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OPTICAL PROPERTIES OF LOW-ENERGY PIONS IN NUCLEI

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A B S T R A C T

A simple non-local potential for low-energy pions in finite nuclei is calculated from the amplitudes for πN scattering and for π production in NN collisions. The potential includes absorption and has no free parameters. The appropriate multiple scattering equations are derived in the co-ordinate representation with nuclear pair correlations included. Owing to the large mass of the scatterers the pion field behaves nearly classically. It is shown that short range pair correlations are important in the multiple scattering owing to the dominant dipole component in the πN scattering. To a good approximation this produces a Lorentz-Lorenz effect analogous to the one occurring in the scattering of electromagnetic waves in dense media. The contributions of Fermi motion to the potential are shown to be small. The isospin part of the potential is derived and is shown to have a tensor component in addition to the ordinary vector one. The former gives rise to direct double charge exchange of pions; its strength becomes important for high momentum pions. The potential is further found to have a term which gives a small but possibly observable hyperfine coupling.

A comparison is made between predictions of the potential and experimental data on level shifts and widths for π mesic atoms. Good agreement is found within the experimental uncertainties, which is evidence for short range anticorrelations between nucleons in the nucleus.

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1. INTRODUCTION

The elastic interaction of pions with finite nuclei and with nuclear matter is of considerable theoretical interest for a number of reasons. We are here dealing with one of the few examples of multiple scattering by a nucleus with an elementary particle projectile, the main other example of which is provided by the nucleon-nucleus interaction. Even as a mere repetition of the latter with a different interaction, the π nuclear multiple scattering would therefore be of interest, particularly since the low energy nucleon optical potential is only obtained moderately well from multiple scattering theory. The π nuclear interaction is, however, of such a qualitatively different structure compared to the nucleon one, that it must be considered to be of interest in its own right. Further, its connection to the basic processes is clearer and more immediate than in the case of nucleons.

More specifically :

- a) the important p wave (dipole) π N scattering amplitude implies an average non-local π nuclear interaction of a simple type. The problem of dipole multiple scattering is well known from the scattering of light in dense classical media. In the classical case, it is quite important to take into account short range correlations between scatterers; this gives rise to non-linear density effects in the refractive index (Lorentz-Lorenz effect). It is therefore of interest to enquire if short range nuclear correlations will play an important role in π nuclear scattering or not. The existence of such effects would justify the expression "pion optical model" in a very literal sense.
- b) the absence of poles in the low energy (πN) scattering, as well as the short range of the (πN) interaction opens the possibility of establishing a very transparent connection between the elementary scattering and the π nuclear scattering, a connection which unfortunately is not so direct in the case of nucleons. The small recoil corrections in the pion case and the absence of a pion spin are also technical advantages compared to nucleons.

- c) the existence of real absorption of pions in nuclear matter at zero energy and even at negative energies (bound orbits) gives rise to a non-vanishing absorptive $\pi\pi$ nuclear interaction in the low energy region, contrasting with the nucleon case. This absorption is intimately connected to $\pi\pi$ absorption in deuterium and to $\pi\pi$ production in (NN) collisions. It should be possible to express it in the amplitudes for these processes.

We may note, in addition to these general points, that the pion may undergo not only single but also double elastic charge exchange within the ground state isospin multiplet, since it has isospin unity. This effect, which is new compared to the nucleon case, occurs either through repeated application of a vector or through a direct tensor coupling. It is furthermore of interest to note the prediction of a $\pi\pi$ nuclear hyperfine coupling which may be observable in $\pi\pi$ mesic atoms, while the corresponding nucleonic effect has not been observed.

Our object in the present article is to investigate the effects mentioned above so as to clearly understand the nature and character of low energy $\pi\pi$ nuclear interactions in finite nuclei. Furthermore, we attempt to establish a quantitative description of these interactions.

The formal problem of multiple scattering in quantal systems has been extensively treated by Watson and collaborators ^{1),2)}; the solution of the problem is expressed in terms of an equivalent potential. Frank et al. ³⁾ applied these methods to pion propagation in an infinite homogeneous medium of uncorrelated nucleons. They found the potential to exhibit a strong momentum dependence with attraction at lower energies, changing into repulsion above 180 MeV through the influence of the (33) (πN) resonance. These results have only qualitative validity for a finite nucleus and only at higher energies due to the neglect of nuclear correlations, as well as of finite nuclear size. The crucial importance of the latter was apparent in the work of Kisslinger ⁴⁾ who allowed the potential to be velocity dependent so as to account properly for the p wave part of the (πN) interaction.

The Kisslinger equation [see Eq. (16)] was derived in the impulse approximation for the (πN) scattering operator in the nucleus. Absorption effects and nuclear correlations were entirely neglected. While this procedure puts strong limits to the predictions of the model in practice, the structure of the equation is affected only in details to a good approximation.

Sternheim ⁵⁾ has recently discussed the high energy pion-nucleon elastic scattering in terms of a modified Glauber approximation applicable to small angle scattering.

Parallel and independent of the above-mentioned work on multiple pion scattering in nuclei, Deser et al. ⁶⁾ and Bethe et al. ⁷⁾ discussed the $1s$ level shift of π mesic atoms. They assumed the nucleus to be an uncorrelated assembly of nucleons confined to a small volume, and they obtained thus the level shift to be proportional to the coherent sum of the (πN) scattering lengths.

It was, however, pointed out by Brueckner ⁸⁾ that the pion absorption process in addition to giving an absorption broadening of the $1s$ level also must give rise to a real level shift of similar magnitude.

The absorption broadening of levels in π mesic atoms was first estimated by Brueckner ⁹⁾ on the basis of the quasi-deuteron model. Recently, Eckstein ¹⁰⁾ has reformulated the basic idea of this model in a convenient way by introducing a phenomenological two-nucleon absorption amplitude for a study of the branching ratios of capture products from ^4He . An approach of this kind which allows for the complicated structure of the short ranged absorption interaction seems presently preferable to a field theoretical perturbation approach with one-body operators. Indeed a recent calculation of π absorption in ^{16}O by Spector ¹¹⁾, using a $(\sigma \cdot \nabla)$ coupling for the (πN) system and shell model wave functions, leads to an absorption rate which is nearly two magnitudes smaller than observed. We have therefore generalized Eckstein's idea to the elastic absorption amplitude from two close nucleons and included the p wave

interaction of the pion. The generalization of the multiple scattering problem to include absorption effects then becomes immediate. In a recent investigation ¹²⁾, one of us (M.E.) has shown that the experimental data on level widths of π mesic atoms are in agreement with a non-local potential of the Kisslinger type with parameters derived from a generalized quasi-deuteron model. A general discussion of the features of π nuclear interactions is given in Ref. ¹³⁾.

In Section 2 we develop the multiple scattering formalism for a finite system under simplified assumptions. This is done through an expansion into nucleon correlation functions which is truncated at the pair correlation function. The resulting integral equations are discussed in various approximations. Section 3 generalizes these results to include several types of scatterers as well as spin and isospin. The influence of excitation energies in intermediate nuclear states, as well as contribution of Fermi motion, are evaluated. Section 4 discusses the absorption amplitudes and the associated multiple scattering. Section 5 relates the multiple scattering parameters to the numerical amplitudes of the (πN) and $(\pi 2N)$ interactions. The magnitudes and relative importance of various contributions and correction factors are discussed. In Section 6 the corresponding parameters are determined from the present material on π mesic X-rays. Measured and predicted quantities are compared.

2. DERIVATION OF MULTIPLE SCATTERING EQUATIONS

In this Section we derive the multiple scattering equations for a pion in the nucleus by a method inspired by Lax's treatment¹⁴⁾ of multiple scattering in classical media. The basic idea is to use a systematic expansion into higher order correlation functions between the scatterers. This gives rise to an infinite hierarchy of equations, which in practice is truncated after a few steps and solved. In order to keep the discussion to essentials, various simplifications will be made below in the derivation; the corresponding generalizations are introduced later.

The (πN) interaction in the nucleus is described in the impulse approximation, namely that the scattering amplitude for pions incident on free and bound nucleons is the same but for trivial kinematical factors. This approximation is expected to be very good for low energy pions, since there are no poles in the (πN) amplitude in this region and since the (πN) interaction is short-ranged. This is in contrast to the case for (NN) interactions. In the low energy region, the (πN) scattering amplitude f_i for a nucleon i is given by

$$\underline{f}_i(r) = \left[\underline{b}_0 + \underline{b}_1 \underline{t} \cdot \underline{\tau}_i + \left(\underline{c}_0 + \underline{c}_1 \underline{t} \cdot \underline{\tau}_i \right) \underline{R}' \cdot \underline{R} - i \left(\underline{d}_0 + \underline{d}_1 \underline{t} \cdot \underline{\tau}_i \right) \underline{\sigma}_i \cdot \left(\underline{R}' \wedge \underline{R} \right) \right] \delta(\underline{r} - \underline{r}_i) \quad (1)$$

Here $\underline{k}'$ and $\underline{k}$ are the momenta of the scattered and of the incident pion, while $\underline{t}$ is its isospin; $\underline{\tau}_i$ and $\underline{\sigma}_i$ are the nucleon isospin and spin matrix vectors. Furthermore, b_0, b_1, c_0, c_1, d_0 and d_1 are constants related to the s wave (πN) scattering lengths a_{2t} , and the p wave (πN) scattering volumes $\alpha_{2t,2j}$ by [t total isospin, j total spin of (πN) system] :

$$\begin{aligned} b_0 &= 1/3 \left(\alpha_{11} + 2\alpha_{33} \right) & ; & & b_1 &= 1/3 \left(\alpha_{33} - \alpha_{11} \right) \\ c_0 &= 1/3 \left(4\alpha_{33} + 2\alpha_{31} + 2\alpha_{13} + \alpha_{11} \right) & ; & & c_1 &= 1/3 \left(2\alpha_{33} + \alpha_{31} - 2\alpha_{13} - \alpha_{11} \right) \\ d_0 &= 1/3 \left(2\alpha_{31} - 2\alpha_{33} + \alpha_{11} - \alpha_{13} \right) & ; & & d_1 &= 1/3 \left(\alpha_{31} - \alpha_{33} - \alpha_{11} + \alpha_{13} \right) \end{aligned} \quad (2)$$

The approximation (1) for the (πN) scattering amplitude is taken to be valid both on and off the energy shell for incident pions in the energy range under discussion $[E_\pi \lesssim 30 \text{ MeV}]$. This limit in the energy is mainly imposed by the influence from the (33) resonance.

The essential character of the multiple scattering problem is preserved if only the constants b_0 and c_0 in Eq. (1) differ from zero. Furthermore, since the pion-nucleon mass ratio is only $1/7$, we can consider the nucleons to be massive in the simplified discussion in this Section.

The total pion-nucleus wave function Ψ corresponding to both elastic and inelastic scattering of the pion can be written as

$$\Psi = \Psi(r) |0\rangle \quad (3)$$

where $|0\rangle$ refers to nuclear ground state and $\Psi(r)$ is an operator depending on both pionic and nuclear co-ordinates. The dependence on the pion co-ordinate r has been indicated explicitly for suggestiveness; we emphasize, however, that $\Psi(r)$ is not the pion field but an operator which connects nuclear excited states to the ground state, for example. The incident wave function Ψ_0 for the pion nuclear system is the product of an incident pion wave $\chi_0(r)$ and the nuclear ground state wave function

$$\Psi_0 = \chi_0(r) |0\rangle \quad (4)$$

The total wave function is obviously the sum of the incident wave and the scattered wave emerging from every one of the scatterers i in the medium. The scattered wave is proportional to the wave incident on the scatterer i , which we denote by $\Psi_i(r) |0\rangle$; observe that this wave function differs from the total wave function Ψ by the diverging wave produced by the scatterer i itself. It is furthermore proportional to the scattering amplitude f_i . Thus,

$$\Psi(\underline{r})|0\rangle = \chi_0(\underline{r})|0\rangle + \sum_i \int g(\underline{r}, \underline{r}') f_i(\underline{r}') \Psi_i(\underline{r}')|0\rangle d\underline{r}' \quad (5.a)$$

The Green's function $g(\underline{r}, \underline{r}_i)$ is an operator describing both the outgoing pion wave and the propagation of nuclear states. An extremely similar equation can be obtained for $\Psi_i(\underline{r})|0\rangle$ by observing that it must be the sum of the incident wave and the scattered wave from all scatterers but i itself. We denote the field incident on a scatterer i , with the knowledge that there is another one at $\underline{r}_i$, by $\Psi_{j;i}(\underline{r})|0\rangle$. Thus,

$$\Psi_i(\underline{r})|0\rangle = \chi_0(\underline{r})|0\rangle + \sum_{j \neq i} \int g(\underline{r}, \underline{r}') f_j(\underline{r}') \Psi_{j;i}(\underline{r}')|0\rangle d\underline{r}' \quad (5.b)$$

We can proceed in this fashion and construct an infinite sequence of equations, the next one being the one for $\Psi_{j;i}(\underline{r})|0\rangle$ expressed in terms of the waves scattered from all scatterers but $\underline{r}_i$. This hierarchy of equations is the basis for all the following discussion. We observe, however, that a solution of this infinite system must be expected to be difficult, since it does not only describe the elastic, but also all inelastic scattering of the pions. Our next step is therefore to eliminate all explicit appearance of excited nuclear states so as to obtain equations depending only on pionic degrees of freedom. To describe pion elastic scattering correctly, it is only necessary to know the correct pion field function at infinity with the nucleus unexcited; as long as this condition is fulfilled, the detailed behaviour of the elastically scattered wave inside the nucleus is irrelevant. It may thus be useful to consider asymptotically correct field functions which differ vastly

from the actual pion field inside the nucleus. A function with correct asymptotic behaviour is certainly the projection $\phi(\underline{r})$ of the total wave function onto the nuclear ground state for any pion co-ordinate

$$\phi(\underline{r}) = \langle 0 | \psi(\underline{r}) | 0 \rangle \quad (6)$$

In other words, $\phi(\underline{r})$ is the average pion field function at the point $\underline{r}$. An equation for $\phi(\underline{r})$ is obtained by taking the ground state component of Eq. (5.a).

Before doing so, we express the total pion-nucleus Green's function for incident pions of energy E in terms of the nuclear states $|n\rangle$ of energy ξ_n

$$g(\underline{r}, \underline{r}') = \sum_n g_n(\underline{r}, \underline{r}') |n\rangle \langle n| \quad (7)$$

Here $g_n(\underline{r}, \underline{r}')$ is the Green's function at an energy $E - \xi_n$. When no external potential [e.g., a Coulomb field] acts on the pion, the propagation is determined by the time independent Klein-Gordon equation. Thus,

$$g_n(\underline{r}, \underline{r}') = (|\underline{r} - \underline{r}'|)^{-1} \exp \{ i k_n |\underline{r} - \underline{r}'| \} \quad (8)$$

where the wave number k_n is given in terms of the pion mass μ by $(E - \xi_n)^2 - \mu^2 = k_n^2$ (*). We have already assumed the nucleons to be massive; it is logical with this assumption to consider the nuclear excitation energies ξ_n to be negligible compared to the energies of pion intermediate states since the corrections are of order $\mu/M \approx 1/7$. This will be discussed in more detail later with more exact conditions established. In this approximation all functions $g_n(\underline{r}, \underline{r}')$ are therefore equal to $g_0(\underline{r}, \underline{r}')$. Eq. (7) then becomes :

*) We will in the following discuss $g_n(\underline{r}, \underline{r}')$ in the absence of an external potential, but our results and derivation remain valid with trivial modifications.

$$g(\underline{r}, \underline{r}') = g_o(\underline{r}, \underline{r}') \sum_n |n\rangle \langle n| \quad (7')$$

which greatly simplifies the following.

It is convenient to introduce the density operator $\hat{\rho}(\underline{r})$ defined by :

$$\hat{\rho}(\underline{r}) = \sum_i \delta(\underline{r} - \underline{r}_i) \quad (9)$$

To avoid to write sums over scatterers explicitly, we have further introduced the operator $\Psi_{\underline{r}}(\underline{r}')$ defined by :

$$\hat{\rho}(\underline{r}) \Psi_{\underline{r}}(\underline{r}') = \sum_i \delta(\underline{r} - \underline{r}_i) \Psi_i(\underline{r}') \quad (10)$$

This way of defining $\Psi_{\underline{r}}(\underline{r}')$ is possible since there is a unique correspondence between $\underline{r}_i$ and the label i in the absence of spin co-ordinates. The scattering amplitude (1) in its simplified version [only b_o and $c_o \neq 0$] is now substituted into Eq. (5.a). The momentum operators $\underline{k}$ and $\underline{k}'$ are replaced by appropriate gradient operators. Using Eqs. (9) and (10), we can then identically rewrite Eq. (5.a) as :

$$\begin{aligned} \Psi(\underline{r})|0\rangle &= \chi_o(\underline{r})|0\rangle \\ &+ \int d\underline{r}' \left[b_o g(\underline{r}, \underline{r}') \hat{\rho}(\underline{r}') \Psi_{\underline{r}'}(\underline{r}') - c_o \nabla g(\underline{r}, \underline{r}') \hat{\rho}(\underline{r}') \left(\nabla'' \Psi_{\underline{r}'}(\underline{r}'') \right) \right] |0\rangle \end{aligned} \quad (5.a')$$

Equation (5.b) is treated in a completely analogous fashion. A density operator $\hat{\rho}_i(\underline{r})$ is introduced for a point $\underline{r}$ with the knowledge that a scatterer is at $\underline{r}_i$ through the definition :

$$\hat{\rho}_i(\underline{r}) = \sum_{j \neq i} \delta(\underline{r} - \underline{r}_j) \quad (9')$$

and we use furthermore a notation similar to (10) :

$$\hat{\rho}_i(\underline{r}) \Psi_{j,i}(\underline{r}') |0\rangle = \sum_{j' \neq i} \delta(\underline{r} - \underline{r}') \Psi_{j',i}(\underline{r}') |0\rangle \quad (10')$$

Equation (5.b) becomes in these notations and using the simplified Eq. (1)

$$\begin{aligned} \Psi(\underline{r}) |0\rangle &= \chi_0(\underline{r}) |0\rangle \\ &+ \int d\underline{r}' \left[\rho_0 g(\underline{r}, \underline{r}') \hat{\rho}_i(\underline{r}') \Psi_{j,i}(\underline{r}') - c_0 \nabla g(\underline{r}, \underline{r}') \hat{\rho}_i(\underline{r}') \left(\nabla \Psi_{j,i}(\underline{r}') \right)_{\underline{r}' = \underline{r}} \right] |0\rangle \end{aligned} \quad (5.b')$$

We are now ready to project Eqs. (5.a') and (5.b') onto the nuclear ground state. It is useful to define the following expectation values : the average density $\rho(\underline{r})$, the average incident pion field $\phi_1(\underline{r})$ and the average gradient of the incident pion field $\underline{E}_1(\underline{r})$ at a point $\underline{r}$. The exact definition of these quantities is

$$\rho(\underline{r}) = \langle 0 | \hat{\rho}(\underline{r}) | 0 \rangle \quad (11.a)$$

$$\rho(\underline{r}) \phi_1(\underline{r}) = \langle 0 | \hat{\rho}(\underline{r}) \Psi_{j,i}(\underline{r}) | 0 \rangle \quad (11.b)$$

$$\rho(\underline{r}) \underline{E}_1(\underline{r}) = \langle 0 | \hat{\rho}(\underline{r}) \left(\nabla \Psi_{j,i}(\underline{r}') \right)_{\underline{r}' = \underline{r}} | 0 \rangle \quad (11.c)$$

One can then immediately express the projected Eq. (5.a') in terms of the quantities defined in Eqs. (6) and (11)

$$\begin{aligned} \langle 0 | \Psi(\underline{r}) | 0 \rangle &\equiv \phi(\underline{r}) = \chi_0(\underline{r}) \\ &+ \int d\underline{r}' \rho(\underline{r}') \left[\rho_0 g(\underline{r}, \underline{r}') \phi_1(\underline{r}') - c_0 \left(\nabla g(\underline{r}, \underline{r}') \cdot \underline{E}_1(\underline{r}') \right) \right] \end{aligned} \quad (12.a)$$

It should be noted that the pion Green's function occurring in Eq. (12.a) is that of a pion of energy E , independent of the assumption of negligible nuclear excitation energies. The problem of finding $\phi(\underline{r})$ has thus been shifted to that of finding $\phi_1(\underline{r})$ and $E_1(\underline{r})$. Equations for these quantities are now derived from Eq. (5.b') in a similar fashion. We first define three new quantities in terms of ground state expectation values, namely the pair correlation function $G(\underline{r}, \underline{r}')$ between two scatterers, the average incident pion field at $\underline{r}'$ with a scatterer at $\underline{r}$, $\phi_{1;\underline{r}}(\underline{r}')$, and the average gradient of the incident pion field at $\underline{r}'$ with a scatterer at $\underline{r}$. Thus :

$$\begin{aligned} \phi(\underline{r}) \phi(\underline{r}') [1 + G(\underline{r}, \underline{r}')] &\equiv \langle 0 | \sum_i \delta(\underline{r} - \underline{r}_i) \hat{\phi}(\underline{r}') | 0 \rangle \\ &\equiv \langle 0 | \sum_i \sum_{j \neq i} \delta(\underline{r} - \underline{r}_i) \delta(\underline{r}' - \underline{r}_j) | 0 \rangle \end{aligned} \quad *) \quad (13.a)$$

*) The pair correlation function $G(\underline{r}, \underline{r}')$ defined in Eq. (13.a) has, or is expected to have, the following important properties in the nucleon case :

- a) $\lim_{|\underline{r} - \underline{r}'| \rightarrow \infty} G(\underline{r}, \underline{r}') = 0$, i.e., no correlations of very long range.
- b) $\lim_{|\underline{r} - \underline{r}'| \rightarrow 0} G(\underline{r}, \underline{r}') = -1$, i.e., two particles do never coincide in space
- c) a normalization condition. The integration of both sides of Eq. (13.a) over $\underline{r}'$ for a system of A particles yields

$$\begin{aligned} \phi(\underline{r}) \left[A + \int \phi(\underline{r}') G(\underline{r}, \underline{r}') d\underline{r}' \right] &= (A-1) \phi(\underline{r}) \\ \text{or} \quad \int \phi(\underline{r}') G(\underline{r}, \underline{r}') d\underline{r}' &= -1. \end{aligned}$$

This condition is in practice weak since it can be fulfilled by a constant $G(\underline{r}, \underline{r}')$ of order A^{-1} extending over the entire system. Such a term would only give rise to A^{-1} corrections in a problem like the one we consider and would be negligible for a large system.

- d) The nucleon pair correlation function is expected to rise from the value -1 for $|\underline{r} - \underline{r}'| = 0$ to the neighbourhood of zero within a range of the order of $1F$; for $|\underline{r} - \underline{r}'|$ larger than this distance, the remaining correlations are small and mainly due to the exclusion principle.

$$g(\underline{r})g(\underline{r}')[1+G(\underline{r},\underline{r}')] \phi_{1;\underline{r}}(\underline{r}') \equiv \langle 0 | \sum_i \delta(\underline{r}-\underline{r}_i) \hat{\rho}_i(\underline{r}') \psi_{\underline{r},\underline{r}'}(\underline{r}') | 0 \rangle \quad (13.b)$$

$$g(\underline{r})g(\underline{r}')[1+G(\underline{r},\underline{r}')] E_{1;\underline{r}}(\underline{r}') \equiv \langle 0 | \sum_i \delta(\underline{r}-\underline{r}_i) \hat{\rho}_i(\underline{r}') \left(\nabla'' \psi_{\underline{r},\underline{r}'}(\underline{r}') \right)_{\underline{r}''=\underline{r}'} | 0 \rangle \quad (13.c)$$

We now multiply Eq. (5.b') by $\delta(\underline{r}-\underline{r}_i)$ and sum over i . Since in Eq. (7') $\sum_n |n\rangle \langle n|$ is just the unit operator on the nuclear coordinates, the projection onto the nuclear ground state gives

$$\begin{aligned} \phi_1(\underline{r}) = & \chi_0(\underline{r}) \\ & + \int d\underline{r}' g(\underline{r}') [1+G(\underline{r},\underline{r}')] \left\{ \rho_0 g_0(\underline{r},\underline{r}') \phi_{1;\underline{r}}(\underline{r}') - \rho_0 (\nabla g_0(\underline{r},\underline{r}')) \cdot E_{1;\underline{r}}(\underline{r}') \right\} \end{aligned} \quad (12.b)$$

where we have used Eqs. (11.b) and (13): we have also suppressed a common factor $\rho(\underline{r})$.

The equation for $E_1(\underline{r})$ is obtained in exactly the same manner from Eqs. (5.b'), (11.c) and (13), the only difference being that the gradient with respect to $\underline{r}$ should first be applied to Eq. (5.b').

Thus

$$\begin{aligned} E_1(\underline{r}) = & \nabla \chi_0(\underline{r}) \\ & + \int d\underline{r}' g(\underline{r}') [1+G(\underline{r},\underline{r}')] \left\{ \rho_0 \nabla g_0(\underline{r},\underline{r}') \phi_{1;\underline{r}}(\underline{r}') - \nabla (\rho_0 g_0(\underline{r},\underline{r}')) \cdot E_{1;\underline{r}}(\underline{r}') \right\} \end{aligned} \quad (12.c)$$

Note that neither in Eq. (12.b) nor in Eq. (12.c) do the gradient operators act on the subscripts $\underline{r}$ in $\phi_{1;\underline{r}}$ and $E_{1;\underline{r}}$.

We could now continue this procedure seeking equations for $\phi_{1;\underline{r}}$ and $\underline{E}_{1;\underline{r}}$ with three-body correlation functions, and so on. This establishes a new and simpler hierarchy of equations than the original equations (5) for Ψ , since the explicit nuclear co-ordinates are eliminated. The obvious way of determining $\phi(\underline{r})$ from such a system of equations in practice is to break the expansion after n steps and to put $\phi_n = \phi_{n+1}$; $\underline{E}_n = \underline{E}_{n+1}$. The finite system of integral equations is then solved exactly for $\phi(\underline{r})$.

The simplest approximation of this type is to stop at the very first step, i.e., to put $\phi_1(\underline{r}) = \phi(\underline{r})$; $\underline{E}_1(\underline{r}) = \underline{\nabla} \phi$ in Eq. (12.a). With this approximation Eq. (12.a) is simply expressed in differential form. Since $g_0(\underline{r}, \underline{r}') \equiv g_0(|\underline{r} - \underline{r}'|)$ we can substitute $-\underline{\nabla}' g_0$ for $\underline{\nabla} g_0$. A partial integration over $\underline{r}'$ then yields

$$\phi(\underline{r}) = \chi_0(\underline{r}) + \int d\underline{r}' g_0(\underline{r}, \underline{r}') \left[\phi_0(\underline{r}') - (\underline{\nabla}' \cdot \underline{c}_0(\underline{r}') \underline{\nabla}') \right] \phi(\underline{r}') \quad (14)$$

Since, by definition, the Green's function $g_0(\underline{r}, \underline{r}')$ obeys

$$(\underline{\nabla}^2 + k^2) g_0(\underline{r}, \underline{r}') = -4\pi \delta(\underline{r} - \underline{r}') \quad (15)$$

the application of $(\underline{\nabla}^2 + k^2)$ to Eq. (14) gives :

$$(\underline{\nabla}^2 + k^2) \phi(\underline{r}) = -4\pi \left[\phi_0(\underline{r}) - (\underline{\nabla} \cdot \underline{c}_0(\underline{r}) \underline{\nabla}) \right] \phi(\underline{r}) \quad (14')$$

For non-relativistic pions, this result can be interpreted in terms of a velocity dependent potential

$$V(\underline{r}) = - \frac{4\pi}{2\mu} \left[b_0 \rho(\underline{r}) - \underline{\nabla} \cdot \underline{c}_0 \rho(\underline{r}) \underline{\nabla} \right] \quad (16)$$

This potential is, in all essentials, the so-called Kisslinger potential ⁴⁾ obtained from Watson's multiple scattering theory ^{1),2)}. Baker et al. ¹⁵⁾ rederived this equation using simplifying elementary methods and generalized it to the Klein-Gordon equation. The most notable feature of Eqs. (14') and (16) is the explicit non-locality in the pion-nucleus interaction, which reflects the importance of the p wave (πN) interaction.

Experimental results indicate that it may be very useful to discuss pion-nuclear interactions in terms of a velocity dependent interaction. The analysis by Baker et al. ^{15),16)} of their experiments on pion scattering at 70-90 MeV on several elements strongly indicates the necessity of a gradient interaction. They were unable to fit their angular distributions with any reasonable local interaction, while no difficulty appeared once velocity dependence was admitted. One of us, (M.E.) has recently analyzed the level widths and shifts of π mesic atoms ^{12),17)}. The data are incompatible with a local interaction, while a quantitative understanding is obtained with a gradient interaction as developed further below. In both these cases, it is apparent that it is more important to include the gradient interaction than to have the detailed form (14') or (16) of the interaction. It should thus be useful to discuss the pion-nuclear interaction in terms of a more general ansatz :

$$\left[\underline{\nabla} \cdot (1 + \alpha(\underline{r})) \underline{\nabla} + K^2(\underline{r}) \right] \Phi(\underline{r}) = 0 \quad (17)$$

where $\alpha(\underline{r})$ and $K^2(\underline{r})$ are more general functions of the nuclear density distribution. Compared to the elastic interaction of nucleons and alpha particles with nuclei, which seem well approximated by local potentials, we conclude that, in contrast, non-locality plays an essential role in pion-nuclear interactions.

Equations (14') and (16) were obtained by interrupting the hierarchy of equations (12) at the earliest possible place. Approximations equivalent to this procedure are very common in the literature, largely due to the ease with which they can be handled. This approximation must, however, be treated with a certain caution, particularly when the scattering amplitude f_1 has important contributions with $\ell \neq 0$. The reason is the following: the replacement of ϕ_1 and E_1 in Eq. (12.a) by ϕ and $\nabla\phi$ implies in particular that on the average the exciting wave at a scatterer has been made to include the scattered wave from the scatterer itself. Since the scattered wave diverges at the scatterer like r^{-1} for s waves, and like r^{-2} for p waves [and even worse for higher ℓ], this implies self-excitation which may give appreciable errors. If the equations are truncated one step later, the strong self-correlation of a scatterer with itself is correctly accounted for. We will therefore replace $\phi_{1;\underline{r}}$ and $E_{1;\underline{r}}$ by ϕ_1 and E_1 in Eqs. (12.b) and (12.c). There is every reason to believe the approximation obtained in this way to be a strong improvement on the previous one.

In principle $\phi(\underline{r})$ should now be determined from $\phi_1(\underline{r})$ and $E_1(\underline{r})$, as determined from Eqs. (12.b) and (12.c). However, for $r \rightarrow \infty$ Eq. (12.a) for $\phi(\underline{r})$ and Eq. (12.b) for $\phi_1(\underline{r})$ become identical since $\lim_{|\underline{r}-\underline{r}'| \rightarrow \infty} G(\underline{r}, \underline{r}') = 0$. Since we only need a correct asymptotic behaviour to describe the elastic pion-nucleus interaction $\phi_1(\underline{r})$ will do as well as $\phi(\underline{r})$, though they may differ vastly in the nuclear interior ^{*)}. The pion-nucleus elastic interaction is therefore given by the solution of the following coupled system of integral equations:

*) We may interpret $\phi_1(\underline{r})$ classically as the wave function seen by a test body at $\underline{r}$. Inside the medium, the test body participates in the scattering and modifies the field it measures. At infinity, however, there is no difficulty in using such a test body to measure the true pion wave function, and therefore $\phi_1 = \phi$ as $r \rightarrow \infty$.

$$\begin{aligned} \Phi_1(\underline{r}) &= \chi_0(\underline{r}) \\ &+ \int d\underline{r}' \varrho(\underline{r}') [1 + G(\underline{r}, \underline{r}')] \left[\epsilon_0 g_0(\underline{r}, \underline{r}') \Phi_1(\underline{r}') - c_0 \nabla g_0(\underline{r}, \underline{r}') \cdot \underline{E}_1(\underline{r}') \right] \end{aligned} \quad (18.a)$$

$$\begin{aligned} \underline{E}_1(\underline{r}) &= \nabla \chi_0(\underline{r}) \\ &+ \int d\underline{r}' \varrho(\underline{r}') [1 + G(\underline{r}, \underline{r}')] \left[\epsilon_0 \nabla g_0(\underline{r}, \underline{r}') \Phi_1(\underline{r}') - c_0 \nabla (\nabla g_0(\underline{r}, \underline{r}') \cdot \underline{E}_1(\underline{r}')) \right] \end{aligned} \quad (18.b)$$

Equation (18.b) is more conveniently expressed in terms of $\nabla \phi_1(\underline{r})$ as obtained from (18.a) :

$$\begin{aligned} \underline{E}_1(\underline{r}) &= \nabla \Phi_1(\underline{r}) \\ &- \int d\underline{r}' \varrho(\underline{r}') \nabla G(\underline{r}, \underline{r}') \left[\epsilon_0 g_0(\underline{r}, \underline{r}') \Phi_1(\underline{r}') - c_0 (\nabla g_0(\underline{r}, \underline{r}') \cdot \underline{E}_1(\underline{r}')) \right] \end{aligned} \quad (18.b')$$

This expression indicates clearly that $\underline{E}_1$ differs from $\nabla \phi_1$ only by the effect of the correlations: the extra terms are contributed from the region where ∇G is large, i.e., for which $|\underline{r} - \underline{r}'|$ is small.

The two equations (18.a) and (18.b), as well as Eqs. (12), can be derived as a special case of the multiple scattering equations associated with a non-local scattering operator $f_i(\underline{r}, \underline{r}')$ of more general type than that of Eq. (1). Define first the average incident wave at a point $\underline{r}$ with the c.m. of the scatterer at $\underline{r}_1$ as well as the average incident wave at $\underline{r}$ with the scatterer at $\underline{r}_1$ and another scatterer at $\underline{r}_2$:

$$\varrho(\underline{r}_1) \Phi(\underline{r}) \equiv \langle 0 | \sum_{\underline{r}_1} \delta(\underline{r}_1 - \underline{r}_i) \psi_i(\underline{r}) | 0 \rangle$$

(A)

$$\varrho(\underline{r}_1) \varrho(\underline{r}_2) [1 + G(\underline{r}_1, \underline{r}_2)] \Phi(\underline{r}) \equiv \langle 0 | \sum_{\substack{\underline{r}_1, \underline{r}_2 \\ i \neq j}} \delta(\underline{r}_1 - \underline{r}_i) \delta(\underline{r}_2 - \underline{r}_j) \psi_{ij}(\underline{r}) | 0 \rangle$$

derived

Within the same approximations as previously used, and by essentially the same arguments, we eliminate the nuclear degrees of freedom and find :

$$\phi(\underline{r}) = \chi_0(\underline{r}) + \iiint g_0(\underline{r}, \underline{r}') \rho(\underline{r}') \int_{\underline{r}_1} (\underline{r}', \underline{r}'') \phi(\underline{r}'') d\underline{r}_1 d\underline{r}' d\underline{r}'' \quad (B)$$

$$\phi_{\underline{r}_1}(\underline{r}) = \chi_0(\underline{r}) + \iiint g_0(\underline{r}, \underline{r}') \rho(\underline{r}') \left(1 + G(\underline{r}_1, \underline{r}_2) \right) \int_{\underline{r}_2} (\underline{r}', \underline{r}'') \phi(\underline{r}'') d\underline{r}_2 d\underline{r}' d\underline{r}'' \quad (C)$$

For an interaction of short range, the approximation $\rho_{\underline{r}_2; \underline{r}_1}(\underline{r}) \simeq \rho_{\underline{r}_2}(\underline{r})$ instead of $\rho_{\underline{r}_1}(\underline{r}) \simeq \rho(\underline{r})$ is, as before, to be expected to be a considerable improvement since the auto-correlation of the scatterers is correctly accounted for.

We note that :

$$\lim_{\underline{r}_1 \rightarrow \underline{r}} \phi_{\underline{r}_1}(\underline{r}) \equiv \phi_1(\underline{r})$$

$$\lim_{\underline{r}_1 \rightarrow \underline{r}} [\nabla_{\underline{r}} \phi_{\underline{r}_1}(\underline{r})] \equiv \nabla_{\underline{r}} \phi_1(\underline{r})$$

Further, the property $\lim_{\underline{r} \rightarrow \infty} \phi_1(\underline{r}) = 0$ remains unchanged as is easily seen from Eqs. (B) and (C).

The particular form :

$$f_i(\underline{r}, \underline{r}') = c_0 \delta(\underline{r} - \underline{r}') \delta(\underline{r}' - \underline{r}_i) - c_0 \nabla_{\underline{r}} \delta(\underline{r} - \underline{r}_i) \cdot \delta(\underline{r}' - \underline{r}_i) \nabla' \quad (D)$$

corresponds to the scattering operator (1) and gives immediately the equations previously obtained.

In general, the integral equations (18) are not easily rewritten into differential form, but they can obviously be solved numerically for any specific case. In practice, the pion wavelength for the energies considered are long compared to the correlation distance between nucleons; therefore it is reasonable to consider the correlations to have zero range to first approximation with finite range corrections considered separately. For this simple case, Eqs. (18) are trivially converted into a differential equation. In terms of $x = |\underline{r} - \underline{r}'|$ we have then ^{*)}

*) As long as the range of the pair correlation function is negligible, the results below remain unchanged, even if G is not spherically symmetrical.

$$\nabla G(x) = \hat{x} \delta(x) \quad ; \quad \lim_{x \rightarrow 0} \nabla g_0(x) = -\hat{x}/x^2 \quad (19)$$

since $G(0)$ is a step function going from -1 for $x=0$ to a term of order A^{-1} for non-zero x , and since $g_0(x)$ is given by Eq. (8). For any smoothly varying vector $\underline{A}(\underline{x})$

$$\int \frac{\hat{x} (\hat{x} \cdot \underline{A}(x))}{x^2} \delta(x) d\underline{x} = \frac{1}{3} 4\pi \underline{A}(0) \quad (20)$$

We thus obtain from Eq. (18.b') by direct substitution of Eqs. (19) and (20) :

$$\left(1 + \frac{1}{3} 4\pi c_0 \rho(r)\right) \underline{E}_1 \simeq \nabla \phi_1 \quad (21)$$

Since the pair correlation function $G(\underline{r}, \underline{r}')$ gives vanishing contributions in the integral equation (18.a) as the range goes to zero, we may neglect it to the same approximation. Expression (21) for $\underline{E}_1$ is substituted into (18.a) and we then go through the same steps of partial integration, etc., which led from Eq. (12.a) to Eq. (14) and to Eq. (14'). This gives $\phi_1(\underline{r})$ in the form of a wave equation :

$$(\nabla^2 + k^2) \phi_1(\underline{r}) = -4\pi \left[\rho_0 \rho(r) - \nabla \cdot \frac{c_0 \rho(r)}{1 + \frac{1}{3} 4\pi c_0 \rho(r)} \nabla \right] \phi_1(\underline{r}) \quad (22)$$

with a corresponding non-relativistic potential :

$$V(\underline{r}) = -\frac{4\pi}{2\mu} \left[\rho_0 \rho(r) - \nabla \cdot \frac{c_0 \rho(r)}{1 + \frac{1}{3} 4\pi c_0 \rho(r)} \nabla \right] \quad (23)$$

The optical model potential $U(r)$ of Eq. (23) is thus seen to differ from those currently employed in other contexts not only by its velocity dependence, but also by its explicit non-linear dependence on the density of scatterers.

We have arrived at Eq. (22) from a consistent quantal formulation of the multiple scattering problem with two important dynamical assumptions :

- a) negligible nuclear excitation energies, and
- b) a very short ranged pair correlation function.

It is interesting to note that for $b_0 = 0$ the result (22) is well known for the dipole scattering of electromagnetic waves through a dense homogeneous medium of polarizable atoms. This gives rise to an observable non-linear dependence of the refractive index on the density (the Lorentz-Lorenz effect). The derivation usually differs appreciably from the one used here. The Kisslinger equation (16), as well as Eq. (14'), were derived by Baker et al.¹⁵⁾ by arguments which were classical [the slab approximation]. Subsequently, Kroll¹⁸⁾ pointed out that a derivation, using such arguments, should generalize to Eqs. (22) and (23) on the grounds of analogy with the electromagnetic case. Our derivation here shows that one may hopefully expect this to be the case approximately, but that the range of the pair correlation function and the influence of the intermediate nuclear excitation energies will be crucial. These effects, as well as the generalizations to several kinds of scatterers, isospin and spin effects, etc., will be discussed in the next Section.

3. GENERALIZATIONS OF AND CORRECTIONS TO THE MULTIPLE SCATTERING EQUATIONS

In this Section we will generalize our results from Section 2 to a realistic nucleus, though the effects of pion absorption will be deferred to Section 4. We will thus discuss the multiple scattering by a system of several kinds of scatterers (A): the effect of non-negligible excitation energies in intermediate nuclear states (B): the effects of binding and Fermi motion of the scatterers (C): and the spin and isospin contributions to the multiple scattering (D). Finally, we will calculate the range corrections to the simple Lorentz-Lorenz effective field equation (22), (E).

A. Several Kinds of Scatterers

For a system consisting of various kinds of scatterers (neutrons and protons, for example) denoted by the index λ , the constants in the scattering amplitude (1) are $b_{k\lambda}$, $c_{k\lambda}$ and $d_{k\lambda}$. The average density of particles λ is denoted by $\rho_\lambda(\underline{r})$. Proceeding exactly as before in the derivation of Eqs. (12), we have a total average wave function $\phi(\underline{r})$ expressed in terms of the average incident wave at a scatterer λ , $\phi_1^\lambda(\underline{r})$ and the corresponding average gradient of the incident wave $\underline{E}_1^\lambda(\underline{r})$.

$$\phi(\underline{r}) = \chi_0(\underline{r}) + \sum_{\lambda} \int d\underline{r}' \rho_{\lambda}(\underline{r}') \left[b_{0\lambda} g_0(\underline{r}, \underline{r}') \phi_1^\lambda(\underline{r}') - c_{0\lambda} \underline{\nabla} g_0(\underline{r}, \underline{r}') \cdot \underline{E}_1^\lambda(\underline{r}') \right] \quad (24.a)$$

where we still neglect the influence of spin and isospin as in Section 2. In analogy to Eqs. (12.b) and (12.c), we express ϕ_1^λ and $\underline{E}_1^\lambda$ in terms of the pair correlation function $G_{\lambda\lambda'}(\underline{r}, \underline{r}')$ between particles λ and λ' . We immediately write the equations in the form (18), corresponding to a break-off of our infinite system of equations at the second step of iteration. The motivation for this is exactly the same as the one leading to equations (18). Thus

$$\begin{aligned} \phi_1^\lambda(r) = & \chi(r) \\ & + \sum_{\lambda'} \int dr' \rho_{\lambda'}(r') \left[1 + G_{\lambda\lambda'}(r, r') \right] \left[\rho_{0\lambda'} g_0(r, r') \phi_1^\lambda(r') - c_{0\lambda'} \nabla g_0(r, r') \cdot \underline{E}_1^{\lambda'}(r') \right] \end{aligned} \quad (24.b)$$

$$\begin{aligned} \underline{E}_1^\lambda = & \nabla \phi_1^\lambda \\ & - \sum_{\lambda'} \int dr' \nabla G_{\lambda\lambda'}(r, r') \left[\rho_{0\lambda'} g_0(r, r') \phi_1^\lambda(r') - c_{0\lambda'} \nabla g_0(r, r') \cdot \underline{E}_1^{\lambda'}(r') \right] \end{aligned} \quad (24.c)$$

Since for $|\underline{r} - \underline{r}'| \rightarrow \infty$ we have $G_{\lambda\lambda'}(\underline{r}, \underline{r}') \rightarrow 0$, the right-hand sides of Eqs. (24.b) and (24.a) for ϕ_1^λ and ϕ become identical at infinity. Since the elastic scattering properties of the system are only determined by the asymptotic behaviour of ϕ , we can use any one of the ϕ_1^λ in its place for the description of the elastic scattering. Furthermore, the difference between two ϕ_1^λ is

$$\begin{aligned} \phi_1^{\lambda_1}(r) - \phi_1^{\lambda_2}(r) = & \sum_{\lambda'} \int dr' \left[G_{\lambda_1\lambda'}(r, r') - G_{\lambda_2\lambda'}(r, r') \right] \rho_{\lambda'}(r') \left\{ \rho_{0\lambda'} g_0(r, r') \phi_1^{\lambda'}(r') - c_{0\lambda'} \nabla g_0(r, r') \cdot \underline{E}_1^{\lambda'}(r') \right\} \end{aligned} \quad (25)$$

This difference is thus due only to the difference of the correlation functions. These differences are due to the different range of the pair correlation functions, in particular due to the exclusion principle between scatterers. In so far as the range is negligible, or alternatively, in so far as all correlation functions are the same, all ϕ_1^λ are the same. Similarly all $\underline{E}_1^\lambda$ are the same under the same circumstances.

Thus :

$$\begin{aligned} \phi_1^\lambda &= \phi_1 \\ E_1^\lambda &= E_1 \end{aligned} \quad \text{for all } \lambda \left\{ \begin{array}{l} \text{if } G_{\lambda\lambda'}(\underline{r}, \underline{r}') \text{ has negligible range} \\ \text{if } G_{\lambda\lambda'}(\underline{r}, \underline{r}') \simeq G(\underline{r}, \underline{r}') \end{array} \right.$$
(26)

When (26) is fulfilled, the only difference with the discussion in the preceding Section is the replacements

$$\begin{aligned} b_0 f(\underline{r}) &\longrightarrow \sum_\lambda b_{0\lambda} f_\lambda(\underline{r}) \\ c_0 g(\underline{r}) &\longrightarrow \sum_\lambda c_{0\lambda} g_\lambda(\underline{r}) \end{aligned}$$
(27)

Anticipating final results, the range corrections are small in π nuclear interactions, so that the replacements (27) are in practice a very good first approximation.

B. Non-negligible Nuclear Excitation Energies

The pion-nuclear Green's function was defined by Eqs. (7) and (8) of the preceding Section. As a consequence of the low pion mass, we argued there that the pion propagation in the presence of an excited nuclear state of energy ϵ_n should be independent of n to a good approximation, i.e., we assumed that $g_n(\underline{r}, \underline{r}')$ equalled $g_0(\underline{r}, \underline{r}')$ [see Eq. (7')]. This approximation greatly simplified the following treatment, and it permitted us in particular to use the pair correlation function $G(\underline{r}, \underline{r}')$ defined in the normal fashion. It is therefore essential to be able to estimate the corrections to this approximation, which will be done now.

For simplicity, we will assume zero pion kinetic energy, i.e., $k_0 = 0$, since we are dealing with low energy pions. The wave number k_n of a pion with the nucleus in the excited state n is thus imaginary, $k_n = iK_n$, with

$$K_n^2 = 2\mu \epsilon_n - \epsilon_n^2 \leq \mu^2 \quad \text{for all} \quad \epsilon_n \leq 2\mu = 280 \text{ MeV} \quad (28)$$

The corresponding Green's function $x^{-1} \exp(-K_n x)$ is thus (compared to $k_n = 0$) damped exponentially by e^{-1} in a distance $K_n^{-1} > \mu^{-1} = 1.4 F$ for all $\epsilon_n \leq 280 \text{ MeV}$. A similar damping occurs also in the gradient of the Green's function, though with an increased damping distance. For $k_0 = 0$, we have

$$\nabla_{\vec{x}} g_n(x) = -x^{-2} \hat{x} (1 + K_n x) \exp(-K_n x) \quad (29)$$

Expression (29) has a damping distance of $2.15 K_n^{-1}$ compared to $K_n = 0$, i.e., the damping distance is over $3F$ for all intermediate states below 300 MeV ; the short range damping will thus be small.

Application of these results to the type of expectation values appearing in the derivation of (12.b) and (12.c) from Eq. (5.b') yields the following type corrections to the pair correlation function from the s wave Green's function :

$$\begin{aligned} & \langle 0 | \sum_{i \neq j} \delta(\vec{r}_i - \vec{r}_j) g(\vec{r}_i, \vec{r}_j) \delta(\vec{r}'_i - \vec{r}'_j) | 0 \rangle \\ &= g_0(\vec{r}, \vec{r}') g(\vec{r}) g(\vec{r}') + \sum_{n \neq 0} g_n(\vec{r}, \vec{r}') \langle 0 | \sum_i \delta(\vec{r}_i - \vec{r}) | n \rangle \langle n | \sum_j \delta(\vec{r}'_j - \vec{r}') | 0 \rangle \\ &\approx g_0(\vec{r}, \vec{r}') g(\vec{r}) g(\vec{r}') + \bar{g}_n(\vec{r}, \vec{r}') g(\vec{r}) g(\vec{r}') G(\vec{r}, \vec{r}') \\ &= g_0(\vec{r}, \vec{r}') g(\vec{r}) g(\vec{r}') \left[1 + \exp(-K_n |\vec{r} - \vec{r}'|) G(\vec{r}, \vec{r}') \right] \end{aligned} \quad (30)$$

A closely analogous expression results from the p wave Green's function (29). It is thus clear that the effect of the intermediate excitations is to decrease the effective long range correlations and to shorten the effective correlation distance, while leaving the short range behaviour nearly untouched. The principal manifestation of the correlations is the important Lorentz-Lorenz effect which is structure independent for any short range correlation function rising rapidly from -1 to 0. The inclusion of the finite range is a correction both in the s and in the p wave scattering. Therefore the non-negligible excitation energies will somewhat modify terms which are of secondary importance themselves.

Our general conclusion is thus that the small pion mass reduces the corrections from intermediate excitation energies to negligible proportions, in striking contrast to the case of nucleon-nuclear scattering.

C. Effects of Binding and of Fermi Motion

- a) Since the scatterers can only approximately be considered to be massive, there is a kinematical correction to be done in going from the scattering amplitude (1) for free nucleons to that of bound nucleons. The origin of the correction is that no energy goes into nucleon recoil in the nuclear case (to order A^{-1}). We should therefore multiply the free scattering amplitude (1) by the well-known Fermi factor $k_{\text{bound}}/k_{\text{free}}$, which for pions of total lab. energy E (including rest mass) is

$$\frac{k_{\text{bound}}}{k_{\text{free}}} = \left[(M^2 + \mu^2 + 2ME) / M^2 \right]^{1/2} \quad (31)$$

where M is the mass of the scatterer. This trivial correction is approximately 14% for low energy pions.

- b) The motion of the scatterers is negligible only in so far as they can be considered to be massive. To estimate the character and magnitude of this correction, we consider the nucleus to be a Fermi gas. The corrections occur in the p wave scattering operator in Eq. (1) which in reality is not proportional to $\underline{k} \cdot \underline{k}'$ but to the translationally invariant operator $\mu(\underline{v}_\pi - \underline{v}_N) \cdot \mu(\underline{v}'_\pi - \underline{v}_N)$, where $\underline{v}_\pi$, $\underline{v}'_\pi$ and $\underline{v}_N$ are pion and nucleon velocity operators. We notice that the Fermi motion contribution must be even in $\underline{v}_N$ because of parity. Going back to the multiple scattering equations (14') and (22), we find that Fermi motion gives rise to an induced s wave interaction $-4\pi(\mu/M)^2 \langle p_N^2 \rangle c_0 g(\underline{r}) [1 + 4\pi/3 c_0 g(\underline{r})]^{-1}$ to be added to the usual term $-4\pi b_0 g(\underline{r})$. Here $\langle p_N^2 \rangle$ is the mean square momentum of a nucleon which approximately is given by $\langle p_N^2 \rangle \simeq \frac{3}{5} p_F^2$, where p_F is the Fermi momentum $[\simeq 250 \text{ MeV}/c]$. The Fermi motion term is basically small of order $(\mu/M)^2 \simeq 1/50$. A detailed estimate of this and the corresponding absorptive term will be given in connection with the numerical discussion.

D. Spin and Isospin Effects

Up to this point we have only discussed the dominant contributions to the multiple scattering and we have neglected terms which give rise to A^{-1} corrections. Such terms are generally only small quantitative corrections which are hard to observe but in very light elements. Spin and isospin effects are, however, also of order A^{-1} and since they give rise to qualitatively new phenomena, they can be observed and merit a more precise treatment. If the scattering system has a total spin $\underline{S}$, a total angular momentum $\underline{J}$ and a total isospin $\underline{T}$, the spin density $\frac{1}{2}\underline{\sigma}(\underline{r})$ and the isospin density $\frac{1}{2}\underline{\tau}(\underline{r})$ are defined by :

$$\sigma(r) = \langle 0 | \sum_i \sigma_i \delta(r - r_i) | 0 \rangle = 2 A^{-1} \int \rho_1(r) = 2 A^{-1} \frac{J(J+1)}{J(J+1)} \rho_1(r) \quad (32.a)$$

$$\tau(r) = \langle 0 | \sum_i \tau_i \delta(r - r_i) | 0 \rangle = 2 A^{-1} \int \rho_2(r) \quad (32.b)$$

The densities $\rho_1(r)$ and $\rho_2(r)$ are like the ordinary density $\rho(r)$ normalized so that the integral over all space is A , the number of nucleons. An important special case is a spin and isospin distribution in the nucleus, proportional to the density, i.e., $\rho_1(r) = \rho_2(r) = \rho(r)$. Similarly, one can define a mixed spin-isospin density [arising from the term d_1 in Eq. (1)], which, however, is of order A^{-2} . It will thus be neglected due to the difficulty of observing its effects.

We now apply the same arguments that have been used several times before [e.g., in Eqs. (24) to (27)]. In principle, we should introduce separate correlation functions for space symmetric and space anti-symmetric two-nucleon states, but in view of the precision of experiments and the smallness of the correction terms, we will use the same pair correlation function for both so as not to introduce additional parameters. In the notation of Eq. (1) for the free scattering amplitude, we should thus make the following substitutions in Eqs. (18.a) and (18.b')

$$b_0 \rho(r) \longrightarrow b_0 \rho(r) + b_1 \tau(r) \quad (33.a)$$

$$c_0 \rho(r) \longrightarrow c_0 \rho(r) + c_1 \tau(r) + i d_0 \sigma(r) \wedge \quad (33.b)$$

The resulting equations are, of course, operator equations since, in general, the ordering of $\sigma(r)$ and of $\tau(r)$ is of importance.

The contributions of these terms are of order A^{-1} and their influence can therefore normally be treated as a perturbation. If we consider the important special case of a purely short range pair correlation, Eqs. (33) should be substituted into Eq. (22). Expansion to leading order in A^{-1} of the spin density term gives :

$$\begin{aligned}
 (\nabla^2 + k^2) \Phi_1(r) = & -4\pi \left\{ b_0 g(r) + b_1 \underline{\underline{t}} \cdot \underline{\underline{z}}(r) \right\} \Phi_1(r) \\
 & + 4\pi \underline{\underline{z}} \cdot \left\{ \left[1 + \frac{4\pi}{3} c_0 g(r) + c_1 \underline{\underline{t}} \cdot \underline{\underline{z}}(r) \right] \left[c_0 g(r) + c_1 \underline{\underline{t}} \cdot \underline{\underline{z}}(r) + c_2 d_0 \underline{\underline{z}}(r) \right] \underline{\underline{z}} \right\} \Phi_1(r)
 \end{aligned} \quad (34)$$

Equation (34) has the following new and important features :

- a) single charge exchange of pions occurs through the non-diagonal elements of the term $\underline{\underline{t}} \cdot \underline{\underline{z}}(r)$ to order A^{-1} . These transitions occur to isobaric analogue states of the ground state.
- b) double charge exchange of pions occurs either through the repeated use of the isospin density term at different points in the nucleus or by the tensor contact interaction induced by the Lorentz-Lorenz term. In both cases, this elastic double charge exchange is of order A^{-2} . The transitions go to isobaric analogue states;
- c) the isospin term introduces a predictable dependence of the energy shifts in π mesic atoms on the neutron excess through its diagonal terms;
- d) the spin density term gives rise to a hyperfine structure in the energy levels of π mesic atoms for $b \neq 0$. This is most easily seen in the case of a spherical spin density distribution [see the right-hand side of Eq. (32.a)]. In this case the entire spin density term on the right-hand side of Eq. (34) becomes :

$$4\pi d_0 \frac{J \cdot \underline{\ell}}{A} \frac{2(S \cdot J)}{J(J+1)} \left\{ \frac{1}{r} \frac{d}{dr} \left(1 + \frac{4\pi}{3} c_0 g(r) \right)^{-1} g(r) \right\} \Phi(r)$$

where $\underline{\ell}$ is the pion orbital angular momentum.

E. Range Corrections to the Lorentz-Lorenz Effect

Though we have derived Eqs. (18) and their generalizations without making the assumption of a short range pair correlation function, we have repeatedly treated this as an important special case. In this Section we have even anticipated that the generalized equation (22) will be a good approximation to the pion multiple scattering equations (18). We will now estimate the leading range corrections to the Lorentz-Lorenz result (22).

For an estimate of the range corrections, we first define the average correlation length ξ by

$$\xi^2 = \int_0^\infty x^2 \frac{dG(x)}{dx} dx = -2 \int_0^\infty x G(x) dx \quad *) \quad (35)$$

As an illustration of the methods that can be used for an estimate we consider the special case of $c_0 = 0$. To second order in ξ we then have from Eqs. (18) and (35), after some rearrangement :

$$\left(1 + \frac{1}{2} \xi^2 4\pi g(r) \rho_0 \right) \Phi(r) = \chi_0 + \int g(r, r') \rho_0 g(r') \Phi(r') dr' \quad (36)$$

*) More generally, the correlation length $\xi_{\lambda\lambda'}$ for two particles λ and λ' is defined in the same way, using $G_{\lambda\lambda'}(x)$.

A new wave function $\Psi_1(\underline{r})$ with the same asymptotic behaviour as $\phi_1(\underline{r})$ can now be defined by

$$\Psi_1(\underline{r}) = \left(1 + \frac{1}{2} \xi^2 4\pi \rho_0 g(\underline{r}) \right) \phi_1(\underline{r}) \quad (37)$$

It is thus clear that for $c_0 = 0$ the only modification of Eq. (22) is the substitution :

$$4\pi \rho_0 g(\underline{r}) \longrightarrow \left(1 + \frac{1}{2} \xi^2 4\pi \rho_0 g(\underline{r}) \right)^{-1} 4\pi \rho_0 g(\underline{r}) \quad (38)$$

valid to terms in ξ^2 . An extension of this result to several kinds of scatterers will be used in Eq. (47).

The general equations are complicated and not very instructive. Since the effect of the correlation function is to exclude a region of effective radius ξ as a contributor to the wave $\phi_1(\underline{r})$ in Eqs. (18), it is clear that the relevant parameter for the corrections is $(\xi K)^2$ as compared to unity, where K is the effective wave number in the medium. For the velocity dependent terms, the parameter is rather $\frac{1}{3}(\xi K)^2$ due to angular average. Thus the structure of the correlation region is negligible for wavelengths appreciably longer than $1F$. Using realistic values of the (πN) scattering parameters and including absorption, the finite correlation length correction is of the order of 10%, typically. These effects are thus normally negligible with the exception of the case discussed in more detail in Eq. (47).

4. ABSORPTIVE EFFECTS ON THE ELASTIC MULTIPLE SCATTERING

Since the pion is a Bose particle, it can undergo real absorption in the nucleus in addition to the multiple scattering. In this respect, the pion differs profoundly from the nucleon. The basic process $\pi + N \rightleftharpoons N$ occurs only virtually for free nucleons due to energy-momentum conservation; the absorption of a zero energy pion on a resting nucleon requires that a momentum $p \sim [2M\mu]^{1/2} \simeq 525 \text{ MeV}/c$ is furnished by some external source. The process above is not strictly forbidden in a complex nucleus due to Fermi motion. However, since the Fermi momentum p_F is only about $250 \text{ MeV}/c$, the one-nucleon process is strongly suppressed. As a consequence, the dominant pion absorption in nuclei involves the participation of at least two nucleons, the basic process being $\pi + N + N \rightleftharpoons N + N$. The relative distances associated with the absorption of a low energy pion on two nucleons are expected to be short, since the relative momentum $p \sim 2(M\mu)^{1/2} \sim 735 \text{ MeV}/c$, corresponding to a distance of 0.3 F . Even if this distance gives an exaggerated idea about the range of the interaction, it is tempting to adopt a formulation of the absorption which does not enquire into the detailed structure of this short-ranged absorption interaction. This philosophy was first applied by Brueckner et al.⁹⁾ in the so-called quasi-deuteron model of π^- absorption, which has been qualitatively successful, though it has never been exposed to any severe tests. A more precise and detailed formulation of the same basic idea was recently used by Eckstein¹⁰⁾ in a study of the branching ratios of final states produced in s wave pion absorption in ^4He . The interaction is considered in the zero range approximation with the two absorbing nucleons in a relative s state. After this, the most general two-nucleon absorption amplitude consistent with these requirements and with s wave pions is introduced as a phenomenological quantity, the constants of which contain all the information on the complicated shortrange absorption process, even including short-range nuclear pair correlations. We have here generalized this approach in two ways: first by introducing an elastic two-nucleon amplitude in the zero-range approximation, since the imaginary part of this amplitude is just the shadow of the absorption; the possibility of a real part of the amplitude is left open, since the absorption process may induce it⁸⁾. Secondly, the absorption of p wave pions will also be allowed for. The

most general amplitude of this kind in the c.m. system is given in Appendix A. To keep notations short, we will use a simplified two-nucleon amplitude, in which only terms giving rise to new effects or giving important contributions have been kept. The extension to the general amplitude and to the general result is easy and straightforward, using Appendices A and B.

The principal new effects are the occurrence of an absorptive interaction at low energy, simulating a non-vanishing contribution from the imaginary part of the one-nucleon amplitudes at zero energy (absorption broadening of energy levels in π mesic atoms) and the appearance of an isospin tensor interaction which can give direct double charge of pions.

The phenomenological two-nucleon scattering amplitude $f_{ij}^{(2)}$ from nucleons i and j is thus [see Appendix A]

$$f_{ij}^{(2)} = \delta(\vec{r}_i - \vec{r}_j) \delta\left(\vec{r} - \frac{1}{2}(\vec{r}_i + \vec{r}_j)\right) \left\{ B_0 + C_0 \frac{\vec{r}' \cdot \vec{r}}{r^2} \right. \\ \left. + (B_5 + C_5 \frac{\vec{r}' \cdot \vec{r}}{r^2}) \left[(\vec{t} \cdot \vec{r}_i)(\vec{t} \cdot \vec{r}_j) + (\vec{t} \cdot \vec{r}_j)(\vec{t} \cdot \vec{r}_i) \right] \right\} \quad (39)$$

Here the complex constants B_k and C_k are linear combinations of the various amplitudes of the angular momentum and isospin states of the $(\pi 2N)$ system. We denote the pion s wave interaction amplitudes with the two nucleons by β_{jt_1} and the pion p wave amplitude by $\gamma_{jt_1}(S, T)$, where j and t_1 are the total angular momentum and the total isospin, S and T the spin and isospin of the nucleon pair. (S and T are suppressed as labels whenever no ambiguity arises.). Then ^{*)}

$$B_0 = 1/48 (9\beta_{11} + 3\beta_{01} + \beta_{00} + \beta_{02}) \\ B_5 = 1/96 (-3\beta_{01} + 2\beta_{00} + \beta_{02}) \\ C_0 = 1/48 (\gamma_{10} + \gamma_{01} + 5\gamma_{21} + 5\gamma_{12} + 3\gamma_{11}(10) + 3\gamma_{11}(01)) \\ C_5 = 1/96 (2\gamma_{10} + 2\gamma_{12} - 3\gamma_{11}(10)) \quad (40)$$

^{*)} We recall that the two nucleons are assumed to be in a relative s state.

The previous discussion of the multiple scattering can now be directly extended to the two-nucleon contributions which are to be added to the one-nucleon terms. The amplitudes β_{jt_1} and δ_{jt_1} are always understood to have been determined from $2N \rightleftharpoons \pi + 2N$ experiments using (39) and neglecting short-ranged correlation effects since these are included in the coefficients. The implicit assumption is thus that the π absorption interaction is only weakly affected by the presence of other nucleons and by nuclear binding. The changes are then the following:

- a) the one-particle density $\rho(\underline{r})$ becomes the two-nucleon density $\rho_2(\underline{r}) \equiv \lim_{\underline{r} \rightarrow \underline{r}'} \rho(\underline{r}, \underline{r}') \simeq \rho^2(\underline{r})$ [note that this refers to the long-range two-nucleon density!]. For convenience the approximate value for randomly distributed particles will be used in the following. In addition, this term should be multiplied by the factor 2 to correctly account for the exchange contributions.
- b) the pair correlation function $G(\underline{r}, \underline{r}')$ should be replaced by the correlation function $G_1(\underline{r}, \underline{r}')$ between the c.m. co-ordinate $\underline{r}'$ of two close nucleons and a nucleon at $\underline{r}$, and the correlation function $G_2(\underline{r}, \underline{r}')$ between the c.m. co-ordinates $\underline{r}$ and $\underline{r}'$ of two pairs of close nucleons. Extremely little is known about the functions G_1 and G_2 . However, we expect that their qualitative behaviour is very similar to that of the pair correlation function $G(\underline{r}, \underline{r}')$, though the ranges of $G_1(\underline{r}, \underline{r}')$ and $G_2(\underline{r}, \underline{r}')$ presumably are somewhat larger. Furthermore, it is probable that the very short-range region in which the ordinary pair correlation function $G(\underline{r}, \underline{r}')$ is an ill-defined object, due to finite nucleon size, extends further out for $G_1(\underline{r}, \underline{r}')$ and $G_2(\underline{r}, \underline{r}')$. Since we previously concluded that the detailed structure of the pair correlation plays a subordinate role, we will adopt the same treatment for $G_1(\underline{r}, \underline{r}')$ and $G_2(\underline{r}, \underline{r}')$ in the absence of any other information. To the precision we are working, it is thus possible to replace G_1 and G_2 by the pair correlation function G .

- c) the tensor isospin term which can give rise to direct double charge exchange corresponds in the approximation a) to a term

$$2(\vec{t} \cdot \vec{t}(\vec{r}))(\vec{t} \cdot \vec{t}(\vec{r})) \simeq 8 \rho^2(\vec{r}) A^{-2} (\vec{t} \cdot \vec{T})(\vec{t} \cdot \vec{T})$$

where the last expression results assuming that the isospin distribution throughout the nucleus is proportional to the density.

- d) the corrections in Section 3.0 for bound scatterers and for Fermi motion should be used with a mass M equal to twice the nucleon mass. Furthermore, the root mean square of the nucleon momentum distribution $\langle p_N^2 \rangle$ should be replaced by the corresponding two-nucleon quantity $\langle p_{2N}^2 \rangle \simeq 2 \langle p_N^2 \rangle$.

With the changes a)-d), the two-nucleon contributions can thus be directly added within the preceding framework as formally corresponding to scattering from additional types of scatterers.

5. DETERMINATION OF MULTIPLE SCATTERING PARAMETERS

We have now finished the formal derivation of the multiple scattering equations. This Section shall be devoted to a discussion of the numerical predictions for the various parameters, while the next Section will compare these predictions with experiments. From the outset, we emphasize that parts of this Section, and much of the next one, are not as complete and quantitative as desirable, due to insufficient experimental information. In the present Section this is particularly true of the real part of the phenomenological two-nucleon amplitudes, which is experimentally unknown, though theoretical arguments for its magnitude exist. Furthermore, the precision of present experiments on complex nuclei, discussed in the next Section, is such that we have felt it necessary to restrict the freedom of parameters as much as possible. So, for example, we will assume an isospin density $\mathcal{Z}(\underline{r})$ proportional to density, although such an assumption is not necessary in principle.

The corrections for the finite range of pair correlation functions are small and will, for the moment, be neglected. The multiple scattering equation for low energy pions then takes the simple form (22) or, more explicitly, according to Eq. (34) :

$$\begin{aligned} & (\nabla^2 + k^2) \Phi_1(\underline{r}) \\ &= \left\{ -m_1(\underline{r}) \cdot m_2(\underline{r}) + \nabla \left[1 + \frac{1}{3} (n_1(\underline{r}) + n_2(\underline{r})) \right] \left[n_1(\underline{r}) + n_2(\underline{r}) + i4\pi d_0 \mathcal{Z}(\underline{r}) \right] \nabla \right\} \Phi_1 \end{aligned} \quad (41)$$

Here $m_1(\underline{r})$, $n_1(\underline{r})$ and d_0 represent single nucleon contributions; $m_2(\underline{r})$ and $n_2(\underline{r})$ are two-nucleon contributions. These functions depend explicitly on the one- and two-nucleon amplitudes: from Eqs. (27), (31) and (39) we have, up to leading terms in A^{-1} *) :

*) We have systematically approximated all two-nucleon density terms by products of one-nucleon density terms, so as to limit parameters. Further, since our application is to kinetic pion energies ≤ 30 MeV the ratio $k_{\text{bound}}/k_{\text{free}}$ of Eq. (31) can, to 3% or better accuracy, be replaced by $(1 + \mu/M)$.

$$\begin{aligned}
n_1(r) &= 4\pi(1+\mu/M) \left[b_0 g(r) + b_1 \underline{k} \cdot \underline{e}(r) \right] + (\mu/M)^2 n_1(r) \left[1 + \frac{1}{3} (n_1(r) + n_2(r)) \right]^{-1} \langle p_N^2 \rangle \\
n_2(r) &= 8\pi(1+\mu/2M) \left[B_0 g(r) + B_5 (\underline{k} \cdot \underline{e}(r)) (\underline{k} \cdot \underline{e}(r)) \right] + (\mu/2M)^2 n_2(r) \left[1 + \frac{1}{3} (n_1(r) + n_2(r)) \right]^{-1} \langle p_{2N}^2 \rangle \\
n_1(r) &= 4\pi(1+\mu/M) \left[c_0 g(r) + c_1 \underline{k} \cdot \underline{e}(r) \right] \\
n_2(r) &= 8\pi(1+\mu/2M) \left[C_0 g(r) + C_5 (\underline{k} \cdot \underline{e}(r)) (\underline{k} \cdot \underline{e}(r)) \right]
\end{aligned}
\tag{42}$$

*)

where the Fermi motion corrections of Section 3.C have been added to $m_1(r)$ and $m_2(r)$.

The constants b_k , c_k , B_k and C_k are previously defined in Eqs. (2) and (40). The π N s wave scattering lengths and the p wave scattering volumes are given by Hamilton and Woolcock²⁰⁾ from the analysis of experiments. Excluding Coulomb terms, they are [in $\hbar = c = \mu = 1$ units]

$$\begin{aligned}
\alpha_1 &= 0.171 \pm 0.005 \quad ; \quad \alpha_3 = -0.088 \pm 0.004 \\
-\alpha_{31} &= -0.038 \pm 0.005 \quad ; \quad \alpha_{33} = 0.215 \pm 0.005 \\
\alpha_{11} &= -0.101 \pm 0.007 \quad ; \quad \alpha_{13} = -0.029 \pm 0.005
\end{aligned}
\tag{43}$$

*) The following notations have been used in an analysis of level shifts¹⁷⁾ and widths¹²⁾ of π mesic atoms and in a preliminary report on the present work¹⁹⁾:

$$u = -(\hbar^2/2\mu)(1+\mu/M)4\pi b_0 \text{ Mev } F^3$$

$$u_1 = -(\hbar^2/2\mu)(1+\mu/M)8\pi b_1 \text{ Mev } F^3$$

$$V = (\hbar^2/2\mu)8\pi \text{Im } B_0 (\hbar/\mu c)^4 \text{ Mev } F^6$$

$$a = -4\pi(1+\mu/M)c_0 (\hbar/\mu c)^3 F^3$$

$$b = -8\pi \text{Im } C_0 (\hbar/\mu c)^6 F^6$$

By Eq. (2) we have thus the following values for the constants b_k , c_k and d_k of Eq. (1) :

$$\begin{aligned} b_0 &= -0.002 \pm 0.003 & ; & \quad b_1 = -0.086 \pm 0.002 \\ c_0 &= -0.208 \pm 0.008 & ; & \quad c_1 = 0.180 \pm 0.005 \\ d_0 &= -0.193 \pm 0.005 & ; & \quad d_1 = -0.060 \pm 0.004 \end{aligned} \quad (44)$$

The corresponding quantities for the two-nucleon amplitudes are not directly available in the literature. According to Eq. (40), there are ten (generally complex) two-nucleon amplitudes, i.e., 20 constants to be determined. The imaginary parts are obtained using the optical theorem for low energy pions. No experimental information is available on the real part of these amplitudes. We will therefore rely on estimates of sign and magnitude for the latter contributions.

For zero energy pions, the inelastic channels are $\pi + (2N) \rightarrow 2N$ and $\pi + (2N) \rightarrow (2N) + \gamma$. We neglect the latter one. Thus only those channels which have quantum numbers compatible with a two-nucleon channel, will have a non-vanishing imaginary amplitude. For the two nucleons in an initial relative s state, there are five such amplitudes, namely β_{01} , β_{11} , γ_{01} , γ_{21} and γ_{10} . They are determined from the measured processes $N+N \rightarrow \pi + D$ and $N+N \rightarrow \pi + N+N$ by detailed balancing, as described briefly in Appendix C. The method is similar to the one used by Eckstein¹⁰⁾ in the analysis of the vertex function.

The real part of the two-nucleon amplitude is generally non-vanishing at zero pion energy in any channel. The imaginary part should, however, provide a guide to the general importance of the real part. We thus expect that the closed channels with vanishing imaginary part play a secondary role. Brueckner⁸⁾ estimated the contributions from intermediate states with no mesons without any attempt to separate various channels. Using the nucleon momentum distribution, he found both the s and p wave pion terms to be about twice the imaginary amplitude and

repulsive. Thouless²¹⁾ included the intermediate two-meson terms in the same picture and found that this leads to a reduction of the p wave term by a factor 2-3 with respect to the s wave one. Another similar type of estimate by Karplus and Halpern²²⁾ gives a several times smaller, but still repulsive, contribution. The real part will therefore be taken to be repulsive both in its s and in its p wave contributions with the magnitude of the imaginary part. It is likely that the p wave contribution is suppressed, compared to the s wave one. More accurate estimates of these terms are highly desirable, since this point introduces the main uncertainty in comparisons to experiments.

Analyzing the pion production experiments according to Appendix C, we find ($\hbar = c = \mu = 1$) :

$$\text{Im } \beta_{01} = 0.054 \pm 0.080 \quad ; \quad \text{Im}(\delta_{01} + 5\delta_{21}) = 9.50 \pm 0.26 \quad (45.a)$$

$$\text{Im } \beta_{11} = 0.039 \pm 0.004 \quad ; \quad \text{Im } \delta_{10} = 1.4 \pm 0.5$$

with the corresponding real parts expected to be similar in magnitude and of opposite sign.

Furthermore, we have according to the above discussion :

$$\text{Re}\{\beta_{00}, \beta_{02}, \delta_{11}(10), \delta_{11}(01), \delta_{12}\} \approx 0 \quad (45.b)$$

$$\text{Im}\{\beta_{00}, \beta_{02}, \delta_{11}(10), \delta_{11}(01), \delta_{12}\} = 0$$

In Eq. (45.a) the contribution $(\delta_{01} + 5\delta_{21})$ has not been separated since it is known to higher accuracy than each term individually.

The constants B_k and C_k of Eq. (40), and of Appendix A Eq. (A.2), can now be derived. In particular :

$$\begin{aligned}
B_0 &= \sim -0.01 + i(0.107 \pm 0.0014) \\
B_5 &= \sim 0.002 - i(0.0017 \pm 0.0006) \\
C_0 &= \sim -0.08 + i(0.089 \pm 0.013) \\
C_5 &= \sim -0.03 + i(0.029 \pm 0.010)
\end{aligned} \tag{46}$$

The outstanding feature of the coefficients b_k , c_k , B_k and C_k , as given by Eqs. (44) and (46), is the enormous cancellation of the singlet and triplet scattering lengths in the term b_0 . Indeed, if these are used in the same combination but with the opposite sign, b_0 would be 0.116 and not -0.002, i.e., there is a cancellation to more than 1:50. The intrinsic accuracy of the multiple scattering is a function of the magnitude of the coefficient without cancellation. There can thus be very significant contributions from terms which basically are small corrections. The most important ones are the real part of the two-nucleon amplitude, the Fermi motion and the finite correlation length. The former ones have been considered previously. For the latter ones, we need a more accurate estimate than that obtained previously in Eq. (38). Neglecting the range corrections to the p wave pion contributions, we find that $m_1(\underline{r})$ of Eq. (42) has an additional term :

$$\Delta m_1(\underline{r}) = -4\pi^2(1+\mu/M) \left[\xi_{nn}^2 (\ell_0^2 + \ell_1^2) + \xi_{np}^2 (\ell_0^2 - \ell_1^2) \right] S(\underline{r})^2 \tag{47}$$

We have here assumed the neutron and the proton pair correlation lengths to be equal, i.e., $\xi_{nn} = \xi_{pp}$. The difference between ξ_{nn} and ξ_{pp} is mainly due to the relatively long-range effect of the exclusion principle. The long-range neutron-proton correlations are small, thus $\xi_{np} \simeq 0$. A straightforward calculation of the long-range neutron pair correlation function, on the basis of the Fermi gas model, gives

$$[x = p_F | \underline{r} - \underline{r}' |] :$$

$$G_{nn} \left(p_F^{-1} x \right) = - (9\pi/4) x^{-3} \left[J_{3/2}(x) \right]^2 \quad (48)$$

Substitution of this function into the definition of ξ in Eq. (35) leads to *)

$$\xi_{nn}^2 - \xi_{np}^2 = (9/4) p_F^{-2} \quad (49)$$

We are now in a position to discuss the expected magnitudes of the various terms in Eqs. (41) and (42). To compare terms of varying powers of $\rho(r)$, we shall, when needed, assume uniform density with $r_0 = 1.10F$ and $p_F \simeq 250 \text{ MeV}/c$ as typical in the nucleus.

A. The Local Interaction Terms $m_1(r)$ and $m_2(r)$

- i) The real part of the isospin independent term is normally dominated by b_0 , which, however, has large cancellations to 1:50. The value of b_0 is -0.002 , i.e., it corresponds to a repulsion. The Fermi motion gives rise to an attractive interaction of $+0.007$ if c_0 , of Eq. (44), is used and the Lorentz-Lorenz effect is neglected. Inclusion of the latter reduces the Fermi motion term to $+0.004$ **). The finite correlation length gives, according to

*) The integral is evaluated, using the relation,

$$\int_0^\infty J_\nu^2(t) t^{-\lambda} dt = \frac{\Gamma(\nu + \frac{1-\lambda}{2}) \Gamma(\lambda)}{2^\lambda \Gamma\left[\frac{(\lambda+1)}{2}\right]^2 \Gamma\left(\nu + \frac{\lambda+1}{2}\right)}$$

**) Wolfenstein 23) has previously estimated this correction presumably by a method similar to ours. His result is close to a factor three larger for the following reasons : 1) the estimate used only α_{33} , 2) the Lorentz-Lorenz effect was not included.

Eqs. (48) and (49), a repulsive term which, for a typical density, is -0.010 . It thus dominates both the previous terms. With the same density, the real part of B_0 also contributes a repulsive term expected to be of the order of -0.10 with a large uncertainty. The "effective b_0 " is thus expected to be -0.018 with a 30-50% uncertainty due to B_0 . Thus, the other terms are ten times more important than b_0 , but compared to the value for b_0 without cancellations, these additional terms are of the order of 15% only.

- ii) Another reflection of the cancellation in b_0 is an unusually strong isospin dependence. The isospin term b_1 contributes $-0.086 A^{-1}$ to be compared to the "effective b_0 " above. For $A=10$, this still represents 40% of the isospin independent term.
- iii) The isospin tensor term B_5 is likely to play a very subordinate role even in the double charge exchange reaction. The reason for this is that the vector interaction simulates its effects to second order. For dimensional reasons, the second order term is of the magnitude $(4\pi b_1)^2 a^2$, where a is a length larger than μ^{-1} . This is to be compared to $8\pi B_5$. The ratio of these terms is larger, and probably much larger than 30. We therefore believe the local isospin tensor term to be extremely hard, if not impossible, to observe.
- iv) The imaginary contribution from B_0 is expected to be about 50% of the "effective b_0 ". Such a large value is normal since they are both largely of the same origin, namely the two-nucleon amplitude.
- v) The two-nucleon contribution to the isospin vector term due to B_3 is of the order of 10% and hence is negligible at present.

B. The Velocity Dependent Terms $n_1(\underline{r})$ and $n_2(\underline{r})$

- i) The non-local interaction is strongly dominated by the spin and isospin independent part, which in turn depends mainly on c_0 . The interaction produced by this term is attractive due to the dominance of α_{33} , which plays an important role in nucleon-nucleon attraction. All the "small" πN scattering volumes α_{11} , α_{13} , and α_{31} , are, however, repulsive and since they contribute coherently, c_0 is reduced by $\sim 30\%$.
- ii) Neglecting all contributions to (n_1+n_2) but the dominant one from c_0 gives $(n_1+n_2) \simeq 1.53$ for a typical density. The inclusion of the Lorentz-Lorenz factor gives thus $(1 + \frac{1}{3}(n_1+n_2))^{-1} \cdot (n_1+n_2) \simeq 1.01$. Inspection of Eq. (41) shows thus that there is a nearly complete cancellation of all gradient terms in the region of a uniform nuclear density. This corresponds to a very short wavelength and a very small wave function in the nuclear centre, due to bad penetration into this region. While such a behaviour does not introduce any difficulties of principle in Eq. (41), the short wavelength could cause some of our previous approximations to be questioned. In reality, the problem never arises due to absorption, which ensures a sizable wavelength. From the value of $\text{Im} C_0$ we conclude that $\text{Im}(n_1+n_2) \simeq 0.62$, which is so large that even approximately singular properties of Eq. (41) do not develop. Absorption thus plays an important regularizing role in the scattering; we further notice that its magnitude is 40% of the real part.
- iii) The absorptive term gives a contribution to the real part of (n_1+n_2) which, according to the estimate (46) for $\text{Re} C_0$, is of order -0.6 , i.e., 40% of the main term. As previously discussed, this is probably an upper limit. The uncertainty in this term is large.

- iv) The isospin vector interaction is, to a very good approximation, given by c_1 . Its importance, compared to the main term c_0 , can be seen from the ratio $A^{-1}(c_1/c_0) \simeq A^{-1}$. This implies a 10% effect for $A=10$, which is a factor four less than the corresponding ratio of the local isospin term to the main local term. Since the velocity dependent interaction sometimes is of much greater importance than the local one, it may still be mainly responsible for isospin effects.
- v) The isospin tensor term due to C_5 may contribute significantly to "elastic" double charge to isobaric analogue states. A comparison to the induced isospin tensor interaction from the Lorentz-Lorenz effect, which is of order $(1 + 4\pi/3 c_0 \rho)^{-2} \frac{1}{3} (4\pi c_1)^2$ shows that the term from C_5 is of equal magnitude. Neither does it seem possible to show this term to be negligible compared to the contributions from the isospin vector term to second order. Its actual importance will have to be established by explicit evaluation in specific cases.
- vi) The hyperfine interaction which arises through d_0 is of order $A^{-1}(d_0/c_0) \simeq A^{-1}$ compared to the main term. For a spherical density distribution it only contributes for $\rho \neq 0$ [see Section 3.D.d]. Its effects are small and hard to observe. Similarly, the spin-isospin interaction due to d_1 is of order A^{-2} with an intrinsically small coefficient. We do not believe this term to be observable in practice.

6. COMPARISON WITH EXPERIMENTS

The sources for experimental information on the various parameters in the multiple scattering equations are in principle the following :

- A. elastic scattering in the wide sense, i.e., including single and double charge-exchange scattering to the ground state isobaric analogue states;
- B. absorption cross-sections;
- C. total cross-sections;
- D. level shifts and level widths of π mesic atoms.

Scattering data relevant to the theory of the preceding Sections should, strictly speaking, come from pion energies for which the scattering length and scattering volume description is valid to a good approximation. This limits the pion energy to $\lesssim 30$ MeV ^{*)}.

Furthermore, in order to use a scattering experiment for parameter determination in the present stage of knowledge of pion-nucleon interactions requires a relatively complete experiment with sufficient precision to permit phase shift analysis, since several parameters enter simultaneously under conditions for which none dominates clearly. This source of information is likely to become of importance once the magnitude of the parameters is roughly fixed. At present, we have renounced to analyze scattering experiments, so much more so since suitable data for analysis do not exist below 69.5 MeV.

*) For higher energies we can generalize the equations without explicitly including the structure of the interaction, by replacing $\alpha_i \rightarrow k^{-1} \sin \delta_i e^{i\delta_i}$ and $\alpha_{ij} \rightarrow k^{-3} \sin \delta_{ij} e^{i\delta_{ij}}$, where k is the incident wave number. This replacement is, however, less justified than our previous treatment, though it probably works fairly well in practice.

The quantitative information at present therefore comes from an analysis of data on π mesic atoms: thus all determined parameters refer to zero (or, rather, negative) energy elastic π nuclear interactions. The advantage of using information from π mesic atoms in our present state of knowledge is that fortuitously one or another of the various parameters dominates any single measured quantity: thus even incomplete information can be usefully employed. For example, the $1s$ level shifts are dominated by the real part of the local interaction, the corresponding width by its imaginary part. Similarly for higher states, $2p$, $3d$, etc., the shifts are dominated by the real part of the non-local interaction, the widths by the corresponding absorptive interaction. The isospin dependence is directly displayed by the different level shifts in nuclei with the same Z but varying neutron number.

Within the accuracy of present data, the energy shifts and widths can be expressed analytically by the following method due to M. Ericson, Refs. 12), 17).

The pion is assumed to move non-relativistically in a bound orbit according to the non-local Schrödinger equation. The nucleus is described by a square well of radius R with equivalent mean square radius to the one measured by electron scattering. Thus the Coulomb potential in the nuclear interior is of the harmonic oscillator type :

$$V_c = \frac{Ze^2}{2R} \left[-3 + (r/R)^2 \right] \quad \text{for} \quad r < R \quad (50)$$

The π nuclear interaction in the interior is described by constant functions $m_k(r)$ and $n_k(r)$ [Eqs. (41) and (42)]. The wave function in the nuclear interior is therefore a confluent hypergeometric function. This has to be matched to that outside solution in a Coulomb field, which goes exponentially to zero at infinity [Whittaker function]. The matching must take into account the discontinuity in the derivative of the pion

wave function at the nuclear surface owing to the non-local interaction. Neglecting the small hyperfine term in Eq. (41), the complex energy shift due to the π nuclear interaction can be expanded as follows for circular orbits :

$$\frac{\Delta E}{E_n} \left\{ 1 + \frac{\alpha \ell}{2\ell+1} + 2K_n R \left[1 + \frac{\alpha}{2}(1 - \delta_{\ell 0}) \right] + (K_0 R)^2 / ((2\ell+1)(2\ell+3)) + \dots \right\} \quad (51)$$

$$= \frac{(2K_n R)^{2\ell+1}}{(\ell+1)(2\ell+1)!} \left[-\alpha \ell - (K_0 R)^2 / (2\ell+1) - 3K_n R (2\ell+3)(2\ell+5) + (1+\alpha)(KR)^4 / (2\ell+3)^2(2\ell+5) + \dots \right]$$

The following notations have been used in the expansion (51) :

Unperturbed energy in Bohr orbit ($n=\ell+1$) :	$E_n \simeq -\mu(Ze^2)^2/2n^2$
Wave number of E_n	$k_n = Ze^2\mu/n$
Non-local interaction parameter	$\alpha = n_1 + n_2 = \text{const for } r < R$
Local interaction parameter	$K_0^2 = m_1 + m_2 = \text{const for } r < R$
Effective wave number inside the nucleus :	$K^2 = (-k_n^2 - K_0^2 + 3\mu Ze^2/R)/(1+\alpha)$

Equation (51) results from an expansion in $k_n R$, i.e., the ratio of the nuclear radius to that of the orbit in question, and in $(KR)^2$, i.e., the square of the ratio between the nuclear radius and the effective wavelength inside the nucleus. The precision of expansion (51) has been estimated to be of the order of 10% for the 1s level shifts in ^{16}O and ^{19}F , the heaviest nuclei for which 1s shifts have been measured. For higher ℓ states the accuracy is expected to be similar [naturally the corrections are largest for those elements of given ℓ with largest radius].

The numbers presented below are essentially the same as previously presented by us as preliminary results ¹⁹⁾, the main improvement in the analysis being in the 1s level width. In the analysis given below, all data have been used simultaneously. The various experimental data ^{*}) are, however, presented as determining the parameter on which they depend dominantly. Errors quoted below do not include theoretical uncertainties, unless explicitly stated.

^{*}) The experimental information on π mesic X-rays is reviewed in Ref. ²⁵⁾.

A. 1s Orbits

The complex energy shifts ΔE_N are dominated by the local interaction.

a) Level shifts

The 1s level shifts in elements from Li to F have been measured by Camac et al. ²⁶⁾, by Stearns and Stearns ²⁷⁾, and by West and Bradley ²⁸⁾. The experiments show considerable spread from nucleus to nucleus and are only partially overlapping. It is at present not clear how much of the spread is experimental and how much of it is a real effect. It is, however, possible to extract the average local multiple scattering parameter and isolate the isospin dependence. For nuclei with $N=Z$, i.e., $T=0$, the level shifts are repulsive. Interpreted according to Eq. (41) with $\text{Re} \{m_2(r)\} = 0$ we determine an effective b_0 : -0.029 ± 0.006 . We recall that the corresponding predicted value for $(b_0)_{\text{eff}}$ is -0.018 with a 50% uncertainty from the two-nucleon terms. Comparing the accuracy of the prediction to the value of b_0 without accidental cancellations (0.116), the multiple scattering description is seen to be accurate to about 10%, which is very satisfactory. Since -0.008 of the contribution to the effective b_0 comes from the one-nucleon terms, there is a contribution -0.021 from the real part of the two-nucleon terms which gives $\text{Re} B_0 \simeq -0.022$, i.e., twice $-\text{Im} B_0$.

The 1s level shift in nuclei of $N=Z+1$, i.e., $T=1$, are more repulsive than those of $N=Z$ for the same Z . The interpretation of these isotope shifts as an effective b_1 gives :

$$(b_1)_{\text{eff}} = -0.10 \pm 0.04 .$$

This is in good agreement with the prediction -0.086 from the πN scattering amplitude (no accidental cancellations occur in the isospin term). The experimental uncertainty is so large, however, that at present the comparison can only be considered to be of qualitative significance.

b) Level widths

To our knowledge, the only measurement of a $1s$ level width is that by West and Bradley²⁸⁾ in ^9Be , for which $\Gamma_{\text{Be}} = (1.2 \pm 0.2) \text{keV}$. Since Be is a very light element, we have applied a -19% correction for A^{-1} effects according to Appendix B, Eq. (B.2). An experimental value of $\text{Im}B_0$ is obtained from Eq. (51), interpreted according to Eqs. (41) and (42). The velocity dependent interaction should contribute a term 0.001 to $\text{Im}B_0$ by Fermi motion; this term has been subtracted. Thus :

$$(\text{Im}B_0)_{\text{exp}} = 0.014 \pm 0.003 .$$

This is to be compared to the value $\text{Im}B_0 = 0.011$ from π production data according to Eq. (46). While this is in very good agreement with the measured value, it should be remembered that the result is based on a single element. It would therefore be very valuable to have this measurement reconfirmed and extended to other cases.

B. 2p and 3d Orbitsa) Level shifts

The $2p$ level shift in ^{27}Al was recently found to be $(240 \pm 80) \text{eV}$ by Astbury et al.²⁹⁾, which means an attractive interaction as expected in contrast to the repulsive $1s$ shifts. The local interaction, determined experimentally above, gives a repulsive contribution of about 25% to the shift. The large remaining attractive contribution must thus be due to the non-local interaction. From the experiment, we determine an effective c_0 (which includes the real two-nucleon contributions) :

$$(c_0)_{\text{eff}} = 0.21 \pm 0.07 .$$

This agrees closely with the value 0.208 from π N phase shifts alone. However, we expect the two-nucleon term to contribute a repulsive contribution of 15-30% to the effective c_0 . We should thus only conclude that $(c_0)_{\text{eff}}$ agrees well and is fully consistent with predictions inside experimental limits.

b) Level widths

Direct measurements of 2p and 3d widths do not exist. The attenuation of the intensity of the (2p-1s) and (3d-2p) transition intensities can be used for indirect width determination, if interpreted as due to the competition between the known electromagnetic transition rate and the absorption rate. As all indirect methods, it relies on a complete understanding of the mechanism (both theoretical and experimental) which makes direct measurements preferable *).

With this caution, the experiments ^{26),27),30)} can be directly analyzed following Huguenin ³¹⁾ and M. Ericson ¹²⁾. The local interaction contributes about 20% to the 2p and 3d widths. We find

$$(\text{Im } C_0)_{\text{exp}} = 0.067 \pm 0.017 .$$

The agreement with the value from π production for $\text{Im } C_0$, 0.082, is once more quite satisfactory.

C. Remarks on the Experimental Parameters

a) Short range correlations

The above analysis was made, taking it for granted that the short range anticorrelations between nucleons produce a Lorentz-Lorenz

*) In particular, the fraction of stopped pions reaching the level in question, i.e., the line intensity in the absence of absorption in the level, is assumed known. Since there is a well-known deficiency in the yield of low energy π mesic X-rays for reasons which appear to be experimental and particle independent, we have excluded data on π mesic yields in the suspect energy range from the analysis.

effect. Evidently, the values deduced from experiments for $(c_o)_{\text{eff}}$

and $\text{Im} C_o$ are quite different in the absence of this effect.

Figure 1 displays the estimated range of variation of $(c_o)_{\text{eff}}$ and $\text{Im} C_o$ and the values deduced from experiments with and without the Lorentz-Lorenz effect. The diagram is strongly suggestive of such an effect.

b) Ratio of local to non-local absorption

The predicted ratio of $\text{Im} B_o / \text{Im} C_o$ is expected to be more accurate than each of these terms individually. Both terms are obtained from π production data and are dominated by the deuteron channel. In the ratio any mistake in the evaluation of the deuteron wave function will tend to cancel. It is, of course, not surprising that the ratio agrees with expectations since the individual terms agree well, but as a matter of principle, the ratio is the more accurately predicted quantity.

It is interesting to visualize the parameters derived in terms of the non-relativistic potential (22), so as to get an impression of magnitudes. For a uniform medium for a density derived from a radius parameter $r_o = 1.1F$ the potential is

$$V = \left(45 - 70 + 100 \frac{E \cdot 3}{A} \right) - (140 + 300) K^2 \text{ MeV}$$

where the wave number K is measured in inverse Fermis. Thus, the local potential is repulsive, while the velocity dependent part is attractive.

It might finally be asked if the whole description of an equivalent interaction may be tested in a parameter independent way. Since the interaction should be typical of nuclear matter rather than of the individual nucleus, the energy shifts and widths should vary smoothly with A from

nucleus to nucleus with a dependence of isospin and spin only. Only minor effects due to individual nuclear effects in density distributions should cause perturbations to the smooth variation. It is therefore clear that fluctuations of shifts and widths away from the common trend will be at variance with an "optical potential" description. Certain irregularities seem to appear in the $1s$ level shifts which could indicate individual nuclear effects. The uncertainties are, however, so large that no conclusion can be drawn at present.

7. CONCLUSIONS

The purpose of our investigation of the pion nuclear elastic interaction has been :

- on the one hand, to obtain a quantitative, parameter free, description of the elastic interaction by an equivalent interaction ["optical potential"] derived directly from the basic πN scattering and from $\pi + 2N \rightleftharpoons 2N$ absorption and production processes. For this, the method has been chosen so as to leave the connection between the elementary process and the potential transparent in the final result and so as to underline the importance of short range anticorrelations between nucleons.
- on the other hand, to test the multiple scattering description by a comparison to experimental data on π mesic atoms and, at the same time, to establish a semi-empirical set of parameters for the equivalent potential to serve as a basis for more detailed future analysis.

We therefore first discussed the general problem of the multiple scattering of a particle with short range s and p wave interactions in a finite quantal system of scatterers. An infinite system of integral equations was derived, which for massive scatterers reduced to a systematic expansion into correlation functions of increasing order. It was explicitly shown that it is a good approximation for low energy pions to neglect the excitation energy of all intermediate nuclear states as compared to the intermediate pion energies. As a consequence, the pion-nuclear interaction becomes nearly classical in the sense that the scatterers may be considered to be massive to a good approximation, a feature further enhanced by the smallness of Fermi motion corrections. The infinite system of integral equations is truncated so as to account correctly for nuclear pair correlations with higher order correlations neglected. Within this approximation, the multiple scattering problem reduces to a system of two coupled integral equations of simple type. For low energy pions, these can be transposed into differential form to a good approximation, since the pion wavelength is much larger than the nuclear correlation length. The average interaction obtained in this fashion is velocity dependent and it is non-linear in the nuclear density, due to nuclear polarization (which enters

by the pair correlation function). The corresponding effect in optics is the well-known Lorentz-Lorenz effect in the density dependence of the refractive index; it appears here owing to the important dipole component in the π N interaction. Indeed, the analogy with optics is so strong that the differential equation for pion-nucleon scattering is just a wave equation for an inhomogeneous medium. One of the main conclusions of the investigation is therefore that the elastic pion interaction with the nucleus is described by an "optical potential" in the most literal sense of the word, the analogy with optics being quasi-complete. A further conclusion is that the structure of the interaction differs strongly from the nucleon-nuclear optical models in current use both by its density dependence and by its velocity dependence.

Terms of order A^{-1} have generally been neglected in the pion-nuclear interaction in so far as they do not give rise to new effects. All isospin and spin terms are of this order. In particular, the isospin effects in the local interaction are found to be large. Of particular interest is the appearance of an isospin tensor interaction which can give rise to a direct double charge exchange reaction on two closely correlated nucleons by the process

Compared to the isovector interaction which gives rise to the same process in two steps, the tensor interaction is shown to be negligible in the local terms, while it is significant in the velocity dependent interaction.

An induced hyperfine coupling appears to order A^{-1} but it is small and should be difficult to observe.

The leading A^{-1} corrections to other terms have generally not been given, but the most important case has been evaluated in Appendix B and the general procedure for obtaining these terms is described there.

The present investigation was made with the intention of deriving the multiple scattering parameters quantitatively to about 20% accuracy for low pion energies. Improvements and extensions may be obtained on the following points among others :

- 1) The s and p wave π N interaction was approximated by a structureless point interaction. Structure effects are of little importance to (and even beyond) the precision we have arrived at for low energy pions. The influence of the structure increases with energy. The effect of structure can be included at the price of less transparent multiple scattering equations.

Similarly, the absorptive interaction was described by a phenomenological point interaction. Arguments can be given in favour of such a description, but the precision to which it is valid is not well known.

- 2) The real parts of the phenomenological two-nucleon amplitudes are at present very poorly known: only crude estimates of sign and magnitude are available. While this is no uncertainty of principle, it introduces the major source of uncertainty in the numerical predictions. It is thus highly desirable to obtain more precise estimates of these terms.
- 3) The coupled integral equations for the multiple scattering may be exactly solved numerically. This would clearly lead to improvements in the treatment of the range of the pair correlation function.
- 4) The major A^{-1} corrections can easily be taken into account systematically as sketched in Appendices A and B. We have excluded these terms so as not to burden the exposition.
- 5) The experimental data on π mesic atoms can be exploited more efficiently by detailed numerical analysis with realistic density distributions. The additional effort will be justified as soon as experiments improve beyond the present level.
- 6) No corrections have been applied to the absorptive amplitudes for the restriction on final state phase space due to nuclear binding.

The five experimental parameters which can be determined from present experiments on π mesic X-rays conform as a whole extremely well with predictions within the fairly large uncertainty. We consider this agreement to provide a semi-quantitative demonstration that π nuclear interactions can be understood directly from the basic interactions. On the other hand, we consider it most unsatisfactory that the experimental information on an interaction typical for nuclear matter, rather than for individual elements, should mainly come from measurements on a single nucleus (1s width, 2p shift), from indirect arguments (2p and 3d widths as deduced from intensities). Under these circumstances, we consider a quantitative experimental test to be missing. In spite of this, the following important conclusions can be drawn :

- The pion-nuclear potential is clearly demonstrated to be non-local ^{*)}; it is not possible to reconcile neither the real energy shifts in the 1s and 2p levels with a local potential, nor is it possible to reconcile the 1s, 2p and 3d widths with such a potential.
- The agreement of s and p wave pion absorption with predictions provides quantitative support to the two-nucleon absorption mechanism in the nucleus. It further implies that the reaction $\pi + 2N \rightleftharpoons 2N$ is nearly unaffected by the presence of other nucleons with the further implication that the short range correlations are very similar for bound and free nucleons.
- The absorption rate and branching ratios for π capture in ^3He ³²⁾ and ^4He ¹⁰⁾ permit, in principle, similar conclusions, but the experimental material only gives a weak qualitative indication at present in this case.
- The 2p level shift and the 2p and 3d level widths taken together point clearly to the existence of short range anticorrelations between the nucleons in the nucleus. While the experimental evidence backing this important conclusion can certainly be improved on, even the present data would require major changes to invalidate it entirely. The main other evidence for short range anticorrelations is the need for a repulsive core in nuclear matter calculations.

^{*)} See also Ref. 15).

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A P P E N D I X A

The $(\pi \ 2N)$ Scattering Amplitude

Under the assumption of the main text that the $(\pi \ 2N)$ absorptive interaction is of short range with the nucleons in a relative s state, we can express the general s and p wave pion elastic two-nucleon amplitude in terms of form factors B_k and C_k (taken as constants) in the following way :

$$f_{ij}^2 = \delta(\vec{r}_i - \vec{r}_j) \delta\left(r - \frac{1}{2}(\vec{r}_i + \vec{r}_j)\right) A_{ij}$$

$$\begin{aligned} A_{ij} = & B_0 + B_1(\vec{\sigma}_i \cdot \vec{\sigma}_j) + B_2(\vec{r}_i \cdot \vec{r}_j) + B_3\left(1 - (\vec{\sigma}_i \cdot \vec{\sigma}_j)\right)(\vec{r}_i + \vec{r}_j) \cdot \vec{r} \\ & + B_4(\vec{\sigma}_i \cdot \vec{\sigma}_j)(\vec{r}_i \cdot \vec{r}_j) + B_5\left(1 - (\vec{\sigma}_i \cdot \vec{\sigma}_j)\right)\left[(\vec{r}_i \cdot \vec{r}_j)(\vec{r}_i \cdot \vec{r}_j) + (\vec{r}_i \cdot \vec{r}_j)(\vec{r}_j \cdot \vec{r}_i)\right] \\ & + \vec{r}_i' \cdot \vec{r}_j \left\{ C_0 + C_1(\vec{\sigma}_i \cdot \vec{\sigma}_j) + C_2(\vec{r}_i \cdot \vec{r}_j) + C_3\left(1 - (\vec{\sigma}_i \cdot \vec{\sigma}_j)\right)\left[(\vec{r}_i + \vec{r}_j) \cdot \vec{r}\right] \right. \\ & \quad \left. + C_4(\vec{\sigma}_i \cdot \vec{\sigma}_j)(\vec{r}_i \cdot \vec{r}_j) + C_5\left(1 - (\vec{\sigma}_i \cdot \vec{\sigma}_j)\right)\left[(\vec{r}_i \cdot \vec{r}_j)(\vec{r}_i \cdot \vec{r}_j) + (\vec{r}_i \cdot \vec{r}_j)(\vec{r}_j \cdot \vec{r}_i)\right] \right\} \\ & + iC_6\left(1 - (\vec{r}_i \cdot \vec{r}_j)\right)\left[(\vec{\sigma}_i + \vec{\sigma}_j) \cdot (\vec{r}_i' \wedge \vec{r}_j)\right] + C_7\left(1 - (\vec{r}_i \cdot \vec{r}_j)\right)\left[(\vec{\sigma}_i \wedge \vec{r}_i')(\vec{\sigma}_j \wedge \vec{r}_j) - (\vec{\sigma}_j \wedge \vec{r}_j')(\vec{\sigma}_i \wedge \vec{r}_i)\right] \\ & + iC_8\left\{\left[(\vec{r}_i - \vec{r}_j) \cdot \vec{r}\right](\vec{\sigma}_i - \vec{\sigma}_j) \cdot (\vec{r}_i' \wedge \vec{r}_j) + i\left[(\vec{r}_i \wedge \vec{r}_j) \cdot \vec{r}\right](\vec{\sigma}_i \wedge \vec{\sigma}_j) \cdot (\vec{r}_i' \wedge \vec{r}_j)\right\} \quad (A.1) \\ & + C_9\left\{\left[(\vec{r}_i - \vec{r}_j) \cdot \vec{r}\right](\vec{\sigma}_i \wedge \vec{\sigma}_j) \cdot (\vec{r}_i' \wedge \vec{r}_j) - \left[(\vec{r}_i \wedge \vec{r}_j) \cdot \vec{r}\right](\vec{\sigma}_i \cdot \vec{\sigma}_j) \cdot (\vec{r}_i' \wedge \vec{r}_j)\right\} \end{aligned}$$

The absence of certain terms which apparently would be possible is due to the condition of relative s state for the two nucleons. The 16 constants B_k and C_k are related to the amplitudes of the various isospin and angular momentum channels β_{jt_1} and $\gamma_{jt_1}(S,T;S',T')$ as follows [see also Section 4] :

$$\begin{aligned}
 B_0 &= 1/48 (9\beta_{11} + 3\beta_{01} + \beta_{00} + 5\beta_{02}) \\
 B_1 &= 1/48 (3\beta_{11} - 3\beta_{01} - \beta_{00} - 5\beta_{02}) \\
 B_2 &= 1/48 (-9\beta_{11} + 3\beta_{01} - \beta_{00} + \beta_{02}) \\
 B_3 &= 1/96 (6\beta_{01} - 2\beta_{00} - 7\beta_{02}) \\
 B_4 &= 1/48 (-3\beta_{11} - 3\beta_{01} + \beta_{00} - \beta_{02}) \\
 B_5 &= 1/96 (-3\beta_{01} + 2\beta_{00} + \beta_{02}) \\
 C_0 &= 1/48 (\gamma_{10} + \gamma_{01} + 5\gamma_{21} + 5\gamma_{12} + 3\gamma_{11}(10) + 3\gamma_{11}(01)) \\
 C_1 &= 1/48 (-\gamma_{10} - \gamma_{01} + \gamma_{21} - 5\gamma_{12} - 3\gamma_{11}(10) + 3\gamma_{11}(01)) \\
 C_2 &= 1/48 (-\gamma_{10} - \gamma_{01} - 5\gamma_{21} + \gamma_{12} + 3\gamma_{11}(10) - 3\gamma_{11}(01)) \\
 C_3 &= 1/96 (-2\gamma_{10} - 7\gamma_{12} + 9\gamma_{11}(10)) \\
 C_4 &= 1/48 (\gamma_{10} + \gamma_{01} - \gamma_{21} - \gamma_{12} - 3\gamma_{11}(10) - 3\gamma_{11}(01)) \\
 C_5 &= 1/96 (2\gamma_{10} + 2\gamma_{12} - 3\gamma_{11}(10)) \\
 C_6 &= 1/96 (2\gamma_{01} + 7\gamma_{21} - 9\gamma_{11}(01)) \\
 C_7 &= 1/96 (2\gamma_{01} + \gamma_{21} - 3\gamma_{11}(01)) \\
 C_8 &= 1/32 (-\gamma_{11}(10;01) - \gamma_{11}(01;10)) \\
 C_9 &= 1/32 (-\gamma_{11}(10;01) + \gamma_{11}(01;10))
 \end{aligned} \tag{A.2}$$

The results (A.1) and (A.2) are most easily obtained by projection operator techniques. The 16 constants are thus interrelated by the 12 constants β and γ . Of these constants, C_8 and C_9 yield identically zero contributions when averaged. These terms therefore never contribute in practice, being much smaller than the corresponding terms in A^{-1} or A^{-2} . Therefore the spin-isospin flip amplitudes $\gamma_{11}(S,T:S',T')$ with $S \neq S'$, $T \neq T'$ can be neglected, leaving us with 10 constants in all.

A P P E N D I X B

Higher Order A^{-1} Corrections to Eq. (42)

In the derivation of Eq. (42) only the leading terms in an expansion in A^{-1} were used for each type of terms. The correction terms are mainly of importance in the two-nucleon terms. We here indicate how they can be obtained under the assumption of spin and isospin distribution proportional to the density distribution. We then give the local contribution from the two-nucleon terms including spin and isospin corrections.

Inspection of Eqs. (39) and (A.1) for the amplitudes $f_{ij}^{(2)}$ shows that typical averages in spin and isospin variables involve $\langle(\underline{\tau}_i \cdot \underline{\tau}_j)\rangle$ or $\langle(\underline{\tau}_i \cdot \underline{\tau}_j)\rangle$. Taking the last case, we have

$$\begin{aligned} \langle \underline{\tau}_i \cdot \underline{\tau}_j \rangle &= \frac{\langle \sum_{i \neq j} (\underline{\tau}_i \cdot \underline{\tau}_j) \rangle}{A(A-1)} = \frac{\langle [(\sum_i \underline{\tau}_i)^2 - \sum_i \tau_i^2] \rangle}{A(A-1)} \\ &= \frac{4T(T+1)}{A(A-1)} - \frac{3}{A-1} \end{aligned} \quad (\text{B.1})$$

Similar expressions result for the spin variables. It is easily seen that the condition $i \neq j$ yields important A^{-1} corrections. Applying the technique above to $m_2(\underline{r})$ of Eq. (42), we find :

$$\begin{aligned}
m_2(\underline{r}) = 8\pi\left(1 + \frac{\mu}{2M}\right) \Bigg\{ & B_0 + \left[\frac{4S(S+1)}{A} - 3\right] \frac{B_1}{A-1} + \left[\frac{4T(T+1)}{A} - 3\right] \frac{B_2}{A-1} \\
& + \left[\frac{4S(S+1)}{A} - 3\right] \left[\frac{4T(T+1)}{A} - 3\right] \frac{B_4}{(A-1)^2} - \left[1 + \frac{3}{A-1} - \frac{4S(S+1)}{A(A-1)}\right] \frac{4B_5}{A-1} \\
& + \left[1 + \frac{3}{A-1} - \frac{4S(S+1)}{A(A-1)}\right] \frac{4(\underline{k} \cdot \underline{T})}{A} (B_3 + B_5) \\
& + \left[1 + \frac{3}{A-1} - \frac{4S(S+1)}{A(A-1)}\right] \frac{8(\underline{k} \cdot \underline{T})^2}{A(A-1)} B_5 \Bigg\} \left(g(\underline{r})\right)^2
\end{aligned} \tag{B.2}$$

A P P E N D I X C

Experimental Determination of Two-nucleon Amplitudes

The absorptive part of the phenomenological two-nucleon amplitudes have been determined from the experimental cross-sections for π production in NN collisions by a method similar to the one used by Eckstein for the vertex function ¹⁰⁾. It is briefly illustrated below by the $p+p \rightarrow \pi^+ + D$ process.

The imaginary part of the πD elastic amplitude for unpolarized deuterons obeys the optical theorem separately for the orbital angular momentum $l = 0$ and $l = 1$:

$$\text{Im} f_{\pi D}^l(\theta=0) = \frac{k}{4\pi} \sigma_{\text{tot}}^l \quad (\text{C.1})$$

At low energy $\sigma_{\text{tot}}^l \simeq \sigma_r^l$ to leading order in $\eta = k/\mu$ since we have real absorption. Since the process $\pi^+ + D \rightarrow p+p$ is dominant $\sigma_r^l \simeq \sigma_{\pi^+ + D \rightarrow p+p}^l$. The two-nucleon operator $f_{ij}^{(2)}$ is the one associated with the last process. Thus

$$\begin{aligned} \text{Im} f_{\pi D}^{l=0}(\theta=0) &= \text{Im} \{ \beta_{11} \} |\Phi_D(0)|^2 \\ \text{Im} f_{\pi D}^{l=1}(\theta=0) &= \mu^2 \eta^2 \text{Im} \left\{ \frac{1}{9} (\sigma_{11} + 3\sigma_{11}(\rho) + 5\sigma_{21}) \right\} |\Phi_D(0)|^2 \end{aligned} \quad (\text{C.2})$$

The channel $(j=1, t=1)$ gives no contribution since it is not compatible with the (pp) channel. In Eq. (C.2), $\Phi_D(0)$ is the deuteron wave function at the origin. It has been evaluated from a square well potential

which correctly reproduces the triplet scattering length and effective range. The very short range behaviour of the deuteron wave function is included in the coefficients β and γ . For further details on the potential parameters and the wave function, see for example Ref. ¹⁰⁾.

The cross-sections $\sigma_{\pi^+D \rightarrow p+p}^{\ell}$ are obtained by detailed balancing from the production process $p+p \rightarrow \pi^+D$:

$$\sigma_{\pi^+D \rightarrow p+p}^{\ell} = \frac{2}{3} \frac{p^2}{\mu^2 \eta^2} \sigma_{p+p \rightarrow \pi^+D}^{\ell} \quad (C.3)$$

with p the proton c.m. momentum. The production cross-section is given by:

$$\sigma_{p+p \rightarrow \pi^+D} = \frac{1}{4} \left\{ |a(^3P_1)|^2 + |a(^1S_0)|^2 + |a(^1D_2)|^2 \right\} \quad (C.4)$$

where the amplitudes $a(^1S_0)$, $a(^3P_1)$, and $a(^1D_2)$ refer to the proton waves; the term $a(^3P_1)$ corresponds to pion orbital angular momentum zero, the other two to pion orbital angular momentum unity. The measured values for these amplitudes are given by Woodruff ³³⁾.

Other combinations of the absorptive two-nucleon amplitudes are obtained by similar analysis of the production processes $N+N \rightarrow \pi + N+N$, Refs. ^{34), 35)}.

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FIGURE CAPTION

Plot of c_0 vs $\text{Im} C_0$. The theoretically expected region is represented by the hatched square area; the indicated uncertainty in $\text{Im} C_0$ is due to experimental uncertainties in π production cross-sections, the one in c_0 to the real part of C_0 . The experiments give point A assuming a Lorentz-Lorenz effect, point B assuming no Lorentz-Lorenz effect. The points A and B are surrounded by equiprobability contours with a 50% probability of the experimental value being inside the contour.

FIG. 1