ON THE USE OF THE IMAGINARY MASS REPRESENTATIONS
OF THE POINCARÉ GROUP IN SCATTERING AMPLITUDES.

PART I: KINEMATICS, PHASE SHIFT ANALYSIS

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A B S T R A C T

The imaginary mass representations of the Poincaré group are used in the S-matrix theory in order to describe two particle elastic scattering amplitudes $F(s,t)$ for fixed momentum transfer $t < 0$. In the first part, I, we give the definitions of the various group theoretical quantities in terms of scattering kinematics. At this stage we can formulate in a precise covariant language a "phase shift analysis in the t channel" using the imaginary mass representations of the Poincaré group. This is the natural starting formula for further theoretical investigations.

We express very simply the fixed t dispersion relations in the group theoretical language. Analyticity of the scattering amplitude is stated in a simple and transparent manner, as a relation between real and imaginary parts.

The asymptotic behaviour at high energies is very simple and Regge pole like.

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I. INTRODUCTION

The irreducible representations of the covering group $\tilde{P}$ of the Poincaré group P have been studied extensively^{1),2),3)}. For convenience of reference we state here some well-known facts. The irreducible representations are partly classified according to the value M^2 of the Casimir operator $P^2 = P_\mu P^\mu$, where P_μ are the self-adjoint generators of the translations. This construction of irreducible representations is based on the method of Frobenius, with the translations as the invariant Abelian subgroup of $\tilde{P}$. The classification becomes complete by further decomposition into the irreducible parts of a little group, namely a group which leaves a vector p_μ invariant. p_μ here is an eigenvalue of the operator P_μ . There are three cases according to whether $p^2 = M^2 > 0, = 0, < 0$.

The irreducible representations which are characterized by $M^2 > 0$, correspond to massive elementary particles. One can see most conveniently in the system where the particle is at rest that the little group is the rotation group. (The group which leaves the four vector $(M, \vec{0})$ invariant.) More exactly, we have to take as little group the SU_2 which is the covering group of O_3 to be able to include half-integer spins. So finally in order to complete the description we must give the value of spin, i.e., the Casimir operator of SU_2 . Thus for $M^2 > 0$ the irreducible representations are fully described with the mass m and the spin j . Of course we can express the spin also covariantly in the form of the Pauli-Lubanski operator $W_\mu W^\mu = M^2 j(j+1)$, where $W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma$.

The representations characterized by $M^2 = 0$, are divided in two, according to whether $p_\mu = 0$ or $p_\mu \neq 0$. The ones with $p_\mu = 0$, describe vacuum states. The relevant little group is the homogeneous Lorentz group. The representation with $p_\mu \neq 0$, describes particles of zero mass (photons, neutrinos, continuous spin, etc.). The little group is homeomorphic to the Euclidean motions in the plane. We leave these representations aside as they do not interest us directly at present and we turn to the imaginary mass representations as to our main subject.

As correctly argued by Wigner⁴⁾ the representations which are characterized by $M^2 < 0$, cannot be interpreted as particles since these particles with imaginary mass would have very unphysical properties. However we want to point out that they can still be very useful. In nature we do not have only elementary particles but also other objects, as for example scattering amplitudes which can serve as a linear space for realizing representations of the Poincaré group.

Consider for simplicity elastic scattering of two scalar particles. The scattering amplitude $F = F(s, t)$ will be for reasons of relativistic covariance a function of the energy squared s , and the momentum transfer squared t . If we fix $s = M^2 > 0$, we have something similar to a massive elementary particle. In the centre of mass system (rest system), the little group is the rotation group. The general group theoretical property of completeness of the irreducible representations of a compact group, (here SU_2) together with the conservation of good quantum numbers lead to the usual partial wave decomposition^{5), 6)}.

We want to investigate $F(s, t)$ for fixed t , starting from a similar group theoretical point of view. In the physical region t is negative and corresponds to imaginary mass. Therefore an expansion of $F(s, t)$ into the imaginary mass representations of the Poincaré group will be the natural basis for investigating the amplitude for fixed t .

In section 2 we give the group theoretical parametrization of the fixed t scattering amplitude in terms of physical quantities describing the kinematics. In sections 3 and 4 we state the decomposition theorem and give the physical interpretation of the irreducible parts, for example, in terms of Feynman diagrams or Regge poles and asymptotic behaviour of the scattering amplitude at high energy. In the s -channel (fixed s), the various waves are characterized uniquely by the angular dependence. We see here that for fixed t it is the asymptotic behaviour of the amplitude as $s \rightarrow \infty$ which characterizes uniquely the various "waves". As in the low energy region we have a simplified picture because of dominance of the low waves, we expect also simplicity in the

extreme relativistic region from dominance of the waves with the greatest Real j on the complex j -plane (the top Regge pole trajectory).

In section 5 we express very simply the fixed t dispersion relations in the group theoretical language. Analyticity is expressed in this language as a simple relation between the real and the imaginary parts of the scattering amplitude. We arrive thus as a natural Regge pole type representation. Further, analyticity, together with unitarity, suggest strongly that we use only the unitarity representations of the $2+1$ Lorentz group. This imposes restrictions on the position of singularities in the complex j -plane. Thus for $t < 0$ all allowed singularities on the right of the background term have to be real and of spin ≤ 1 . This result is well known from models; however a general proof is still lacking.

The results of this paragraph might also be of some group theoretical interest. They demonstrate explicitly the use of the supplementary series in extending the completeness of the unitary representations of the $2+1$ Lorentz group so that one is able in this way to express a wider class of functions, which are not square integrable.

II. GROUP THEORETICAL PARAMETRIZATION OF THE t -CHANNEL AMPLITUDE

In order to bring up analogy we consider first the elastic scattering amplitude $F(s, t)$ for two scalar particles of equal mass (μ) in the s -channel⁶⁾. $F = F(s, t)$ is a function of $s = (p_1 + p_2)^2$ and $t = (p_1 - p_1')^2$. For fixed s , the best suited system is the centre of mass. In this system $P = p_1 + p_2 = (\sqrt{s}, \vec{0})$ is a constant four vector and $P^2 = s$, is the analogue of the $(\text{mass})^2$. Only the components of the relative momentum $p = p_1 - p_2 = (0, \vec{p})$ or $p' = (0, \vec{p}') = (p_1' - p_2')$ can have various values. We have, however, $p^2 = p'^2 = s - 4\mu^2 = \text{constant}$.

Fig. 1 Kinematics for fixed s .

The little group, the rotation group, operating on $p' = |p'| (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ leaves the p^2, p'^2 invariant. θ , and ϕ , the relevant group theoretical parameters are the polar and azimuthal angles of p' with respect to p which we assume along the Z -axis. The angle θ is related to the invariant momentum transfer as follows: $-t = (s - 4\mu^2) \sin^2 \theta/2$. Now the Peter-Weyl theorem provides us with a complete basis for representing any continuous (square integrable) function on the unit sphere in terms of the matrix elements $D_j^{m, m'}(R)$ of the irreducible representations of SU_2 . The relevant complete basis consists of the spherical functions $d_j^{m, 0}(\phi, \theta, \chi) = \sqrt{4\pi/2\ell+1} Y_j^m(\theta, \phi)$. Assuming that the physical law which gives the scattering admits the symmetry of the group we can write:

$$F(s, \hat{p}') = \sum_{\ell=0}^{\infty} (2\ell+1) \alpha_{\ell}(s) Y_{\ell}^0(\theta, \phi) = \sum_{\ell=0}^{\infty} (2\ell+1) \alpha_{\ell}(s) P_{\ell}(\cos \theta). \quad (2.1)$$

For the reduction formula of two particle plane waves and of the S-matrix into irreducible representations, see appendix III.

We should remind you here that the importance of this group-theoretical formula, which is an elementary application of the Wigner-Eckart theorem⁵⁾, is not merely the completeness of the representations (we could use any complete system of functions). It is mainly the fact that the formula (2.1) factorises the amplitude into dynamics and symmetry. Thus now the $\alpha_\ell(s)$ contain the dynamics and the $P_\ell(\cos \theta)$ reflect the symmetry. Clearly this is possible only with the use of the representations of the symmetry group which leaves the Hamiltonian invariant.

In this way we achieve also a great practical economy. In our example we need only a subset of the complete system the $\{Y_\ell^m \text{ with } m = 0\}$. Further, if for some dynamical reasons only one wave is involved in a scattering amplitude, we need only one parameter to describe it, the corresponding $\alpha_\ell(s)$. With an arbitrary complete system we could not discover this simplicity so easily.

Well in the s-channel all this is our bread and butter and we do it most of the time unconsciously, but in the t-channel it will be very important for us to realize it.

Let us look now at the t-channel (cross channel) or more exactly to the same physical scattering amplitude for fixed t. We want to factor out the "wave" dependence on s for fixed t, in complete analogy to what we have done previously for fixed s where we have factored out the wave dependence on $\cos \theta$, i.e. on t.

As we shall see soon explicitly the relevant little group is the three-dimensional Lorentz group L_3 . More precisely we shall deal with the $SU(1,1)$ group which for the imaginary mass representations of $\tilde{P}$ plays the role which the SU_2 plays for the representations with $M^2 > 0$. This group has been studied exhaustively by Bargman²⁾ whom we follow very closely.

Since the L_3 group is not compact, instead of a discrete spectrum of ℓ we expect also a continuous spectrum and the sum of formula (2.1) is replaced by a general linear sum containing integrals. The irreducible representations corresponding to a fixed value of the Casimir operator will be in general infinite dimensional (for example all the unitary ones). Thus the waves of the $SU(1,1)$ group will be of infinite polarizability instead of the $2\ell + 1$ polarizability of the SU_2 waves.

We shall give first the group theoretical parametrization of L_3 in terms of the familiar kinematic nomenclature in the t-channel.

In the t-channel, instead of the c.m. system the convenient system is the brick-wall (Breit system). Here $t = \text{constant}$, and the "waves" are in s , i.e. (s is distributed.)

Fig. 2 Kinematics for fixed t .

P now is $p_1 - p_1' = (0, 0, \sqrt{-t})$, and $P^2 = t = -4\alpha^2$.

The relative momentum $p = p_1 + p_1' = (2\omega, 2\alpha \tan \theta \cos \phi, 2\alpha \tan \theta \sin \phi, 0)$ where $2\omega = 2\sqrt{p^2 + \mu^2}$, $p = \alpha/\cos \theta$. The components of this momentum are constrained by:

$$4m^2 + \frac{4\alpha^2}{\cos^2 \theta} - 4\alpha^2 \tan^2 \theta \cos^2 \phi - 4\alpha^2 \tan^2 \theta \sin^2 \phi = -t + 4\mu^2.$$

Thus

$$p_0^2 - p_x^2 - p_y^2 = -t + 4\mu^2 = \text{const}$$

which shows that the "relative" momentum p transforms according to the $2+1$ Lorentz group L_3 .

To conform with the group theoretic notation of Bargman's paper we introduce new, more convenient variables.

$$\left. \begin{matrix} p_0 \\ p_x \\ p_y \end{matrix} \right\} = \sqrt{4\mu^2 - t} \begin{cases} \cosh 2\zeta \\ \sinh 2\zeta \cos \phi \\ \sinh 2\zeta \sin \phi \end{cases} \quad (2.2)$$

where $0 \leq \zeta < \infty$, $-\pi \leq \phi_{1,2} < +\pi$.

In terms of these parameters every element $\alpha \in L_3$, $SU(1,1)$ can be exponentiated in the form:

$$\alpha = e^{i2\phi_1 H_{12}} e^{2\zeta H_{10}} e^{i2\phi_2 H_{12}}, \quad (2.3)$$

where H_{12} is the generator of ordinary rotations (the compact sub-group SU_1 of the $SU(1,1)$) and H_{10} of proper Lorentz transformations. The ϕ_1 , ϕ_2 and ζ correspond to the Euler angles which enter in the description of the ordinary group of rotations, SU_2 ⁵⁾. The invariant volume element (both left and right invariant) is given by:

$$d\Omega = \frac{1}{4\pi^2} \sin 2\zeta d\zeta d\phi_1 d\phi_2. \quad (2.4)$$

Since the generator of the proper Lorentz transformation is purely imaginary the $\cos \theta$ of the usual rotation group is replaced here by $\cosh 2\zeta$.

8.

In terms of the Euler angle ζ the s is given by

$$\cos 2\zeta = -1 + \frac{2s}{4\mu^2 - t} \quad (2.5)$$

which is exactly the cosinus of the scattering angle in the cross channel as it appears in the Regge pole theory.

In appendix I we review quickly the definitions of the group and of the Hilbert spaces for its unitary representations, modified appropriately when needed for our purposes. In appendix III we reduce the tensor product of two particle states and decompose the S-matrix accordingly.

III. DECOMPOSITION THEOREM AND PHYSICAL INTERPRETATION OF THE IRREDUCIBLE PARTS

The decomposition into the irreducible representations of the $SU(1,1)$ group is carried out according to the eigenvalue spectrum of the Casimir operator $Q = H_{01}^2 + H_{02}^2 - H_{12}^2$, and according to the range of eigenvalues m of H_{12} , which is the generator of rotations in the Euclidean 1,2 plane^{7),8)}. The H_{01} and H_{02} , which appear in the above expression of the Casimir operator Q generate proper Lorentz transformations along the x and y axis respectively. We remind the reader here that the L_3 group has infinite connectivity since it is topologically homeomorphic to the direct product of the circle and the Euclidean plane^{2),4)}. Thus its covering group unfolds ∞ -sheets, and we find in it also multi-valued representations with continuous (not integer or half integer) third component of the spin. However here we concentrate our attention only on single or double valued representations, i.e. on the representations of $SU(1,1)$. Each irreducible representation is conveniently described in the canonical basis, which consists of eigenvectors $|qm\rangle$ of Q and H_{12} (spin) with eigenvalues q and m respectively. The spectrum q of Q which characterizes the unitary representations consists of a discrete and a continuous part:

$$0 < q < \infty \text{ and } q = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\dots\dots$$

The continuous spectrum is divided into two intervals. The interval $1/4 < q < \infty$ corresponds to the principal series and the exceptional interval $0 < q < 1/4$ to the supplementary series.

There are two series of irreducible representations which correspond to discrete q , the D_k^+ and D_k^- where $q = k(1-k)$ and $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\dots\dots$. The D_k^+ contains $m = k, k+1$. The D_k^- contains $m = -k, -k-1, -k-2, \dots\dots$. The representations D_k^+ and D_k^- are complex conjugate to each other, they differ only in the range of eigenvalues of m .

It is convenient to introduce a new variable in order to bring up the analogy of Q to the $-\vec{J}^2$, the total angular momentum which is the Casimir operator of SU_2 . Thus let $q = -j(j+1)$. The spectrum of Q in the complex j -plane is given in Fig. 3, which strikingly reminds us of the Regge pole plane. The principal series occupies the imaginary axis and the supplementary series occupies the interval $0 < j + \frac{1}{2} < \frac{1}{2}$ of the complex $(j + \frac{1}{2})$ plane. We should remark that two points in the $j + \frac{1}{2}$ imaginary axis which are conjugate correspond to the same q and therefore belong to the same irreducible representation. Thus in Fig. 3 we split each Hilbert space H_j in two H_j^+ and H_j^- , according to whether $m \geq 0$ or $m < 0$ in analogy to D_k^+ and D_k^- . The points of the (unitary) discrete spectrum are double corresponding to D_k^+ and D_k^- .

These representations are all unitary and of course infinitely dimensional since the $SU(1,1)$ group is not compact. The matrix elements $d_j^{m,m'}(z) = \langle jm | D(\phi, 2\zeta, \phi') | jm' \rangle$ of the group transformations $D(\phi, 2\zeta, \phi') \in SU(1,1)$, taken between two vectors $|jm\rangle$ and $|jm'\rangle$ of the canonical basis were given in analytic form by Bargman^{2),10)}:

$$d_j^{m,m'}(z) = \left(\frac{\Gamma(j+m+1)\Gamma(j-m+1)}{\Gamma(j+m'+1)\Gamma(j-m'+1)} \right)^{\frac{1}{2}} \left(\frac{1+z}{2} \right)^{-\frac{1}{2}(m+m')} \left(\frac{1-z}{2} \right)^{\frac{1}{2}(m-m')} F(j+m, -j, 1+m-m', \frac{1-z}{2}) \quad (3.1)$$

$$m \geq m'$$

where $F(j+m, -j, 1+m-m', \frac{1-z}{2})$, $m \geq m'$ are hypergeometric functions. In particular the $d_j^{0,0}(z) = P_j(z)$ are the ordinary Legendre functions. For the discrete series the $d_j^{m,m'}(z)$ can be expressed in terms of the Jacobi polynomials⁹⁾

$$d_j^{m,m'}(z) = \frac{1}{(m-m')!} \left(\frac{\Gamma(j+m+1)\Gamma(m-j)}{\Gamma(j+m'+1)\Gamma(m'-j)} \right)^{\frac{1}{2}} \left(\frac{1+z}{2} \right)^{-\frac{m+m'}{2}} \left(\frac{1-z}{2} \right)^{\frac{m-m'}{2}} P_{m+j}^{(m-m', m-m')} \left(\frac{1+z}{2} \right). \quad (3.2)$$

$$m \geq m'$$

Fig. 3

The spectrum q of the Casimir operator $Q, q = -j(j+1)$ in the j -plane.

- The boson discrete spectrum D^+, D^- .
- × The fermion $>> > >$
- | The principal series
- ... The supplementary series
- Elementary particles, bosons
- × Elementary particles, fermions.

The m, m' indicate ordinary Fourier series projections corresponding to real angles ϕ_1, ϕ_2 of formula (2.3). They are either integer or both half integer numbers and we have to restrict ourselves to $|m|, |m'| \geq -j$ as mentioned earlier. Similar formulae are valid also for $m < m'$. The $Z = \text{ch } 2\zeta$ where 2ζ is the imaginary Euler angle of (2.3).

In appendix II we list a few properties of these functions, some of which are very similar to familiar relations of Legendre functions. For more information consult refs. 2), 10), 11), 12) and other references which are cited there. A very interesting property of these functions is the asymptotic behaviour when $Z \rightarrow \infty$, i.e., according to formula (2.4) when $s \rightarrow \infty$. Indeed these functions are uniquely characterized by the asymptotic behaviour for large Z , as the $P_j(\cos \theta)$ are characterized by their angular dependence.

We have²⁾:

$$d_j(Z) \underset{Z \rightarrow \infty}{\sim} Z^j \quad (3.3)$$

This shows that the $d_j^{\pm}(Z)$ of the discrete spectrum are proper square integrable functions over the group manifold. When j lies in the continuous spectrum of the imaginary axis, i.e. in the principal series, the $d_j(Z) \rightarrow Z^{-\frac{1}{2} + i\rho}$, so they are not square integrable, they are distributions. However, because of their imaginary phase one can make out of them square integrable wave packets. This means that we can use these representations as a basis for expressing square integrable functions, i.e. wave packets as we see below (see also appendix III). The $d_j(Z)$ with j in the exceptional interval are not square integrable and since they do not have a phase like the representations of the principal series we cannot make wave packets with them.

Now we can state the analogue of the Peter-Weyl theorem^{2), 10), 11)}. Every square integrable function over the group manifold can be expressed as a linear infinite sum of the matrix elements $d_j^{m, m'}(Z)$:

$$f^{m,m'}(z,t) = \sum_{\substack{j=1, \\ \text{or } j=\frac{1}{2}, \frac{3}{2}, \dots}}^{M-|m-m'|} \tilde{f}_j(t) d_j^{m,m'}(z) + \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d_j(2j+1) \tilde{f}_j^{m,m'}(t) d_j^{m,m'}(z) \quad (3.4)$$

where

$$\tilde{f}_j(t) = \int_{-\infty}^{\infty} dz d_j^{m,m'}(z) f(z,t), \quad j = -\frac{1}{2}, -1, -\frac{3}{2}$$

and

$$\tilde{f}_j(t) = \int_{-\infty}^{\infty} dz e_j^{m,m'}(z) f(z,t), \quad \text{Real } j = -\frac{1}{2}$$

The $e_j^{m,m'}(z)$ are the functions of the second kind which are related to the $d_j^{m,m'}(z)$ as the Legendre functions of the second kind $Q_j(z)$ are related to the $P_j(z)$ ¹¹⁾. Together with (3.4) we have also the Plancherel formula, $\int_{-\infty}^{\infty} dz |f|^2 = \int_{-\infty}^{\infty} d_j |f_j|^2 + \sum_j |\tilde{f}_j|^2$, which assures the completeness. Since the dependence on ϕ, ϕ' is very simple we have suppressed it. Actually, by making use of the Wigner-Eckart theorem we can restrict ourselves to specific values of the m, m' , for example in the case of scattering of scalar particles to $d_j^{0,0}(z) = P_j(z)$ which correspond to the $P_j(\cos \theta)$ of formula (2.1). We have introduced in (3.3) the extra parameter t having in mind the application of the completeness to the description of scattering amplitudes at fixed t . Indeed the formula (3.4) with $m = m' = 0$, as we shall see in section V, decomposes the scattering amplitude $F(s, t)$ for fixed t into "partial waves" of $\cos h 2\zeta$ as the formula (2.1) for fixed s decomposes the $F(s, \cos \theta)$ in partial waves of $\cos \theta$.

The sum over the discrete class in (3.4) extends from $|j| = \frac{1}{2}$ up to the maximum helicity flip involved, $j_{\max} = M - |m - m'|$, where $M = \max \{ |m|, |m'| \}$. This is due to the fact that each irreducible space $H_j^{\pm}$ is spanned with vectors of quantum number $m = \pm |j|, \pm (|j| + 1), \pm (|j| + 2), \dots$. Thus, if for example we consider scattering of spin zero particles - for spin $\frac{1}{2}$, non spin flip amplitudes - we can restrict ourselves to $m = 0$, and thus see that no "waves" of the discrete series will appear. For spin flip amplitudes in pion nucleon scattering the minimum possible $m = 1$. Therefore in this case we shall have $|j| = 1$ contributions.

Analogous statement can be made when the particle 1, (Fig. 2) is a photon and 1' is a pion as in the example (iii) of chapter IV. Cases with m equal to a half integer occur when the particle 1 is a fermion and the particle 1' is a boson as in pion-nucleon scattering looking at the u channel. For nucleon scattering we have $|j| = 1$ and $|j| = 2$ contributions. Therefore (with Wigner time reversal) we need six independent parameters to describe the helicity amplitudes. One uses charge symmetry to reduce them to five. The supplementary series does not show up in formula (3.4) since it is a set of non-square integrable functions.

The Peter-Weyl theorem, as stated in (3.3), is very interesting but not enough for the most general elastic scattering amplitude. Indeed the scattering amplitudes do not have to be square integrable in s (in the asymptotic region according to (2.4) we have $dZ \sim ds$). We are bound only by unitarity, so we can go as far as the Froissart bound²¹⁾ $F(s,t) \sim s$ without trouble. Let us therefore look at what we have left out, i.e., at the supplementary series and even at the non-unitary representations. The simplest non-unitary representations are the finite dimensional ones to which we now turn since they give a good insight on physics.

IV. THE FINITE DIMENSIONAL REPRESENTATIONS OF THE L_3 LORENTZ GROUP

These representations are the finite dimensional, spinorial ones. They are just polynomials in $\cosh 2\zeta$, and of course they are not unitary since the L_3 group is not compact. In the language of S-matrix theory they correspond to scattering amplitudes with exchange of one elementary particle. In the following we give some simple examples where these representations appear as Born terms of Feynman diagrams.

- (i) $g\phi^3$ theory, exchange of one spin zero particle.

The scattering amplitude is given by:

$$F(s,t) = \frac{g^2}{t - \mu^2} = \frac{g^2}{t - \mu^2} P_0(\cosh 2\zeta), \quad (4.1)$$

where g is the coupling constant and μ is the mass of the exchanged scalar particle. The factor $g^2/t - \mu^2$ contains the dynamics and the zeroth order Legendre polynomial $P_0(\cosh 2\zeta)$ expresses the symmetry.

- (ii) Pion-pion scattering through exchange of a massive vector boson coupled to conserved current.

The scattering amplitude $F(s,t)$ is given by:

$$\begin{aligned} F(s,t) &= \frac{g^2}{t - M^2} (p_1 + p_1')^\mu (g_{\mu\nu} - \frac{q_\mu q_\nu}{M^2}) (p_2 + p_2')^\nu = \\ &= \frac{g^2}{t - M^2} (t - 4p^2 + 2s) = \frac{g^2(t - 4p^2)}{t - M^2} P_1(\cosh 2\zeta) \end{aligned} \quad (4.2)$$

The conservation of the current is essential in order to keep only the spin one part. We have an analogous case in Coulomb scattering. The factor $t - 4\mu^2 = q_t^{2\ell}$, $\ell = 1$, is the threshold behaviour (in the t -channel), and $g^2/t - M^2$ describes the strength and shape of the potential. In a way we can state that the Born terms of scattering amplitudes are determined from the finite dimensional representations, with additional dynamical ingredients, the coupling constant, the mass of the exchanged particle and the threshold behaviour.

Finally we give an example to exhibit also the appearance of the angle ϕ associated with the quantum number m .

(iii) $\gamma + \pi \rightarrow \pi + \pi$. The Born term amplitude corresponding to a ρ exchange is given by:

$$F(s, t, \phi) = \frac{gf}{t - M^2} \epsilon_{\mu\nu\rho\sigma} k^\mu \epsilon^\nu(p_1, p_1') \sigma^\rho =$$

$$= \frac{gf}{t - M^2} (q_1)_t (k)_t P_1^1(\cos \theta_t) e^{i\phi} \quad (4.3)$$

where $(q_1)_t = \sqrt{t/4 - \mu^2}$, $(k)_t = (t - \mu^2)/2\sqrt{t}$ are the momenta of the pion 1 and of the photon respectively, calculated from the cross channel. The $\cos \theta_t$ in this particular case is given by:

$$\cos \theta_t = \frac{\sqrt{t(t - 4\mu^2)}}{-t + \mu^2} \left(-1 + \frac{2s}{4\mu^2 - t} + \frac{3\mu^2 - 2t}{4\mu^2 - t} \right), \left(k + \sqrt{k^2 + \mu^2} \right)_t = \sqrt{-t}.$$

We have a similar case for the spin flip amplitude in π -N scattering.

Apparently, if the coupling constants in the above Born terms are replaced by form factors $g(t)$, these formulae will describe exact contributions from exchange of one elementary particle. We remark that

the $P_j(Z)$, as they appear in the spinorial representations with $Z > 1$ do not constitute a complete set in the physical region. However, analytically continued in the unphysical region $|Z| < 1$, they become a complete set for the cross channel where they are transformed into the unitary representations of the SU_2 group. Indeed, with crossing $F(s,t,u) \rightarrow F(t,s,u)$ the $P_j^m(\cos h 2\zeta, \phi)$ of the above diagrams are replaced by $P_j^m(\cos \theta, \phi)$, i.e., by the irreducible representations of the SU_2 group. The S-matrix through crossing connects analytically the representations of the rotation group with the representation of the L_3 Lorentz group.

Aside from these finite dimensional non-unitary representations the whole complex j plane outside $\text{Re } j = -\frac{1}{2}$, $-\frac{1}{2} < j < 0$ and $j = -n/2$, $n = 1, 2, 3$ is filled up with non-unitary representations which are however infinitely dimensional.

V. FIXED t DISPERSION RELATIONS IN THE LANGUAGE OF THE IMAGINARY MASS REPRESENTATIONS OF THE POINCARÉ GROUP

Consider for simplicity an amplitude $F(s, t, u)$ for elastic scattering of two scalar particles of equal mass. For fixed physical negative momentum transfer $t < 0$, $F(s, t, u)$ satisfies a one-dimensional dispersion relation of the form¹³⁾:

$$F(s, t, u) = P(s, t, u) + \int_{4\mu^2}^{\infty} ds' \frac{F_s(s', t)}{s' - s} + \int_{4\mu^2}^{\infty} du' \frac{F_u(u', t)}{u' - u}, \quad (5.1)$$

where the "absorptive parts" F_s, F_u give the discontinuity of $F(s, t, u)$ along the right and left hand cut respectively. The s, t and u are the familiar Mandelstam variables connected by $s+t+u = 4\mu^2$ ¹⁴⁾. The absorptive parts F_s and F_u are related by crossing symmetry. In our example $F(s, t, u) = F(u, t, s)$ and we can use only one function $F_s(s, t) = -F_u(u, t) = F$. The $P(s, t, u)$ expresses terms which do not contribute to the discontinuities on the cuts¹⁵⁾. In order to be able to write the dispersion integral representation (5.1) without subtractions, we have assumed that the spectral functions vanish at infinity; otherwise we might need two subtractions. In the following we shall replace $F(s, t, u)$ in (5.1) with the normalized amplitude $\phi(s, t, u) = \frac{F(s, t, u)}{s}$ which is more convenient for the group theoretical reduction of the S -matrix (see appendix III). Analyticity of course is simpler in F but in any case we can easily go from F to ϕ and vice versa. The dispersion relation (5.1) with F replaced by ϕ might need one subtraction.

Let us restrict our attention first to the right hand cut of the complex s -plane. The integral from $4\mu^2$ to $+\infty$ consists of two parts, the cut of the physical region, $(4\mu^2 - t) < s < +\infty$, and the cut $4\mu^2 < s < (4\mu^2 - t)$ of the unphysical region. When $t \rightarrow 0$ the latter cut disappears and the physical cut extends from $4\mu^2$ to $+\infty$. In the following it is more convenient to work with the variable $Z = \cos h 2\zeta = -1 + \frac{2s}{4\mu^2 - t}$. In the Z -plane the physical right hand cut is from $+1$ to

$+\infty$ and the unphysical from $(4\mu^2 + t)/(4\mu^2 - t)$ to $+1$. Similarly the u -cut (from the crossed channel) gives the left hand cut on the Z -plane with a physical part from $-\infty$ to -1 and an unphysical part from -1 to $-(4\mu^2 + t)/(4\mu^2 - t)$.

Introducing the variable Z in the dispersion relation (5.1) the integral over the physical right hand cut takes the form:

$$\int_{4\mu^2 - t}^{\infty} ds' \frac{\varphi(s', t)}{s' - s + i\epsilon} = \int_1^{\infty} dz' \frac{f(z', t)}{z' - z + i\epsilon} \quad (5.2)$$

where

$$f(z, t) \equiv \varphi\left(\frac{(z+1)(4\mu^2 - t)}{2}, t\right).$$

Further, with the use of Mehler's identity¹⁶⁾

$$\frac{1}{z' - z + i\epsilon} = \frac{-1}{2i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} dj \frac{(2j+1)}{\sin \pi j} P_j(-z + i\epsilon) Q_j(z') = \frac{i}{2} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} dj (2j+1) \cot \pi j P_j(z') Q_j(z) \quad (5.3)$$

we express the dispersion integral (5.2) in the form,

$$\int_1^{\infty} dz' \frac{f(z', t)}{z' - z + i\epsilon} = \pi \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} dj (2j+1) f_j(t) P_j(-z + i\epsilon) \quad (5.4)$$

where

$$f_j(t) = \frac{i}{2\pi} \int_1^{\infty} dz Q_j(z) f(z, t)$$

The formula (5.4) is very important as it expresses the scattering amplitude in a general linear superposition of irreducible energy waves. This is what we have called earlier a "phase shift analysis in the t channel". A representation of this type was proposed also by Fivel¹⁷⁾ for dispersion relations in non-relativistic potential scattering. What we show here is that this representation is the most natural one for completely relativistic scattering. More precisely we want to assume that they are the absorptive parts

$$f(Z,t) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj(u+i) f_j(t) P_j(Z) \quad (5.5)$$

which can be expressed as a general sum over "imaginary mass waves" $P_j(Z)$. The coefficients $f_j(t)$ are the analogues of the partial wave phase shifts and express dynamics. When the $f_j(t)$ are given one can of course use (5.5), which is Mehler's inversion formula, to recover the total $f(Z,t)$, but then the dynamics overlap and are obscure since $f(Z,t)$ is not irreducible.

Formulae similar to (5.4) and (5.5) are valid also for the contributions from the unphysical region. In this case the appropriate Mehler's identity is the last expression to the right in (5.3). We then have

$$\int_{-1}^1 dZ' \frac{f(Z',t)}{Z'-Z+i\epsilon} = -2 \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj(u+i) f_j(t) Q_j(Z-i\epsilon) \quad (5.6a)$$

$$f_j(t) = \frac{i}{4} \cot \pi j \int_{-1}^1 dZ f(Z,t) P_j(Z) \quad (5.6b)$$

$$f(z,t) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj (j+1) f_j(t) P_j(z), \quad |z| < 1 \quad (5.6a)$$

It is clear that in (5.6a) we can include also the contributions from the unphysical part of the left hand cut. With this in mind the integral in (5.6b) has been extended from $Z = -1$ to $Z = +1$. An analogous statement can be made also for the physical region of the left hand cut in relation to the formulae (5.4).

The representation of the dispersion relations in the form (5.4) and (5.5) is also remarkable for expressing the analytic properties of the scattering amplitude in a simple form. Thus in (5.4) the amplitude is a continuous linear superposition of $P_j(Z)$ and the absorptive parts are again $P_j(Z)$, the discontinuities of $P_j(Z)$ for $Z > 1$. In (5.5) we have a superposition of $Q_j(Z)$ to give absorptive parts $P_j(Z)$ which are the discontinuities of $Q_j(Z)$ for the unphysical region $|Z| < 1$. Knowing the absorptive parts one can construct the whole amplitude with simple analyticity. We should notice here that the representations of the group provide directly a basis only for the physical region where Z is real and larger than one. For the unphysical region we have used analytic continuation of the waves to $|Z| < 1$.

The formulae (5.4), (5.5) correspond to the background term $\text{Real } j = -\frac{1}{2}$ in the Regge pole language. They express contribution from absorptive parts f_s which vanish faster than $1/\sqrt{s}$ at high energies. If instead of $f(Z,t)$ we want to express F_s in a similar way the background is displaced on the axis $\text{Real } j = \frac{1}{2}$. However, physically, the most interesting part is not this since the high energy experiments seem to indicate an asymptotically constant total cross section,

$$\sigma_{\text{TOT}} \sim \frac{F(s,0)}{s} \underset{s \rightarrow \infty}{\sim} \text{constant}^{18)}.$$

These other contributions are to be expressed either by the supplementary series or the exchange of elementary particles (spinorial representations) or the other non-unitary representations (complex Regge poles). In any case while we remain within the unitary representations we have a powerful mathematical machinery at hand and we should try first to see what they can give at the most.

The principal series appears in the generalization of the Peter-Weyl theorem (3.4) as a basis for expressing square integrable functions. Nevertheless we can ask the question what is the most general class of functions for which the $P_{-\frac{1}{2} + ip}(Z)$ form a complete basis, for example, in analogy to the fact that the generalized eigenfunctions e^{ipx} of the translation operator which appear as the kernel of the ordinary Fourier transform form a complete basis not only for L_2 functions but also for tempered distributions¹⁹⁾. The answer is easy to guess. The functions $P_{-\frac{1}{2} + ip}(Z)$ behave asymptotically as $Z^{ip}/\sqrt{Z}$ which is a kernel of Mellin transform²⁰⁾ and therefore we expect the principal series to be complete also for distributions of asymptotic behaviour $(\ln Z)^N/\sqrt{Z}$, any N . To express a wider class of functions we need to go out of the principal series.

The supplementary series is like a Regge cut from $-\frac{1}{2}$ to 0, as in the case of potential scattering with marginally singular potentials $V(r)$ which are attractive near the origin and behave like $V(r) \sim g^2/r^2$, $g^2 < 1/4$. The value $g = \frac{1}{2}$, which is the bound beyond which the unitarity of the S-matrix is violated corresponds to $j = 0$ which again is the boundary point for the unitary representations of the group L_3 . It is indeed probable that we need only the unitary representations in order to express a physical scattering amplitude. Besides the experimental evidence of constant high energy cross sections that we have mentioned earlier there is also theoretical support for such a conjecture. The unitarity of the S-matrix together with an assumption of Regge like behaviour imply that $F(s, t)$ is bounded by $|F(s, t)| < s^{21}$. As it is the assertion of the present paper that Regge behaviour is a result of relativity, we have the same bound.

This already limits us to Real $j < 1$. We have yet to prove that for $t < 0$ the singularities on the strip $\frac{1}{2} < \text{Real } j < 1$ of the j -plane are real, as in potential scattering²²⁾ and in relativistic theories with Mandelstam analyticity²³⁾. This is expected to be true also from plausibility arguments. In particular for the leading trajectory at $t = 0$ it follows from unitarity, positive cross section. We would like to prove this in general using probably causality and TCP but we have not been able to do so.

Let us recapitulate briefly what completeness we have established using all the unitary representations of the L_3 group. We can expand any function f which belongs to the Hilbert space $H = \int^{\oplus} H_j$, the direct integral of all the H_j of the unitary representations. To realize this concretely, all the H_j can be imbedded simultaneously on the Hilbert space H' of L_2 functions on the unit circle (see appendix I) or functions on the interval $1 \leq Z < +\infty$ (not L_2 in Z). Again we remember that for physical scattering amplitudes it is enough to establish completeness for functions with cylindrical symmetry around the third axis so that from each H_j we choose only one vector, the $|j, m = 0\rangle = P_j(Z)$. From the point of view of absolute square integrability the Hilbert spaces H and H' contain functions with the same asymptotic behaviour of moduli as $Z \rightarrow \infty$. However H is a proper subspace of H' , $H' - H \neq 0$. The principal series coincides with the subspace of H' which consists of functions f such that $|Zf(Z)|_{Z \rightarrow \infty} \rightarrow 0$; therefore the content of the principal series is essentially described by a bound on the asymptotic behaviour of the modulus $|f(Z)|$. The supplementary series is only that part of the rest $H' - H_{\text{Princ.}}$, which consists of functions, physically speaking scattering amplitudes, with singularities only on the real interval $-\frac{1}{2} < j < 0$ of the j -plane. For example it does not include functions like $P_{\sigma + ip}(Z)$, $-\frac{1}{2} < \sigma < 0$, $p \neq 0$.

Unfortunately, if we widen the space of functions over the group manifold in order to include non-square integrable functions we lose the nice orthogonality properties of the principal series. If we need only the supplementary series in addition to the principal series we can

still handle the situation simply by defining appropriately the contour of integration in the j -plane (see appendix II). For more general cases we know only very crude methods. Nevertheless the uniqueness of the expansion of the scattering amplitude in terms of the irreducible representations must be valid and useful again. Suppose, for example, that we need to include spinorial representations or complex Regge-poles. The coefficients $f_j(t)$ cannot be determined by scalar products but rather by looking at the asymptotic behaviour of the scattering amplitude as $s \rightarrow \infty$. If there is only a finite number of these additional terms we can determine them step by step starting from the one with the greatest real j and subtracting till we exhaust all of them. Then we can express the rest by the unitary representations.

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APPENDIX I

Definition of the L_3 group and the Hilbert spaces for its unitary representations

For complete information about the L_3 group we refer to standard references^{2),3),10)}. In the following we repeat some of the basic definitions which we need to clarify the group theoretic parametrization of the scattering amplitude.

It is simple and useful to consider the L_3 group as a subgroup of the Lorentz group L_4 . The L_4 is isomorphic to the group of linear transformations of a complex variable,

$$\chi' = \frac{\delta\chi + \gamma}{\beta\chi + \alpha} \quad (\text{I.1})$$

Its covering group (covers L_4 twice) is the spinor group $G_4 = SL(2, \mathbb{C})$, which consists of the 2×2 complex unimodular matrices:

$$W = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1 \quad (\text{I.2})$$

The elements W of the group $SL(2, \mathbb{C})$ define linear transformations $W^* X W = X'$ on the space of the matrices

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \quad (\text{I.3})$$

leaving the form $x_0^2 - x_1^2 - x_2^2 - x_3^2$ invariant.

The spinor group $SU(1,1)$ is the sub-group of $SL(2, \mathbb{C})$ which consists of the matrices:

$$A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \alpha\bar{\alpha} - \beta\bar{\beta} = 1 \quad (\text{I.4})$$

that leave x_3 and therefore $x_0^2 - x_1^2 - x_2^2$ invariant. Corresponding to the transformations (I.1) which are isomorphic to the L_4 group we now have the conformal transformations on the plane

$$x' = \frac{\bar{\alpha}x + \bar{\beta}}{\beta x + \alpha}, \quad (\text{I.5})$$

isomorphic to the L_4 group. These transform the unit disc $|x| \leq 1$ into itself leaving the circle $|x| = 1$ and its interior $|x| < 1$ separately invariant. The transformations of the circle $|x| = 1$ into itself can be considered as projective transformations induced by (I.5) on $n_1 = x_1/x_0$, $n_2 = x_2/x_0$ with $n_1^2 + n_2^2 = 1$.

The Hilbert spaces for the representations of the principal and the supplementary series are realized on functions defined on the unit circle $x = e^{i\theta}$, $-\pi < \theta < \pi$. The classes of functions which belong to the Hilbert spaces H_j , $j = -\frac{1}{2} + ip$ of the principal series are square integrable (L_2 functions) over the group. The classes of functions which belong to the Hilbert spaces H_j , $-\frac{1}{2} < j < 0$ of the supplementary series are not square integrable. For these H_j one uses an appropriately modified positive definite inner product. The regular analytic functions inside $|x| < 1$ with appropriately defined inner product form the Hilbert spaces H_j , $j = -\frac{1}{2}, -1, -\frac{3}{2}, \dots$ for the representations of the discrete series²⁾. For our purposes it is convenient to map the unit disc $|x| \leq 1$ on the upper half x' -plane, $x' = 2 \frac{x+i}{ix+1}$ and then on the Z -plane, $Z = 1 + x'^2$ cut along the real axis $(1, +\infty)$. The unit circle $x = e^{i\theta}$ is then mapped on both sides of the cut $(1, +\infty)$ according to $\theta = 2 \tan^{-1} \sqrt{\frac{Z-1}{2}}$, and the invariant volume element (2.4) is given by $d\phi_1 d\phi_2 dZ$. We should remark here that although the Hilbert spaces H_j on which are realized the unitary representations are distinguished in general by different inner products, all H_j can be imbedded simultaneously in the Hilbert space of L_2 functions on the circle, with measure $d\theta$.

Indeed the basic $e^{im\theta} = e^{im2 \tan^{-1} \sqrt{\frac{z-1}{2}}}$, $m = 0, \pm 1, \pm 2, \dots$ which is used for ordinary Fourier series expansion on the circle is dense in all H_j .²⁾ This will be very useful in the next appendix where we can use the topology of this last Hilbert space in order to define an extended completeness of all generalized eigenfunctions $|q, m\rangle$ of Q, H_{12} , including the supplementary series.

The variables α, β of (I.4) are related to those of (2.3) as follows:

$$\begin{aligned} \alpha \bar{\alpha} &= 1 + \beta \bar{\beta} = \frac{1}{2}(\cos 2\zeta + 1) \\ \arg \alpha &= (\phi_1 + \phi_2), \quad \arg \beta = (\phi_2 - \phi_1) \end{aligned} \quad (\text{I.6})$$

For example the matrix element of the discrete class

$$U_{k2}^{+mn}(\alpha) = \frac{1}{(m-n)!} \sqrt{\frac{\Gamma(m+1-k)\Gamma(m+k)}{\Gamma(n+1-k)\Gamma(n+k)}} \alpha^{-m} \beta^{m+n} F(k-m, 1-n-k, 1+m-n, -\beta\bar{\beta}) \quad m > n$$

given by Bargman (see reference 2, Eq. 10.27d) in the notation using the variables $(2\phi_1, 2\zeta, 2\phi_2)$ of (2.3) is written as

$$\begin{aligned} U_{k2}^{+mn}(\alpha) &= d_j^{+mn}(\cosh 2\zeta) = \frac{1}{(m-n)!} \sqrt{\frac{\Gamma(m+1+j)\Gamma(m-j)}{\Gamma(n+1+j)\Gamma(n-j)}} e^{2im\phi_2} e^{2in\phi_1} \\ &\quad \left(\frac{\cosh 2\zeta + 1}{2} \right)^{-\frac{(m+n)}{2}} \left(\frac{\cosh 2\zeta - 1}{2} \right)^{\frac{m-n}{2}} P_{n+j}^{(m-n), -(m+n)}(\cosh 2\zeta) \end{aligned} \quad (\text{I.7})$$

A P P E N D I X II

Completeness and orthogonality relations for the representations of the L_3 group

We remind one here that a set of functions $\{f_m\}$ in an L_2 Hilbert space is a complete orthonormal basis if:

$$\sum_n |f_n(x)\rangle \langle f_n(y)| = \delta(x-y) \quad (\text{completeness}) \quad (\text{II.1a})$$

and

$$\langle f_m | f_n \rangle = \int dx f_m^*(x) f_n(x) = \delta_{mn} \quad (\text{orthonormality}) \quad (\text{II.1b})$$

Completeness means that any $f \in L_2$ can be expanded linearly in the form

$$|f(x)\rangle = \sum_n a_n |f_n(x)\rangle = \sum_n \langle f_n | f \rangle |f_n(x)\rangle \quad (\text{II.2})$$

and if $a_n = 0$ for all n then $f \in \{0\}$. The convergence of the above series expansion is defined with the L_2 topology.

Complete orthonormal bases are generated in a natural way by self-adjoint operators A or unitary operators e^{iA} . Indeed the spectral decomposition theorem states the completeness of the set of all the generalized eigenfunctions of a self-adjoint (unitary) operator²⁴⁾. If the operator A is completely continuous then its spectrum is discrete and the eigenfunctions correspond to properly normalizable vectors in H .

Completeness is then stated in terms of discrete sums as in formula (II.1a). In case of continuous spectrum the eigenfunctions are improper vectors (generalized functions). Completeness still holds. In order to formulate it in a precise mathematical language one can use either the method of von Neumann, spectral decomposition, or the Gel'fand triplet space where an improper eigenfunction corresponds to a continuous functional in a Rigged Hilbert space²⁴⁾. The end result is that (II.2) is again valid but $\sum_n$ is replaced by $\int dn$. A classical example is the Fourier transform. The unitary operator is the operator of translations, the self-adjoint operator $A = p$ the momentum. The generalized eigenfunctions are the $1/\sqrt{2\pi} e^{ipx}$ and the completeness is stated in terms of Fourier transform expansion,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int dp e^{ipx} \tilde{f}(p), \quad \tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} f(x),$$

and the Plancherel formula $\int dx |f(x)|^2 = \int dp |\tilde{f}(p)|^2$.

We turn now to the Legendre functions that interest us. The ordinary Legendre polynomials and associated Legendre functions (more precisely the spherical harmonics) are coming from the rotation group. The self-adjoint operator is the spin and its spectrum is discrete. Completeness is then stated as a series expansion in terms of Legendre polynomials as used for example in the partial wave decomposition of the scattering amplitude. For the L_3 group we have, as we saw in chapter 3, a discrete as well as a continuous spectrum. We can formulate completeness in abstract Hilbert space language. We want a basis for functions $f(\theta)$ defined on the unit circle $-\pi < \theta < \pi$, or after the transformation $\theta = \tan^{-1} \sqrt{(Z-1)/2}$, on functions $F(Z) = f(\theta)$ (scattering amplitudes) defined on the real axis $(1, +\infty)$. The completeness states that,

$$F = \int_{\theta} d(I_{\theta} F) = \int_Z d(I_Z F), \quad (II.3)$$

where I_0, I_j are the identity projections of the Casimir operator $Q = -\vec{J}^2$. However, in the following we are interested in analytic expressions to apply in fixed t dispersion relations and we shall formulate completeness using well-known formulae from analysis. As mentioned earlier, we can restrict ourselves to $P_j(Z)$ without loss. For orthogonality properties within each Hilbert space H_j and other general formulae for any m, m' , see references 2), 10).

We start with the usual completeness of the Legendre functions $P_\ell(Z)$, $\ell = 0, 1, 2, \dots$. The basic formula is Heine's identity,

$$\frac{1}{\zeta - Z} = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(Z) Q_\ell(\zeta). \quad (\text{II.4})$$

For ζ on any ellipse with foci at -1 and $+1$ and Z fixed point inside this ellipse the convergence is uniform with regard to ζ . From (II.4), using the identity

$$\frac{1}{2} \left(\frac{1}{\zeta - Z - i\varepsilon} - \frac{1}{\zeta - Z + i\varepsilon} \right) = i\pi \delta(\zeta - Z), \quad (\text{II.5})$$

and

$$Q_\ell(\zeta - i\varepsilon) - Q_\ell(\zeta + i\varepsilon) = 2\pi i P_\ell(\zeta), \quad (\text{II.6})$$

we obtain the completeness:

$$\sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(Z) P_\ell(\zeta) = \delta(Z - \zeta). \quad (\text{II.7a})$$

From (II.4) and

$$Q_\ell(\zeta) = \frac{1}{2} \int_{-1}^1 \frac{P_\ell(y)}{\zeta - y} dy,$$

we obtain the conditions of orthonormality,

$$\int_{-1}^1 dx \left(\ell + \frac{1}{2}\right) P_\ell(x) P_{\ell'}(x) = \delta_{\ell\ell'}. \quad (\text{II.7b})$$

Now, any function $f(x)$ square integrable in $-1 \leq x \leq 1$ can be expressed as follows,

$$f(x) = \sum_{\ell=0}^{\infty} (\ell+1) \alpha_\ell P_\ell(x)$$

where

$$\alpha_\ell = \int_{-1}^1 dx P_\ell(x) f(x)$$

(II.8)

and

$$\int_{-1}^1 dx |f(x)|^2 = 2 \sum_{\ell=0}^{\infty} |\alpha_\ell|^2$$

For the L_3 group we have to establish the completeness of the $P_j(Z)$, Real $j = -\frac{1}{2}$. The basic formula is Mehler's identity¹⁶⁾. Applying the Watson transform on (II.4) we have:

$$\begin{aligned} \frac{1}{\zeta - \bar{z}} &= \sum_{\ell=0}^{\infty} (\ell+1) P_\ell(z) Q_\ell(\zeta) = \frac{i}{2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj (j+1) \frac{P_j(-z) Q_j(\zeta)}{\sin \pi j} = \\ &= \frac{i}{2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj (j+1) \cot \pi j P_j(z) Q_j(\zeta). \end{aligned} \quad (\text{II.9})$$

Using (II.9), the identity (II.5), and (II.6) we obtain the completeness for the interval $(-1, +1)$. We have:

$$\delta(z-\zeta) = \frac{i}{4} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d_j(\nu+1) \cot \pi j P_j(z) P_j(-\zeta)$$

and for any $f(x)$, (L_2 summable in $(-1,1)$)

$$f(x) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d_j(\nu+1) f_j P_j(x) \quad (\text{II.10a})$$

where

$$f_j = \frac{i}{4} \cot \pi j \int_{-1}^1 dx P_j(x) f(x) \quad (\text{II.10b})$$

For $|z|, |\zeta| > 1$ we can use the discontinuity of P_j

$$P_j(-z-i\epsilon) - P_j(-z+i\epsilon) = 2i \sin \pi j P_j(z)$$

to obtain

$$\delta(z-\zeta) = \frac{i}{2\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d_j(\nu+1) P_j(z) Q_j(\zeta)$$

Therefore for the interval $1 \leq x < \infty$, we can express $f(x)$ as follows:

$$f(x) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d_j(\nu+1) f_j P_j(x) \quad (\text{II.11a})$$

where

$$f_j = \frac{i}{2\pi} \int_1^\infty dx Q_j(x) f(x) \quad (\text{II.11b})$$

If $f(x)$, $1 \leq x < \infty$ is not square integrable but still in the space of the unitary representations of L_3 we can write,

$$f(x) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj (j+1) f_j P_j(x) + \int_{-\frac{1}{2}}^0 dj (j+1) \Delta f_j P_j(x) \quad (\text{II.12})$$

where

$$\Delta f_j = \lim_{\epsilon \rightarrow 0} (f_{j+i\epsilon} - f_{j-i\epsilon})$$

The orthogonality properties are contained in the formula,

$$\int_1^\infty dx P_{j'}(x) Q_{j+i\epsilon}(x) = \frac{1}{(j+i\epsilon-j')(j+i\epsilon-j'-1)}$$

This corresponds to taking the contour of integration in the complex j -plane as indicated in the next figure. The situation is similar to the Regge pole theory with singular potential of g/r^2 behaviour near the origin.

A P P E N D I X III

Reduction formulae for the two particle states and of the S-matrix for two particle elastic scattering

We sketch here an elementary way of reducing two particle states and calculating the Clebsch-Gordan coefficients. We start from the familiar reduction in the s-channel (little group SU_2). For simplicity the model will be again that of two neutral scalar particles of equal mass. We give the reduction of two particle plane waves $|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle$. Since by superposition of plane waves we can make any square integrable wave packet and thus span the whole Hilbert space of the two particle states, this solves the problem completely. We have to reduce the state

$$|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle = \frac{e^{-ip_1 x_1}}{\sqrt{2\omega_1}} \frac{e^{-ip_2 x_2}}{\sqrt{2\omega_2}} = \frac{e^{-iP \cdot X}}{\sqrt{2\omega_1 2\omega_2}} e^{-ip \cdot x} \quad (\text{III.1})$$

where $P = p_1 + p_2$ is the total momentum, $X = \frac{1}{2}(x_1 + x_2)$ the centre of mass (c.m.), $p = p_1 - p_2$ the relative momentum and $x = (x_1 - x_2)/2$ the relative co-ordinates of the two particles, 1 and 2. The reduction is simplest in the c.m. system, with $\vec{p}_1 = \vec{p}_2 = \vec{p}$ along the Z-axis. We have:

$$|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle = \frac{e^{-iET}}{E} e^{i\vec{p} \cdot \vec{x}} = \frac{e^{-iET}}{E} \sum_{\ell=0}^{\infty} (2\ell+1) i^{\ell} j_{\ell}(p) P_{\ell}(\cos \theta), \quad (\text{III.2})$$

the familiar formula of text-books in Quantum Mechanics²⁵⁾. E is the total energy $E = \sqrt{s}$ in the c.m., T the time, $\varphi = |\vec{p}| |\vec{x}|$, and $\cos \theta = \frac{\vec{p} \cdot \vec{x}}{\varphi}$.

The expansion (III.2) states that

$$|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle = \sum_{\ell=0}^{\infty} \oplus |\ell\rangle, \quad (\text{III.3})$$

while the radial wave functions $j_{\ell}(\rho)$ are parts of Clebsch-Gordan coefficients. The relativistic wave function normalization $1/\sqrt{2w_1 2w_2}$ enters in (III.2) as a trivial factor since in the c.m. system it is a scalar under SU_2 . In a different system however this factor will affect the Clebsch-Gordan coefficients.

For the reduction of "waves at fixed t " we have to consider tensor products of the form $|\vec{p}_1\rangle \otimes \langle \vec{p}'_1|$, where particle 1 belongs to the Hilbert space of the incoming particles H_1 , and particle 1' to the Hilbert space of the outgoing particles H'_1 . Since the Hilbert spaces are reflexive we can consider this form either as a vector in $H = H_1 \otimes H'_1$ (identifying $\langle \vec{p}'_1|$ with $|\vec{p}'_1\rangle^*$) or as an operator on H_1 . We have:

$$|\vec{p}_1\rangle \otimes \langle \vec{p}'_1| = \frac{e^{-ip_1 x}}{\sqrt{2w_1}} \frac{e^{ip'_1 x}}{\sqrt{2w'_1}} = \frac{e^{-iPx}}{\sqrt{2w_1 w'_1}} e^{-ipx} \quad (\text{III.4})$$

where

$$P = (p_1 - p'_1), \quad X = \frac{x_1 + x'_1}{2}, \quad p = p_1 + p'_1, \quad x = \frac{x_1 - x'_1}{2}.$$

If we go to the convenient system which here is the brickwall system of the particles 2, 2' of Fig. 2 (the particles 2, 2' can also be fictitious) and use an appropriate choice of axis,

$$\begin{aligned} |\vec{p}_1\rangle \otimes \langle \vec{p}'_1| &= \frac{e^{i\sqrt{-t}Z}}{\sqrt{2w_1 w'_1}} e^{-i(p_0 x_0 - p_x x - p_y y)} = \frac{e^{i\sqrt{-t}Z}}{\sqrt{4\mu^2 - t}} e^{-ip \cosh 2\zeta} \\ &= \frac{e^{i\sqrt{-t}Z}}{\sqrt{4\mu^2 - t}} \int_{-\frac{1}{2}-i\omega}^{-\frac{1}{2}+i\omega} d_j(u+1) g_j(p) P_j(\cosh 2\zeta). \end{aligned} \quad (\text{III.5})$$

where $\rho = \|p\| \cdot \|x\|$; $\|p\|^2 = p_0^2 - p_x^2 - p_y^2 = 4t^2 - t$, $\|x\|^2 = x_1^2 - x^2 - y^2$.

The radial wave functions $g_j(\rho)$ are given by

$$g_j(\rho) = \frac{i}{\pi} \int_1^\infty dx e^{i\rho x} Q_j(x).$$

Therefore the two particle plane wave states for fixed t reduce as follows:

$$|\vec{p}_1\rangle \otimes \langle \vec{p}'_1| = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d_j P_j$$

where P_j are projection operators. It is remarkable that for this we need only the principal series.

Consider now the reduction of the S-matrix. In order to treat both cases on the same footing we write the S-matrix in the following alternative forms:

$$\begin{aligned} \langle \vec{p}'_1 \vec{p}'_2 | S | \vec{p}_1 \vec{p}_2 \rangle &= \langle \vec{p}'_1 | \otimes \langle \vec{p}'_2 | S | \vec{p}_1 \rangle \otimes | \vec{p}_2 \rangle = \\ &= \text{Trace} \{ | \vec{p}_1 \rangle \otimes \langle \vec{p}'_1 | S | \vec{p}_2 \rangle \otimes \langle \vec{p}'_2 | \} \end{aligned} \quad (\text{III.8})$$

The first expression to the right of (III.8) together with (III.2) and the Wigner-Eckart theorem leads to the familiar partial wave decomposition of the S-matrix for fixed s .

$$S(s, z) = \sum_{\ell=0}^{\infty} (\ell+1) q_\ell(s) P_\ell(z), \quad z = \cos\left(\frac{\vec{p} \cdot \vec{p}'}{p^2}\right). \quad (\text{III.9})$$

The second expression, with (III.5) will lead to

$$S(z, t) = \varphi(z, t) = \left[\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} + \int_{-\frac{1}{2}}^0 \right] \left(d_j(j+1) \alpha_j(t) P_j(z) \right) \quad (\text{III.10})$$

As we discussed in Chapter 5, the properties of the S-matrix suggest that we need in (III.10) to use only the unitary representations of the group, the principal plus at most the supplementary series.

We can go from $\varphi(z, t)$ to $F(z, t) = z \varphi(z, t)$ by the simple reduction formula

$$(2j+1) P_j(z) \otimes P_j(z) = (j+1) P_{j+1}(z) - j P_{j-1}(z), \quad (\text{III.11})$$

provided from the recurrence formulae for Legendre functions. (A generalization for reduction of $P_n(z) \times P_j(z)$, n integer is obvious by induction).

If $j = -\frac{1}{2} + ip$ as in our case the result of this reduction in the complex j -plane is to displace the principal series from the line $\text{Re } j = -\frac{1}{2}$ to the line $\text{Re } j = \frac{1}{2}$. The supplementary series will occupy now the cut $\frac{1}{2} < j < 1$.

$$P_1 \otimes P_{-\frac{1}{2}+ip} = P_{\frac{1}{2}+ip} \oplus P_{\frac{1}{2}-ip}$$

$$P_1 \otimes P_{-\frac{1}{2}+\sigma} = P_{\frac{1}{2}+\sigma} \oplus P_{\frac{1}{2}-\sigma}, \quad \sigma < \frac{1}{2}$$

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