epsilon)$ in the domain where V is > 0 ($\epsilon \neq 0$).

1. First, we want to show that for r sufficiently small $\phi(\lambda, r, \epsilon)$ is < 0 (independently of $\epsilon \neq 0$). Because $V(\lambda, r, \epsilon)$ is not singular when $r \rightarrow 0$, (A.2), we get that $\phi(\lambda, r, \epsilon)$ is integrable when $r \rightarrow 0$.

We note that for $0 < \epsilon \leq \epsilon_0$ and $0 \leq r < R_0$, because of (A.3), Ψ vanishes only at $r=0$ such that $\exp \int_r^\infty \phi dr'$ cannot be zero: on the other hand $V(r, \epsilon)$ being real, Ψ and Ψ' are real and $\phi = \Psi'/\Psi - \frac{1}{r}$ is real. Then ϕ in the above domain in (r, ϵ) is real and has no poles. From (13) we get ^{*)} :

$$\frac{1}{\frac{1}{r} - \phi(r, \epsilon)} > C(R, r, \epsilon) \quad C(R, r, \epsilon) = R \left[\frac{1}{1 - \phi(R, \epsilon)R} - 1 \right] + r \quad (14)$$

where $r < R < R_0$.

We now consider only $\epsilon \neq 0$, because ϕ is integrable when $r \rightarrow 0$ and has no poles we can find R sufficiently small $< R_0$, such that both $\frac{1}{r} - \phi(r, \epsilon) > 0$ and $\frac{1}{R} - \phi(R, \epsilon) > 0$ for $r < R$. Then we get :

^{*)} For simplicity, in the following, we shall write $\phi(\lambda, r, \epsilon)$, defined in (13), as $\phi(r, \epsilon)$.

- i) if $\phi(R, \epsilon) > 0$, then $C > 0$ and $\phi(r, \epsilon) > \frac{1}{r} - \frac{1}{C}$: thus $\phi(r, \epsilon)$ is > 0 and non-integrable when $r \rightarrow 0$;
- ii) if $\phi(R, \epsilon) < 0$, then $C < 0$ and $\phi(r, \epsilon) < \frac{1}{r} - \frac{1}{C}$: no contradiction;
- iii) if $\phi(R, \epsilon) = 0$, then for $r < R$, $\phi(r, \epsilon) > 0$ and we can still use the argument i) for $R=r$ and find ϕ non-integrable when $r \rightarrow 0$.

In conclusion we have $\phi(r, \epsilon) < 0$ for any $r < R < R_0$, independently of $0 < \epsilon \leq \epsilon_0$. Thus, for any $r < R$ if $\lim_{\epsilon \rightarrow 0} \phi(r, \epsilon)$ exists then $\lim_{\epsilon \rightarrow 0} \phi(r, \epsilon) \leq 0$.

2. Secondly, we want to prove that $\lim_{\epsilon \rightarrow 0} \phi(r, \epsilon)$ exists and is equal to $\phi_R(\lambda, r)$ the corresponding quantity for Ψ_R . We write Ψ_R as $\Psi_R = r \exp \int_r^\infty \phi_R(\lambda, r') dr'$ where of course $\phi_R(\lambda, r)$ is now non-integrable when $r \rightarrow 0$. From the usual combination of the Schrödinger equation for Ψ and Ψ_R , we get :

$$\phi(\lambda, r, \epsilon) - \phi_R(\lambda, r) = \frac{1}{\Psi \Psi_R} \int_0^r [V(\lambda, r') - V(\lambda, r', \epsilon)] \Psi(r', \epsilon) \Psi_R(r') dr'$$

where we have used the relation $\Psi_R' \Psi - \Psi' \Psi_R = (\phi - \phi_R) \Psi \Psi_R$.

We consider $0 < r < R < R_0$ where from the previous results

$\phi(r, \epsilon) < 0$ for $\epsilon \neq 0$. We recall that for $r \neq 0$ because of

(A.3) Ψ does not vanish..

Then $\Psi' = (\frac{1}{r} - \phi) \Psi$ has the same sign as Ψ and for r sufficiently small, independently of $\epsilon \neq 0$, we have $|\Psi'| > 0$. Now we consider

$V(\lambda, r) \Psi_R(r) \Psi(r, \epsilon)$. We recall, a), that the most singular part of V is repulsive near the origin, then we can find a range $0 \leq r \leq x_0$, where $V \Psi_R \Psi$ has always the same sign, is increasing in modulus and has its maximum at x_0 ; we get :

$$\phi - \phi_R = V(r) \int_0^{r_1} \left[1 - \frac{V(\lambda, r', \epsilon)}{V(\lambda, r')} \right] dr' \quad 0 \leq r_1 \leq r \leq x_0$$

r is fixed in this relation: we want to show that the right-hand side vanishes for r fixed and $\epsilon \rightarrow 0$. We choose r_2 arbitrarily small and consider $[0, r_1] = [0, r_2] + [r_2, r_1]$. In $[r_2, r_1]$, $V^{-1}(\lambda, r) > \text{const}$ and

$$\left| \int_{r_2}^{r_1} \left[1 - \frac{V(\lambda, r', \epsilon)}{V(\lambda, r')} \right] dr' \right| < \text{const} \int_{r_2}^{r_1} |V(\lambda, r', \epsilon) - V(\lambda, r')| dr' \xrightarrow{\epsilon \rightarrow 0} 0$$

following (A.2).

On the other hand, $(V(\lambda, r, \epsilon))/V(\lambda, r)$ is > 0 and goes to zero when $r \rightarrow 0$, such that

$$\int_0^{r_2} \left[1 - \frac{V(\lambda, r', \epsilon)}{V(\lambda, r')} \right] dr' < r_2$$

then $\lim_{\epsilon \rightarrow 0} |\phi(\lambda, r, \epsilon) - \phi_R(\lambda, r)| = 0$.

Thus (A.1), (A.2), [a] and (A.3) provide sufficient conditions in order that $\lim_{\epsilon \rightarrow 0} \psi(r, \epsilon) = \psi_R(r)$.

g) $V(\lambda, r, \epsilon)$ attractive near the origin ($k=0$)

We assume always the conditions of a), (A.1) but for simplicity we replace (A.2) by the strongest one, that $V(\lambda, r, \epsilon)$ is continuous in (r, ϵ) for $r > 0$ and $0 \leq \epsilon \leq \epsilon_0$. We add now the following assumption :

$$(A.4) \quad V(\lambda, r, \epsilon) < 0 \quad \text{for } 0 < \epsilon \leq \epsilon_0, 0 < r < R_0(\epsilon)$$

$$R_0(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$$

$V(\lambda, r, 0)$ being repulsive near the origin, then necessarily the range in r where V is attractive goes to zero when $\epsilon \rightarrow 0$. We assume $V(\lambda, R_0(\epsilon), \epsilon) = 0$ and that there is only one root $R_0(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$ for V . Now, because V is attractive for $r < R_0(\epsilon)$, $\Psi(r, \epsilon)$ can have roots $r_j = D_j(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$ ($j=1, 2, \dots, n(\epsilon)$). At the right of $R_0(\epsilon)$, for ϵ fixed, we can have at most one of these roots. We call $D_{n\epsilon}(\epsilon)$ the position of the highest root for given ϵ . [There are no roots at the right of $D_{n\epsilon}(\epsilon)$, and $D_{n\epsilon}(\epsilon)$ can be at the left or at the right of $R_0(\epsilon)$].

Contrary to the previous case (where V was > 0), we take ϵ fixed, $\epsilon = \epsilon_1$ and we try to find a range for r such that these roots of $\Psi(r, \epsilon)$ [which are poles for $\phi(r, \epsilon)$] are at the right of the range. In Appendix B it is shown that :

- i) there always exists $R_1(\epsilon)$ such that there are no such roots of Ψ at the left of $R_1(\epsilon)$ and $R_1(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$;
- ii) for any finite ϵ , the number $n(\epsilon)$ of these roots is finite;
- iii) when $\epsilon \rightarrow 0$ we have two different cases following whether $n(\epsilon)$ is not increasing or whether $n(\epsilon)$ is increasing;
- iv) this number $n(\epsilon)$ is of the order of magnitude of

$$\int_{R_1(\epsilon)}^{R_0(\epsilon)} \sqrt{|V(r, \epsilon)|} dr$$

[we can put 0 or $R_1(\epsilon)$ at the lower bound of the integral].

We now consider the range $0 < r < R_1(\epsilon)$. From (13), if we assume (A.4), we get a similar relation to (14) :

$$\frac{1}{r - \phi(r, \epsilon)} < C(R_1, \epsilon) \quad 0 < r < R < R_1(\epsilon) < R_0(\epsilon) \quad (14')$$

First $\phi(r, \epsilon_1)$ has no poles because r and R are at the left of $R_1(\epsilon_1)$, secondly $\phi(r, \epsilon_1)$ is integrable when $r \rightarrow 0$, then as in the

preceding case ($V > 0$), we still have that $\frac{1}{r} - \phi(r, \epsilon_1) > 0$ and $\frac{1}{R} - \phi(R, \epsilon_1) > 0$. Similarly to the $V > 0$ case, we assume $\phi(R, \epsilon_1)$ higher, equal or less than zero and we get that the only possibility is $\phi(r, \epsilon_1) > 0$.

We take r fixed and we let ϵ_1 decrease. We try to see what happens for the sign of $\phi(r, \epsilon)$ ($\epsilon \leq \epsilon_1, \epsilon \rightarrow 0$). We must note that due to the existence of the attractive range $R_0(\epsilon)$, ϕ has not only poles $D_j(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$, but also roots $N_i(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$. Because $\phi = -\left(\frac{d}{dr}\left(\frac{\Psi}{r}\right)\right) / \frac{\Psi}{r}$, these roots $N_i(\epsilon)$ are oscillatory points for Ψ/r . Thus when ϵ_1 decreases at least all these $N_i(\epsilon_1)$ and $D_j(\epsilon_1)$ will cross r and can give a change of the sign of ϕ :

On the one hand, we remark that the multiplicity of these poles $D_j(\epsilon)$ of ϕ is not arbitrary. In fact, the multiplicity is one because for instance a pole of order 2 or 3, for ϕ , give rise in $\Psi(r, \epsilon)$ to essential singularities. This is not possible with the assumptions made about $V(\lambda, r, \epsilon)$. On the other hand, also $N_i(\epsilon)$, roots of ϕ , are of the first order except in the exceptional case where a root of $\phi(r, \epsilon)$ is also a root of $V(\lambda, r, \epsilon)$. [We recall that $V = \phi^2 - \phi' - \frac{2\phi}{r}$.] Then we have two cases.

1. $n(\epsilon)$ is non-increasing when $\epsilon \rightarrow 0$. We can find ϵ_2 such that for all $\epsilon < \epsilon_2$, $n(\epsilon)$ remains fixed. In this case, for $\epsilon < \epsilon_2$ (and if $R_0(\epsilon)$ is not a root of ϕ), the number of changes of the sign of ϕ is very simple. It is the sum of $n(\epsilon)$ [number of $D_j(\epsilon)$, roots of Ψ , poles of ϕ] and $m(\epsilon)$ [number of $N_i(\epsilon)$, oscillatory points of Ψ/r , roots of ϕ].

If we assume that $\lim_{\epsilon \rightarrow 0} \phi(r, \epsilon)$ exists, then according to whether $n(\epsilon) + m(\epsilon)$ is odd or even, then $\lim_{\epsilon \rightarrow 0} \phi(r, \epsilon)$ is ≤ 0 or ≥ 0 and $\lim_{\epsilon \rightarrow 0} \psi(r, \epsilon)$ gives ψ_R or ψ_{sing} . The important points are first that all these $D_j(\epsilon)$, $N_i(\epsilon)$ are in finite number and go to zero when $\epsilon \rightarrow 0$ and, secondly, r is fixed and finite, then we can always find ϵ_3 ($\epsilon_3 < \epsilon_2 < \epsilon_1$) such that $D_j(\epsilon) < r$, $N_i(\epsilon) < r$ for all $\epsilon \leq \epsilon_3$, for all i , for all j .

this, whatever is the position of $D_n(\epsilon)$ at the left or at the right of $R_0(\epsilon)$. For all $\epsilon \leq \epsilon_3$, $\phi(r, \epsilon)$ cannot change sign. In this case $\phi(r, \epsilon)$ has a finite number of discontinuities and roots.

2. $n(\epsilon)$ increases when $\epsilon \rightarrow 0$. For $\epsilon < \epsilon_1$ new roots and poles of ϕ can now appear. The number of oscillations of ψ or of ψ/r increases when ϵ decreases and also the number of changes of ϕ .

We want to show that ϕ and ψ have no limits when $\epsilon \rightarrow 0$.

This cannot come only from the fact that the number of oscillations increases. For instance, let us assume that all the roots of ψ appear only between $R_1(\epsilon)$ and $R_0(\epsilon)$, or even between $R_1(\epsilon)$ and some other finite function $G(\epsilon) > R_0(\epsilon)$, $G(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$. In this case, for any fixed finite r , we can still find ϵ_3 such that for $\epsilon \leq \epsilon_3$ all the roots and poles of ϕ are at the left of r . After an infinite number of oscillations, ϕ does not change sign. But this is not possible. We note that for $\epsilon_2 < \epsilon_1$ a new root of ψ cannot come from a shift of a root being present for ϵ_1 because the poles of ϕ (roots of ψ) are only of

multiplicity one. In fact when ϵ decreases, the new roots appear from ∞ and decrease with ϵ . This can be seen directly with examples [see after h] satisfying the assumption of this section or even for square well potentials $[V = -A(\epsilon) \text{ for } r < R_0(\epsilon), V = 0 \text{ for } r > R_0(\epsilon)]$, see Appendix B]. But a general argument can be found by considering a regular solution normalized at $r=0$. Instead of $\Psi(r, \epsilon)$ normalized at ∞ , we consider

$$u(r, \epsilon) = \frac{\Psi(r, \epsilon)}{e^{\int_0^r \phi(r, \epsilon) dr}} \quad r \rightarrow 0$$

normalized at the origin by a condition independent of ϵ . We recall that in this section $V(r, \epsilon)$ is assumed continuous in (r, ϵ) for $r > 0$, $0 \leq \epsilon \leq \epsilon_0$. For any finite r , $u(r, \epsilon)$ is continuous in ϵ . We assume that for ϵ_1 , u has N roots and for $\epsilon_2 < \epsilon_1$, u has $(N+1)$ roots, all these roots being at finite distance. We take r finite, higher than $D_{N\epsilon_1}(\epsilon_1)$ and $D_{N\epsilon_2}(\epsilon_2)$, then $u(r, \epsilon_1)$ and $u(r, \epsilon_2)$ have opposite signs. This means that between ϵ_1 and ϵ_2 , there exists ϵ_3 such that $u(r, \epsilon_3) = 0$. Then $D_{N\epsilon_3}(\epsilon_3)$ is higher than $D_{N\epsilon_1}(\epsilon_1)$ and $D_{N\epsilon_2}(\epsilon_2)$ *).

Now $\Psi = u \cdot \text{const}(\epsilon)$, this constant can have some discontinuities (poles) but the roots of Ψ and u are the same. Then for any finite r , when $\epsilon \rightarrow 0$, an increasing number of roots of Ψ at the right of r will appear. When $\epsilon \rightarrow 0$, all the roots of Ψ can never cross r , ϕ will always have discontinuities with values $+\infty$ or $-\infty$ and roots. For any fixed ϵ as small as we want, $\phi(r, \epsilon)$ considered as a function of ϵ has an infinity of discontinuities for smaller values of ϵ (not fixed) going to zero. Then $\lim_{\epsilon \rightarrow 0} \phi(r, \epsilon)$ or $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon)$ does not exist. Because there exists an infinity of families of attractive cut-off potentials, such that $n(\epsilon)$ increases when ϵ decreases, we see that it is wrong ** that $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon)$ always gives a solution of the Schrödinger equation.

*) We wish to thank Dr. A. Martin for this result.

**) Contrary to the belief expressed by Calogero 4).

h) An example for the attractive case

In order to illustrate these features in the attractive case, we give now an example :

$$V(\lambda, r, \epsilon) = \frac{\lambda}{(r+\epsilon)^4} - \frac{B\epsilon^\gamma}{r(r+\epsilon)^3} \quad \begin{cases} \lambda > 0, \gamma > 0, B > 0 \\ B < \lambda \text{ if } \gamma = 0 \end{cases}$$

$$\Psi(r, \epsilon) = r \exp(\lambda^{1/2}/r + \epsilon) \frac{\Phi(1-A, 2, z)}{\Phi(1-A, 2, 2\frac{\lambda^{1/2}}{\epsilon})}$$

where $z = 2\lambda^{1/2}r/\epsilon(r+\epsilon)$, $A = B\epsilon^{\gamma-1}/2\lambda^{1/2}$ and

$$\phi(\lambda, r, \epsilon) = \frac{\lambda^{1/2}}{(r+\epsilon)^2} \left[1 + \frac{(A-1)\Phi(2-A, 3, z)}{\Phi(1-A, 2, z)} \right]$$

From the well-known behaviour of the confluent hypergeometric function $\Phi(a, c, z)$ when $z \rightarrow \infty$ it is easy to show that if $\gamma > 1$ and $\gamma = 1$ ($A \neq \text{integer } 1, 2, \dots$) then $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_R$ and if $\gamma = 1$, and A integer $1, 2, \dots, n$, then $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_{\text{sing}}(r)$. On the other hand,

$$\int_0^{R_0(\epsilon)} \frac{R_0(\epsilon)}{V\Psi(r, \epsilon)} dr \simeq \text{const } \epsilon^{\gamma-1}$$

such that for $\gamma \leq 1$, $n(\epsilon)$ does not increase when $\epsilon \rightarrow 0$.

1. For $\gamma > 1$ we find that ϕ has no pole and only one root, then $n(\epsilon) + m(\epsilon) = 1$.

2. $\gamma = 1$, then A is a constant and by studying the roots of the numerator and of the denominator in ϕ we get ^{*)} :

*) For the number of roots of $\Phi(a, c, z)$ when $a < 0$ see Bateman "Higher transcendental functions" (Vol. I, p. 289). [For instance $\Phi(1-A, c, z)$ has n roots if $n < A \leq n+1$, n integer.]

A	$m(\epsilon)$	$n(\epsilon)$ roots of $\psi(r, \epsilon)$	$n+m$	$\lim_{\epsilon \rightarrow 0} \psi(r, \epsilon)$
$0 < A < 1$	1	0	1	ψ_R
1	0	0	0	ψ_{sing}
$1 < A < 2$	2	1	3	ψ_R
2	1	1	2	ψ_{sing}
$2 < A < 3$	3	2	5	ψ_R
3	2	2	4	ψ_{sing}

We see that the discrete ^{*)} values of $A = 1, 2, 3, \dots$ correspond to the special values where the sum of the number of roots of ψ and the number of stationary points of $\frac{\psi}{r}$ changes of parity (or the sum of the changes of signs of $\emptyset$).

3. $\gamma < 1$. In this case $n(\epsilon)$ and $m(\epsilon)$ increase when $\epsilon \rightarrow 0$. We can verify that the number of roots of $\tilde{\Phi}(1-A, 2, z)$, (poles of $\emptyset$), increases as ϵ^{-1} . When ϵ decreases, the new roots of $\tilde{\Phi}$ appear from ∞ . From the results of the above section g), we know that $\emptyset$ and ψ have no limits when $\epsilon \rightarrow 0$. For instance, in the case $\gamma = 0$, we put $\lambda = g(1+x)$, $B = gx$, $x > 0$, then

*) Calogero has given a similar example with the cut-off $e^{-\epsilon/r} V(\lambda, r, \epsilon)$ [see Eqs. (3.1) and (3.2) of Ref. 4]. The method we give here [section g)] explains as well the example of Calogero, where also the discrete values of a parameter (similar to A here) play a special role. But contrary to Calogero, we find that these discrete values of $A = 1, 2, \dots$ have a direct physical significance. These values are the critical values of the family of potentials considered. When γ decreases, the potentials become more and more attractive. There is a transition case $\gamma = 1$ where the number of roots does not increase for each potential but when A increases this small family ($\gamma = 1, A$) becomes more and more attractive. Then for these special values of the parameter the number of roots of the wave functions changes.

$$\frac{g}{\epsilon^4} \left[\frac{1+x}{\left(1+\frac{r}{\epsilon}\right)^4} - \frac{X\epsilon}{r\left(1+\left(\frac{r}{\epsilon}\right)\right)^3} \right] \xrightarrow{\epsilon \rightarrow 0} \frac{g}{r^4}$$

is a potential of the family considered by Khuri and Pais ^{*)}, but $\lim_{\epsilon \rightarrow 0} V(r, \epsilon)$ does not exist.

i) $V(\lambda, r, \epsilon)$ has a finite number of changes of signs near the origin

In f) we have considered $V > 0$ for $r < R_0$, (R_0 independent of ϵ), and in g) where V was < 0 for $r < R_0(\epsilon)$ we have assumed that $V(\lambda, r, \epsilon)$ has only one root $R_0(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$.

More generally, we can consider $V(\lambda, r, \epsilon)$ with the same assumptions as in g), but where (A.4) is replaced by the hypothesis that $V(\lambda, r, \epsilon)$ has for any $\epsilon \leq \epsilon_0$ always a same finite number of roots $V(\lambda, R_0^{(i)}(\epsilon), \epsilon) = 0$, such that $R_0^{(i)}(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$ and $R_0^{(1)}(\epsilon) < R_0^{(2)}(\epsilon) \dots < R_0^{(p)}(\epsilon)$, (p finite). Now V can have a finite number of changes of signs near the origin. We still have that there exists $R_1(\epsilon)$ such that for all i $R_1(\epsilon) < R_0^{(i)}(\epsilon)$, and we can still consider r in the range $0 < r < R_1(\epsilon_1)$, (ϵ_1 fixed $< \epsilon_0$). With the same analysis as in f) and g) we first study $\phi(r, \epsilon_1)$ for $r < R_1(\epsilon_1)$

*) This does not affect the main result of the work of Khuri and Pais ³⁾ which was to show the validity of the peratization procedure for $gV(r)$ of the power type. In fact, from the family considered by Khuri and Pais we can find an infinity of potentials satisfying the sufficient conditions given above in section f), a) [$V > 0$ and (A.1), (A.2)] such that, at least for these cut-off potentials, all the results of Khuri and Pais are valid.

and we get that, according to whether $V(\lambda, r, \epsilon_1)$ is > 0 or < 0 , we have $\phi < 0$ or > 0 . After we take r fixed and let $\epsilon_1 \rightarrow 0$, then as in g) all the poles of ϕ (roots of Ψ) and all the roots of ϕ will cross r and can give a change in the sign of ϕ . We have to investigate for each interval $[R_0^{(i)}(\epsilon), R_0^{(i+1)}(\epsilon)]$ the number of corresponding roots of Ψ , if V is attractive we call

$$n_i(\epsilon) \simeq \int_{R_0^{(i)}(\epsilon)}^{R_0^{(i+1)}(\epsilon)} \sqrt{|V(r, \epsilon)|} dr$$

if V is repulsive we have at most one root and we see that we get for $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon)$:

1. if all $n_i(\epsilon) \xrightarrow{\epsilon \rightarrow 0} \text{const}$, then we get either Ψ_R or Ψ_{sing} and we have to study, as in g), the oscillations of Ψ in order to distinguish between the two possibilities:
2. if at least one of the $n_i(\epsilon)$ increases when $\epsilon \rightarrow 0$, with the same type of proof as in g) we see that $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon)$ does not exist.

P A R T T W O

FREDHOLM FORMULATION OF THE SOLUTIONS

For cut-off potentials $\lambda V(r, \epsilon)$, we now assume always in this section that properties (1) and one of (9), (9.a) or (9.b) are satisfied. Then we recall

$$\lim_{\epsilon \rightarrow 0} \frac{T(k, 0, \epsilon)}{R(-k, 0, \epsilon)} = \lim_{\epsilon \rightarrow 0} \frac{T(k, \epsilon, 0)}{R(-k, \epsilon, 0)} = \lim_{\epsilon \rightarrow 0} \frac{T(k, \epsilon)}{R(-k, \epsilon)} \quad (15)$$

We want to study the Fredholm formulation of the solutions. Similarly to what was made in Refs. 1) and 2) for $\theta(r-\epsilon)$, we shall obtain the solutions in two different manners. The first one will be the Fredholm formulation and the other one will be obtained from the Jost solutions $R(\pm k, r, \epsilon)$. We give briefly the results because the method is the same as in Refs. 1) and 2).

a) Physical solution in co-ordinate space $\Psi(k, r, \epsilon)$

From Eqs. (6) and (7), we get

$$\Psi(k, r, \epsilon) = \frac{\sin kr}{k} + \frac{N(\lambda, k, r, \epsilon)}{D(\lambda, k, \epsilon)} \quad (16)$$

where

$$\left\{ \begin{aligned} N(\lambda, k, r, \epsilon) &= \left(T(k, r, \epsilon) - \frac{\sin kr}{k} \right) R(-k, 0, \epsilon) - R(-k, r, \epsilon) T(k, 0, \epsilon) \\ &= \sum_0^{\infty} \lambda^{q+1} N_q(k, r, \epsilon) \\ D(\lambda, k, \epsilon) &= R(-k, 0, \epsilon) = 1 + \sum_1^{\infty} \lambda^q D_q(k, \epsilon) \end{aligned} \right. \quad (17)$$

In order to obtain D_q and N_q we consider the iterate solutions $T = \sum \lambda^q T_q$ and $R = \sum \lambda^q R_q$ of the corresponding Volterra equation (8) and then we substitute these expansions in N and D . We get, for instance, for D_q

$$D_q(k, \epsilon) = k^{-q} \int_0^{\infty} dr_1 \int_{r_1}^{\infty} dr_2 \dots \int_{r_{q-1}}^{\infty} dr_q \prod_{j=1}^q e^{ikr_j} \sin k(r_j - r_{j-1}) V(r_j, \epsilon) \quad (18)$$

Now we consider the Fredholm formulation of (2) :

$$\left. \begin{aligned} Y(k, r, \epsilon) &= \frac{\sin kr}{k} + \frac{B(\lambda, k, r, \epsilon)}{\Delta(\lambda, k, \epsilon)} \\ B(\lambda, k, r, \epsilon) &= \sum_0^{\infty} \lambda^{q+1} B_q(k, r, \epsilon) \\ \Delta(\lambda, k, \epsilon) &= 1 + \sum_1^{\infty} \lambda^q A_q(k, \epsilon) \end{aligned} \right\} \quad (19)$$

where the Fredholm determinants B_q and A_q are ^{1), 2)} :

$$B_q(k, r, \epsilon) = \frac{(-1)^q}{q!} \int_0^r dr_1 \int_0^r dr_2 \dots \int_0^r dr_q \frac{\sin kr}{k} \left| \begin{array}{c} K(r_1, r_1, \epsilon) K(r_1, r_2, \epsilon) \dots K(r_1, r_q, \epsilon) \\ K(r_2, r_1, \epsilon) K(r_2, r_2, \epsilon) \dots K(r_2, r_q, \epsilon) \\ \vdots \\ K(r_q, r_1, \epsilon) K(r_q, r_2, \epsilon) \dots K(r_q, r_q, \epsilon) \end{array} \right|$$

$$A_q(k, \epsilon) = k^{-q} \int_0^r dr_1 \dots \int_0^r dr_q \mathcal{D}_q(r_1, \dots, r_q, k) \prod_{j=1}^q V(r_j, \epsilon) \quad (20)$$

$$\mathcal{D}_q = \left| \begin{array}{c} g(r_1, r_1, k) \dots g(r_1, r_q, k) \\ \vdots \\ g(r_q, r_1, k) \dots g(r_q, r_q, k) \end{array} \right| = e^{ikr_q \prod_{j=1}^q \sin k(r_j - r_{j-1})}$$

From (18) and (20) we see that $D_q = A_q$ or $D(\lambda, k, \epsilon) \equiv \Delta(\lambda, k, \epsilon)$. It is also easy to verify, like in Refs. 1) and 2), that $N_q = B_q$ for $q=1, 2, \dots$. Then we have identified the numerator and the denominator of the two formulations N/D and B/Δ . Because of this identification and of the fact that (15) is assumed, we get :

- i) the numerator and the denominator of the Fredholm solutions diverge in the same way when $\epsilon \rightarrow 0$, their ratio being the right physical solution

$$\Psi_R(k, r) = \lim_{\epsilon \rightarrow 0} \Psi(k, r, \epsilon) = \frac{\sin kr}{k} + \lim_{\epsilon \rightarrow 0} \frac{B(\lambda, k, r, \epsilon)}{\Delta(\lambda, k, \epsilon)}$$

- ii) for $\lambda V(r+\epsilon)$ we can connect $^{*)} \epsilon \rightarrow 0$ and the order of the determinants going to ∞ such that

$^{*)}$ The extensions from the corresponding results obtained in Refs. 1) and 2) are straightforward for $\lambda V(r+\epsilon)$ because of the relations (10). But for other cut-offs, where (9) is satisfied, we think that we can still connect $\epsilon \rightarrow 0$ and $p \rightarrow \infty$.

$$\Psi_R(k, r) = \frac{\sin kr}{k} + \lim_{p \rightarrow \infty} \frac{\sum_{q=0}^p \lambda^{q+1} B_q(k, r, \epsilon(p))}{\sum_{q=0}^p \lambda^q A_q(k, \epsilon(p))} \epsilon(p) \rangle, \epsilon_L(p)$$

$\epsilon_L(p)$ being the same as in Refs. 1) and 2).

iii) for $\lambda V(r+\epsilon)$ similar properties for the S matrix, $\lim_{\epsilon \rightarrow 0} F(k, \epsilon)$, the Jost function and the determination of bound states.

b) Lippman-Schwinger equation

We consider

$$H(p, k) = -\lambda \int_0^\infty \sin pr V(r) \Psi_R(k, r) dr \quad (21)$$

where $\Psi(k, r)$ is the physical solution studied above, $\lim_{\epsilon \rightarrow 0} \Psi(k, r, \epsilon) = \Psi_R(k, r)$ and such that $H(k, k)$ is the usual amplitude $e^{i\delta} \sin \delta$. Now, with the cut-off solution $\Psi(k, r, \epsilon)$ of (2), we define

$$H(p, k, \epsilon) = -\lambda \int_0^\infty \sin pr V(r, \epsilon) \Psi(k, r, \epsilon) dr = \frac{\bar{N}(\lambda, p, k, \epsilon)}{\bar{D}(\lambda, p, k, \epsilon)} \quad (22)$$

$$\bar{N}(\lambda, p, k, \epsilon) = -\lambda \int_0^\infty \sin pr V(r, \epsilon) [T(k, r, \epsilon) R(-k, 0, \epsilon) - R(-k, r, \epsilon) T(k, 0, \epsilon)] dr$$

$$\bar{N}(\lambda, p, k, \epsilon) = \sum_{m=0}^{\infty} \lambda^{m+1} \bar{N}_m(p, k, \epsilon)$$

$$\bar{D}(\lambda, p, k, \epsilon) = R(-k, 0, \epsilon) = D(\lambda, k, \epsilon) = 1 + \sum_{m=1}^{\infty} \lambda^m \bar{D}_m(k, \epsilon)$$

where $\bar{\bar{D}}_m(k, \epsilon) = D_m(k, \epsilon)$. From (21) and (22) we get :

$$\lim_{\epsilon \rightarrow 0} \frac{\bar{N}}{\bar{D}} = \lim_{\epsilon \rightarrow 0} -\lambda \int_0^\infty \sin pr V(r, \epsilon) \psi(k, r, \epsilon) dr = -\lambda \int_0^\infty \sin pr \lim_{\epsilon \rightarrow 0} [V(r, \epsilon) \psi(k, r, \epsilon)] dr$$

and the last integral exists, following (9), and is equal to $H(p, k)$. In order to get $\bar{\bar{D}}_m$ and $\bar{\bar{N}}_m$ we still have to consider the expansion given by (8) and to substitute it in the corresponding expressions for $\bar{D}$ and $\bar{N}$.

We define

$$V(p, q, \epsilon) = -\int_0^\infty \sin(pr) \frac{\sin qr}{q} V(r, \epsilon) dr \quad (23)$$

From the integral equation (2) of $\psi(k, r, \epsilon)$ and with (22) and (23), we get an integral equation for $H(p, k, \epsilon)$:

$$H(p, k, \epsilon) = \lambda V(p, k, \epsilon) + \lambda \int_0^\infty \frac{2q}{\pi} \frac{V(p, q, \epsilon)}{q^2 - k^2 - i\eta} H(q, k, \epsilon) dq \quad (24)$$

We consider the Fredholm formulation of (24)

$$H(p, k, \epsilon) = \lambda V(p, k, \epsilon) + \frac{\sum_0^\infty \lambda^{m+2} C_m(p, k, \epsilon)}{1 + \sum_1^\infty \lambda^m \bar{D}_m(k, \epsilon)}$$

where C_m and $\bar{D}_m$ are the corresponding Fredholm determinants of the numerator and denominator. But here the Born term $V(p, k, \epsilon)$ in (24) diverges when $\epsilon \rightarrow 0$ if $V(r)$ is more singular near the origin than $r^{-3+\epsilon'}$ (ϵ' arbitrarily small > 0). Then we can rewrite the Fredholm solution in the following form :

$$F(p, k, \epsilon) = \frac{\bar{N}(\lambda, p, k, \epsilon)}{\bar{D}(\lambda, k, \epsilon)} = \frac{\sum_0^{\infty} \lambda^{m+1} \bar{N}_m(p, k, \epsilon)}{1 + \sum_1^{\infty} \lambda^m \bar{D}_m(k, \epsilon)}$$

where $\bar{N}_0 = V(p, k, \epsilon)$, $\bar{N}_1 = C_0 + \bar{D}_1 V, \dots, \bar{N}_m = C_{m-1} + \bar{D}_m V$.

We want now to show that $\bar{D} = \bar{\bar{D}}$ or $\bar{D}_m = \bar{\bar{D}}_m = A_m$. We define :

$$\bar{V}_{\epsilon}(q_i, q_j) = \int_0^{\infty} \sin q_i r \sin q_j r V(r, \epsilon) dr = \bar{V}_{\epsilon}(q_j, q_i)$$

Then we have ²⁾ :

$$\bar{D}_m(k, \epsilon) = \left(\frac{2}{\pi}\right)^m \int_0^{\infty} dq_1 \dots \int_0^{\infty} dq_m \prod_{j=1}^m \frac{1}{q_j^2 - k^2 - i\eta} \left| \begin{array}{c} \bar{V}_{\epsilon}(q_1, q_1) \dots \bar{V}_{\epsilon}(q_1, q_m) \\ \vdots \\ \bar{V}_{\epsilon}(q_m, q_1) \dots \bar{V}_{\epsilon}(q_m, q_m) \end{array} \right|$$

We use the relation

$$\frac{2}{\pi} \int_0^{\infty} \frac{\bar{V}_{\epsilon}(q_i, q_i) dq_i}{q_i^2 - k^2 - i\eta} = \int_0^{\infty} \frac{g(r_i, r_i, k) V(r_i, \epsilon) dr_i}{k}$$

and we get $\bar{D}_1(k, \epsilon) = A_1(k, \epsilon)$. With the use of similar relations [see Ref. 2] we can show that for any m we have $\bar{D}_m = A_m$. Similarly as in Ref. 2), it is also easy to identify $\bar{N}_m = \bar{\bar{N}}_m$ for $m=1, 2, \dots$. Then always when the most singular part of $V(r)$ is repulsive, and assuming that the relations (1) and (9) are satisfied, we get for the Fredholm formulation of the (cut-off dependent) Lippman-Schwinger equation :

$$i) F(p, k) = \lim_{\epsilon \rightarrow 0} \frac{\bar{N}(\lambda, p, k, \epsilon)}{\bar{D}(\lambda, k, \epsilon)}$$

ii) for $\lambda V(r+\epsilon)$ we get :

$$F(p, k) = \lim_{q \rightarrow \infty} \frac{\sum_0^q \lambda^{m+1} \bar{N}_m(p, k, \epsilon(q))}{\sum_0^q \lambda^m \bar{D}_m(k, \epsilon(q))} \quad \epsilon(q) \geq \epsilon_L(q)$$

where $\epsilon_L(q)$ is the same as $\epsilon_L(p)$ for $\Psi_R(k, r)$ and which have been investigated ^{1), 2)} for many types of singular potentials.

Application :

In the case of the potential $\lambda/(r+\epsilon)^4$ we get :

$$F(p, k=0, \epsilon) = \frac{-\lambda \int_0^\infty \sin pr (r+\epsilon)^{-3} \operatorname{sh}\left(\frac{\lambda^{1/2}}{\epsilon} - \frac{\lambda^{1/2}}{r+\epsilon}\right) dr}{\operatorname{sh}\left(\frac{\lambda^{1/2}}{\epsilon}\right)} \xrightarrow{\epsilon \rightarrow 0} -\lambda \int_0^\infty \frac{\sin pre^{-\lambda^{1/2}/r}}{r^3} dr$$

and $\lim_{\epsilon \rightarrow 0} F(p, k=0, \epsilon) = F(p, k=0)$.

c) Case of singular potentials $\lambda V(r)$ where $\lambda = |\lambda| e^{i\theta}$ is complex and $V(r) > 0$

We want to see if for this particular choice of complex potentials we can still use real cut-off potentials $\lambda V(r, \epsilon)$. For simplicity, we consider $\lambda V(r+\epsilon)$. From (10) we see that the use of the cut-off is justified and the properties of cut-off dependent Fredholm determinants (for the equation in co-ordinate space as well as for the Lippman-Schwinger equation) are still valid if the Jost solutions are such that

$$S(\lambda, k) = \lim_{r \rightarrow 0} \frac{R(\lambda, k, r)}{R(\lambda, -k, r)}$$

still exists. But

$$\frac{R(\lambda, k, r)}{R(\lambda, -k, r)} \underset{r \rightarrow 0}{\sim} \frac{R(\lambda, k) + \lim_{r \rightarrow 0} \frac{Y_{\text{sing}}(\lambda, r)}{Y_{\text{reg}}(\lambda, r)}}{R(\lambda, -k) + \lim_{r \rightarrow 0} \frac{Y_{\text{sing}}(\lambda, r)}{Y_{\text{reg}}(\lambda, r)}}$$

where $Y_{\text{sing}}(\lambda, r)$ and $Y_{\text{reg}}(\lambda, r)$ are the singular and regular parts of solutions of the Schrödinger equation near the origin.

First :

$$\frac{Y_{\text{sing}}}{Y_{\text{reg}}} \underset{r \rightarrow 0}{\sim} \exp \left[-2|\lambda|^{\frac{1}{2}} e^{i\frac{\theta}{2}} \int_r^\infty \sqrt{V(r')} dr' [1 + O(r)] \right] \rightarrow 0 \text{ if } |\theta| < \pi$$

Secondly, we have also the condition that $R(\lambda, -k)$ must have no zeros for k real. If we consider potentials of finite range, $V(r) = 0$ for $r > r_0$, we get by the usual combination of the equation for Ψ_R and Ψ_R^* , if $R(\lambda, -k) = 0$ for k real :

$$k |\Psi_R(r_0)|^2 = \text{Im} \lambda \int_0^{r_0} V |\Psi_R(r)|^2 dr$$

This relation holds also if V is non-singular near the origin. Thus if we impose $\text{Im} \lambda < 0$ or $-\pi < \theta < 0$ then the condition is satisfied. Now for potentials with exponential decreasing behaviour, we get the same result ^{*)}. The interesting thing is that we can have the real part of the potential attractive if the imaginary part is < 0 .

^{*)} This study is similar to the case when $V(r)$ is non-singular near the origin [R. Nataf and H. Cornille, Journal de Physique 24, 591 (1963)].

CONCLUSIONS

We have seen in this paper that the cut-off $\theta(r-\epsilon)$ is not the only one that we can use in potential scattering when we consider singular potentials where the most singular part is repulsive. We do not think that the existence of counter examples when the cut-off does not work is a very important fact against the use of cut-off, if the cut-off is not present in the theory, being introduced only as an intermediate step and also if we are free, as in potential scattering, to choose arbitrary types of cut-off. But this is an indication that in other fields, like field theory or in Bethe-Salpeter equation, if we introduce cut-off we must, as in potential scattering, first show the validity of the procedure. But here we have shown that at least a great family of real cut-off [in fact all usual cut-offs physically reasonable like $\theta(r-\epsilon)$, $e^{-\epsilon/r^\beta} V(r)$, $V(r+\epsilon)$, ...] gives the correct result when $\epsilon \rightarrow 0$. This means that formally the problem can certainly be formulated without specific use of cut-off⁵⁾. But practically, the cut-off is of great help. First it gives the possibility to use the usual Green functions for any type of singularities [see Eq. (2)]. On the other hand when we have proved that a physical quantity is really the $\lim_{\epsilon \rightarrow 0} (A(x_i, y_j, \epsilon)) / (B(x_i, y_j, \epsilon))$, we are in the same situation as for the S matrix equal to $\lim_{r \rightarrow 0} (R(k, r)) / (R(-k, r))$ and then we can test if approximate methods for the S matrix are also valid in the cut-off case. We want also to emphasize the great interest of our method of connecting two limiting procedures (cut-off and order of determinants) because it works not only for $\theta(r-\epsilon)$, but certainly also for other cases where the introduction of the cut-off is valid, as we have shown explicitly for $V(r+\epsilon)$.

At the end, we want to add a few words about the distinction between the validity of the use of cut-offs $[\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) \stackrel{?}{\rightarrow} \Psi_R]$ physical solution, $\lim_{\epsilon \rightarrow 0} \sum_q a_q(\lambda, \epsilon) \stackrel{?}{\rightarrow} A$ [scattering length] and the validity of the approximation of summing up the first leading singularities.

We recall that already in Ref. 1) this distinction was clearly established because it was shown for $\lambda V(r)\theta(r-\epsilon)$ that :

- 1) $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_R$, $\lim_{\epsilon \rightarrow 0} \sum_q \lambda^q a_q(\epsilon) = A$.
- 2) The first leading singularities $\lim_{\epsilon \rightarrow 0} \sum_q \lambda^q \bar{a}_q(\epsilon)$ give a result if the most singular part of $V(r)$ is of the power type. If the most singular part is not of the power type, for instance of the logarithmic type or of the exponential type, then $\lim_{\epsilon \rightarrow 0} \sum_q \lambda^q \bar{a}_q(\epsilon)$ gives a wrong and nonsensical answer.

In the present paper we show that these results are more general.

- 3) $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_R$, $\lim_{\epsilon \rightarrow 0} \sum_q a_q(\lambda, \epsilon) = A$ at least if $V(\lambda, r, \epsilon)$ satisfies sufficient conditions given in Part One [sections a) and f)].
- 4) For $\lambda V(r, \epsilon)$ satisfying the assumptions a) of Part One, if $\lim_{\epsilon \rightarrow 0} \Psi(r, \epsilon) = \Psi_R$ or if $\lim_{\epsilon \rightarrow 0} \sum_q \lambda^q a_q(\epsilon) = A$, then $\lim_{\epsilon \rightarrow 0} \sum_q \lambda^q \bar{a}_q(\epsilon)$ gives the same results as for $\lambda V(r)\theta(r-\epsilon)$ and we have the same features according to whether the most singular part of $V(r)$ is of the power type or not.

In particular the class considered by Khuri and Pais ³⁾ includes a subclass of potentials $\lambda V(r, \epsilon)$ having the properties 3) and 4).

ACKNOWLEDGEMENTS

We thank Dr. G. Burdet and Dr. M. Giffon for discussions and their help in order to verify a part of the calculations of the present paper.

We thank also Dr. A. Martin and Dr. T.T. Wu for very useful discussions and suggestions. We thank Prof. N.N. Bogoliubov and Prof. D.I. Blokhintsev for their kind hospitality at the theoretical department of the Joint Institute.

We thank Dr. A.V. Efremov and Dr. B.A. Arbuzov for their interest in this work.

A P P E N D I X A

We want to show that if the potentials satisfy (A.1) and (A.2) then we have for the Jost solutions :

$$|R(\pm k, r, \epsilon) - R(\pm k, r)| \xrightarrow{\epsilon \rightarrow 0} 0 \quad (r \neq 0)$$

$R(k, r, \epsilon)$ is a solution of a Volterra equation (A.1)

$$R(k, r, \epsilon) = e^{-ikr} + \int_r^\infty \frac{\sin k(r-r')}{k} V(\lambda, r', \epsilon) R(k, r', \epsilon) dr'$$

$$R(k, r, \epsilon) = \sum_0^\infty R_n(k, r, \epsilon) \quad R_0 = e^{-ikr}$$

We consider the difference between the n th term for $\epsilon \neq 0$ and $\epsilon = 0$

$$R_n(k, r, \epsilon) - R_n(k, r) = \int_{r_0}^\infty dr_1 \int_{r_1}^\infty dr_2 \dots \int_{r_{n-1}}^\infty \frac{e^{-ikr_n}}{k^n} \left[\prod_{j=1}^n \sin k(r_j - r_{j-1}) \right] \left[\prod_{j=1}^n V(\lambda, r_j, \epsilon) - \prod_{j=1}^n V(\lambda, r_j) \right]$$

where $r_0 = r < r_1, \dots, r_{n-1} < r_n$.

We get

$$|R_n(k, r, \epsilon) - R_n(k, r)| < \frac{1}{n!} \int_r^\infty dr_1 \int_{r_1}^\infty dr_2 \dots \int_{r_{n-1}}^\infty \left| \prod_{j=1}^n V(\lambda, r_j, \epsilon) - \prod_{j=1}^n V(\lambda, r_j) \right|$$

We have the following identity

$$\prod_{j=1}^n V(\lambda, r_j, \epsilon) - \prod_{j=1}^n V(\lambda, r_j) =$$

$$= \sum_{i=1}^n \left[V(\lambda, r_i, \epsilon) - V(\lambda, r_i) \right] \prod_{j=i+1}^n V(\lambda, r_j, \epsilon) \prod_{k=1}^{i-1} V(\lambda, r_k)$$

With this identity we get :

$$\begin{aligned}
 |R_n(\epsilon) - R_n(\epsilon=0)| &< \frac{1}{n!} \int_r^\infty r' |V(\lambda, r', \epsilon) - V(\lambda, r')| dr' \sum_{i=0}^{n-1} \left[\int_r^\infty r' |V(\lambda, r', \epsilon)| dr' \right]^{n-1-i} \\
 &\quad \cdot \left[\int_r^\infty r' |V(\lambda, r')| dr' \right]^i \\
 &< \frac{\int_r^\infty r' |V(\lambda, r', \epsilon) - V(\lambda, r')| dr' \left[\int_r^\infty r' [|V(\lambda, r', \epsilon)| + |V(\lambda, r')|] dr' \right]^{n-1}}{n!}
 \end{aligned}$$

and finally :

$$|R(\pm k, r, \epsilon) - R(\pm k, r)| <$$

$$\int_r^\infty r' |V(\lambda, r', \epsilon) - V(\lambda, r')| dr' \left[\frac{\exp \left\{ \int_r^\infty r' [|V(\lambda, r', \epsilon)| + |V(\lambda, r')|] dr' \right\} - 1}{\int_r^\infty r' [|V(\lambda, r', \epsilon)| + |V(\lambda, r')|] dr'} \right]$$

A P P E N D I X B

It is a well-known thing that if the potential becomes more attractive, the number of roots of the wave function increases. We want to show that for $V(\lambda, r, \epsilon)$ attractive near the origin ($r < R_0(\epsilon)$), with the assumptions made in g) there exists always $R_1(\epsilon)$ such that there are no roots (except $r=0$) at the left of $R_1(\epsilon)$ and that $n(\epsilon)$ is fixed for ϵ fixed.

1. First, we consider two attractive potentials $-W(r)$ and $V(\lambda, r, \epsilon)$ and two of their corresponding solutions u and $\bar{\Psi}$ such that :

a) $W+V > 0$ for $0 < r \leq a$;

b) $\bar{\Psi}$ is a regular^{*)} solution $\bar{\Psi} \underset{r \rightarrow 0}{\sim} r$,
 $\lim_{r \rightarrow 0} u'(r) \bar{\Psi}(r) \rightarrow 0$;

c) $u > 0$ for $0 < r \leq a$.

We want to show that $\bar{\Psi}$ has no root for $0 < r < a$. If we assume that $\bar{\Psi}(r_0, \epsilon) = 0$ for $0 < r_0 < a$ then by the usual linear combination of the Schrödinger equation for u and $\bar{\Psi}$ we get :

$$\bar{\Psi}'(r_0, \epsilon) u(r_0) = \int_0^{r_0} (V+W) \bar{\Psi} u dr'$$

Because the right-hand side is > 0 then necessarily $\bar{\Psi}'(r_0, \epsilon) > 0$ and we get a contradiction.

Secondly, we remark that $V(r, \epsilon)$ is less attractive than $-\gamma/r^2$ where $0 < \gamma < \frac{1}{4}$. If we consider $W = \gamma/r^2$ and $u = r^{(1+\sqrt{1-4\gamma})/2}$ then a), b) and c) are satisfied and $\Psi(r, \epsilon)$ has no roots $D_j(\epsilon)$ in the range $\gamma/r^2 + V(r, \epsilon) > 0$. But when $\epsilon \rightarrow 0$, $V(r, \epsilon)$ is > 0 then necessarily this relation holds at least for some $r < R_1(\epsilon)$.

^{*)} In fact $\bar{\Psi}(r, \epsilon) = \Psi(r, \epsilon) \cdot \text{const}(\epsilon)$.

2. We now consider ϵ fixed, $\epsilon = \epsilon_1$ and $V(\lambda, r, \epsilon)$ sufficiently attractive in order to have some roots $D_j(\epsilon_1)$. Because of the assumptions made in g), $V(\lambda, r, \epsilon)$ is continuous in (r, ϵ) for $r > 0$, then between $R_1(\epsilon_1)$ and $R_0(\epsilon_1)$ there exist $m(\epsilon_1)$ and $M(\epsilon_1)$ such that $m^2(\epsilon_1) \leq |V(\lambda, r, \epsilon_1)| \leq M^2(\epsilon_1)$ and we know ^{*}) that the number of roots of Ψ is finite (depending on ϵ_1).
3. Now we want to give an order of magnitude of $n(\epsilon)$. If we substitute $V(\lambda, r, \epsilon)$ for $r < R_0(\epsilon)$ by an attractive square well [range $R_0(\epsilon)$ and depth $-A(\epsilon)$], then $n(\epsilon) \simeq \text{const} \sqrt{|A(\epsilon)|} R_0(\epsilon)$. Now as usual between $R_1(\epsilon)$ and $R_0(\epsilon)$ we approximate $V(\lambda, r)$ by a sum of square well [range Δr and depth $\sqrt{|V(r, \epsilon)|}$] and we get :

$$n(\epsilon) \simeq \int_0^{R_0(\epsilon)} \sqrt{|V(r, \epsilon)|} dr$$

We can replace at the lower bound 0 by $R_1(\epsilon)$.

4. We now study the trajectory of the last root of Ψ to the right, for a square well $V = -A(\epsilon)$ for $r < R_0(\epsilon)$, $V = 0$ for $r > R_0(\epsilon)$ and $R_0(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0$. We get $\Psi = (1/\sqrt{A})(\sin \sqrt{A} r)/(\cos \sqrt{A} R_0)$ for $r \leq R_0$, $\Psi = r - R_0 + (1/\sqrt{A}) \text{tg} \sqrt{A} R_0$ for $r > R_0$. For $r > R_0$, the last root is $X = R_0 - (1/\sqrt{A}) \text{tg} \sqrt{A} R_0$.

i) If $A = \text{const}$, then $X_\epsilon \xrightarrow{\epsilon \rightarrow 0} -A R_0^3$, there is no root at the right of R_0 for ϵ sufficiently small.

ii) If $\sqrt{A(\epsilon)} = c/R_0(\epsilon)$, ($c = \text{const}$), we are in the transition case. Then $X = R_0 [1 - c^{-1} \text{tg} c]$. If the constant c increases and crosses the values $(2m+1)\frac{\pi}{2}$ (m integer), we see that X goes from $-\infty$ to $+\infty$. We compare two potentials of the family corresponding to values of c near $(2m+1)\frac{\pi}{2}$ but the first $c < (2m+1)\frac{\pi}{2}$ and for the second $c > (2m+1)\frac{\pi}{2}$. Then we see that in the second case Ψ has a supplementary root coming from $+\infty$.

^{*}) W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Functions of Mathematical Physics", New York, p.161 (1954).

iii) If $\sqrt{A(\epsilon)} = c/(R_0(\epsilon))^\delta$, $\delta > 1$, then

$$X = R_0 \left[1 - \frac{R_0^{\delta-1}}{c} \operatorname{tg} \left(\frac{c}{R_0^{\delta-1}} \right) \right]$$

For a given potential of the family (δ and c fixed), we see that when ϵ decreases, $c/[R_0(\epsilon)]^{\delta-1}$ cross values of the type $(2m+1)\frac{\pi}{2}$ and then new roots appear, coming from ∞ .

REFERENCES

- 1) H. Cornille, Nuovo Cimento, 38, 1243 (1965).
- 2) H. Cornille, "Limiting procedures for singular potentials, II",
CERN preprint TH. 544.
- 3) N.N. Khuri and A. Pais, Rev.Mod.Phys. 36, 590 (1964).
- 4) F. Calogero, "Singular potentials and regularization",
Istituto di Fisica, G. Marconi, Preprint Roma (Nota Interna
No. 73).
- 5) W. Güttinger, R. Penzl and E. Pfaffelhuber, Annals of Physics 33,
246 (1965).