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BOOTSTRAP SOLUTIONS TO THE BETHE-SALPETER EQUATION

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A B S T R A C T

A bootstrap dynamics based on the properties of the Bethe-Salpeter equation is developed. Bootstrap conditions are derived from the Bethe-Salpeter bound state wavefunction equation and the normalization condition on the wave function under the assumption that the interaction kernel is given by the exchange of the bound state. As in N/D bootstraps, tractable equations result only if crossing symmetry is violated. It is shown that the bootstrap conditions generalize in a very simple manner to the case of interacting Lie group multiplets of particles. Some applications of the bootstrap dynamics are given for the case of meson bound states in the fermion-antifermion channel ; for reasons of mathematical convenience, only massless mesons are treated here. Arguments are given to suggest that this model might provide a relativistic description of meson bound states in a massive quark model.

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1. INTRODUCTION

Ever since the bootstrap idea was suggested ¹⁾ as a possible dynamical mechanism to explain the existence and properties of the hadrons, there has been much effort and some success in converting the idea into a useful tool. The customary framework for such bootstrap calculations has been the N/D coupled integral equations ²⁾. In this paper we shall present an alternative approach to the dynamics of the bootstrap mechanism based on the properties of the Bethe-Salpeter equation ³⁾.

Along with its successes, the N/D approach has been beset by certain ambiguities and difficulties ⁴⁾; some of these difficulties may be a result of the approximations used in practical calculations. Since the formulation of the bootstrap idea in a different dynamical framework will generally lead to new approximations, the search for different bootstrap dynamics hardly needs justification. For example, in the approach to bootstraps which we will develop here, the primary approximation in practical calculations will be the restriction to ladder graphs. This approximation is unique to a field theoretic formulation of the bootstrap problem and its consequences are expected to be different from those resulting from the customary approximations to the N/D bootstrap equations.

Our bootstrap assumption is that the force that produces a certain particle (or multiplet of particles) arises from the iterated exchange of the same particle (or multiplet of particles). We shall use the Bethe-Salpeter wave function to describe the two-particle system and derive a bootstrap condition which utilizes the normalization condition on the wave function. Our interest is in solutions to the Bethe-Salpeter wave function equation and therefore the range of problems we can treat exactly is severely limited by the well-known mathematical complications associated with the relativistic aspects of this equation. In particular, exact solutions can be obtained only for the case of massless particle exchange ⁵⁾.

Exact solutions to the Bethe-Salpeter equation for fermion-fermion^{6),7)} and boson-boson^{8),9),10)} massless bound states have been investigated by many authors. Approximate solutions in the weak binding, non-relativistic limit have also been discussed¹¹⁾. In this paper, we shall confine our applications of the bootstrap mechanism to the solutions in the large binding limit. Thus, while our applications may be unrealistic, they do exhibit the relativistic features and perhaps shed some light on some formal aspects of the Bethe-Salpeter equation. For reasons that will be made clear later, we will restrict our applications to fermion-antifermion systems that interact via meson exchange. A possible physical realization of our application might be meson bootstraps in a massive quark model, since for sufficiently heavy constituent particles it may be a reasonable approximation to neglect the mass of the bootstrapped meson.

We wish to emphasize that the bootstrap scheme we propose here is quite general and is applied in this paper to this somewhat drastic model for reasons of mathematical convenience. More realistic applications will require approximation techniques that have not as yet been developed.

In Section 2 we shall derive the bootstrap conditions. A discussion is given of the number of equations versus number of unknowns and comparisons are drawn with the N/D bootstrap. A multichannel generalization of these conditions is presented in Section 3. This generalization is suited for the bootstrapping of Lie group multiplets of particles. In the next Section [using the methods of Gourdin¹²⁾ and of Baumann, Freund and Thirring¹³⁾] we give a general spin, parity, and charge parity analysis of the wave functions for the fermion-antifermion system. In Section 5, we consider the pseudoscalar and vector meson bootstrap. The Bethe-Salpeter equation we study here is a generalization of that treated by Goldstein⁶⁾ in the unrealistic limit of vanishing centre-of-mass four-momentum. A heuristic eigenvalue condition for the coupling constant is discussed and a bootstrap solution with this eigenvalue is exhibited. Finally, we study more general bootstraps in which the bound state mass is small but not zero and discuss approximation schemes that lead to bootstrap solutions. Some mathematical details are given in an Appendix.

2.

THE BOOTSTRAP MODEL

Since our approach to bootstraps is based on the Bethe-Salpeter equation, let us look briefly at the Green's function description of a two-body elastic scattering process. We consider only spinless particles for the present. As indicated in Fig. 1, p_1, p_2 and q_1, q_2 designate the four-momentum of the two incoming and outgoing particles. We introduce new variables by the definitions

$$p_1 + p_2 = q_1 + q_2 = P \quad (2.1)$$

$$\frac{p_1 - p_2}{2} = p, \quad \frac{q_1 - q_2}{2} = q.$$

In momentum space, the Bethe-Salpeter Green's function equation reads ¹⁴⁾

$$G(P, p, q) = G_0(P, p) \delta(p - q) + \\ + G_0(P, p) \int d^4k I(P, p, k) G(P, p, q) \quad (2.2)$$

where G_0 is just the product of the propagators of the scattered particles and I is the interaction kernel. Equation (2.2) is illustrated in Fig. 1. In ladder approximation, I (renormalized) is given by

$$I(P, p, k) = \frac{ig^2}{(2\pi)^4} \frac{1}{(k - p)^2 - \mu^2} \quad (2.3)$$

where g is the coupling constant for the interaction and μ is the mass of the exchanged particle. In this approximation G_0 is given by

$$G_0(P, p, k) = \frac{1}{[(\frac{P}{2} + p)^2 - m_1^2][(\frac{P}{2} - p)^2 - m_2^2]} \quad (2.4)$$

All masses are understood to have small negative imaginary parts. It was shown by Gell-Mann and Low ¹⁴⁾ and Bethe and Salpeter ¹⁴⁾ that if the interaction results in a bound state of mass M_B , then the Green's function, G , may be written

4.

$$G(P, k, z) = \frac{i \chi(P, k) \bar{\chi}(P, z)}{P^2 - M_B^2} + R(P, k, z) \quad (2.5)$$

where R can be neglected in comparison with the pole term for $P^2 \approx M_B^2$. χ and $\bar{\chi}$ in co-ordinate space are given by the expressions

$$\chi(x_1, x_2) = \langle 0 | T \{ \psi_1(x_1) \psi_2(x_2) \} | B \rangle \quad (2.6a)$$

$$\bar{\chi}(x_1, x_2) = \langle B | T \{ \psi_1^\dagger(x_1) \psi_2^\dagger(x_2) \} | 0 \rangle \quad (2.6b)$$

where $|B\rangle$ is a state vector for the bound state, and the ψ_i are the Heisenberg field operators for the constituent particles. Substitution of Eq. (2.5) into Eq. (2.2) yields the Bethe-Salpeter wave function equations for χ and $\bar{\chi}$:

$$\chi(P, k) = G_0(P, k) \int d^4k' I(P, k, k') \chi(P, k') \quad (2.7a)$$

$$\bar{\chi}(P, k) = \int d^4k' \bar{\chi}(P, k') I(P, k, k') G_0(P, k') \quad (2.7b)$$

valid for $P^2 \approx M_B^2$. Our interest in this paper will be centered on the solutions and properties of Eqs. (2.7).

Stated in the context of the ladder approximation, our bootstrap hypothesis is twofold; first we require that

$$N^2 = M_B^2 \quad (2.8)$$

and secondly, that the coupling constant, g , in Eq. (2.3) is equal to the effective coupling constant, f , for the bound state to couple into the constituent particle channel. Diagrammatically, this second condition which reads

$$g = f \quad (2.9)$$

is clarified in Figs. 2, 3. In Fig. 3 the constituent particles are on their mass shell, and we have assumed that particle 1 is the anti-particle of particle 2 as is necessary for the one-channel bootstrap. Our procedure will be to relate the T matrix for the process shown in Fig. 3 to the Bethe-Salpeter wave function, χ . We will take as the definition of the coupling constant, f^2 , the value of the residue of the T matrix when all legs are on the mass shell. Thus we shall obtain an equation for f as function of g , since χ depends on g via Eq. (2.7).

Consider, then, the T matrix element for the scattering of particles 1 and 2 with centre-of-mass momentum squared equal to the bound state mass M_B . Using the integral equation satisfied by the T matrix

$$T = I + I G T \quad (2.10)$$

and Eq. (2.2), we have

$$T = I + I G I \quad (2.11)$$

The bound state pole in G will give rise to a pole in T , and hence for $P^2 \simeq M_B^2$, we obtain

$$T \simeq I G I \quad (2.12)$$

Substitution of Eq. (2.5) into Eq. (2.12) yields

$$T(P, \epsilon) = \frac{i \left[\int d^4k I(P, k) \chi(P, k) \right] \left[\int d^4k' \bar{\chi}(P, k') I(P, k', \epsilon) \right]}{P^2 - M_B^2} \quad (2.13)$$

Now using the Bethe-Salpeter wave function equation [Eqs. (2.7)] we obtain

6.

$$T(P, k, q) = \frac{i G_0^{-1}(P, k) X(P, k) \bar{X}(P, q) G_0^{-1}(P, q)}{P^2 - M_B^2} \quad (2.14)$$

On the other hand, we know that this T matrix element is given by the expression

$$T(P, k, q) = \frac{i f^2}{(2\pi)^4 (P^2 - M_B^2)} \quad (2.15)$$

when the constituent particles are placed on the mass shell. Equating the expressions for T given in Eqs. (2.14), (2.15), we obtain the result

$$f^2 = (2\pi)^4 G_0^{-1}(P, k) X(P, k) \bar{X}(P, q) G_0^{-1}(P, q) \Big|_{\substack{k_1^2 = m_1^2 \\ k_2^2 = m_2^2 \\ P^2 = M_B^2}} \quad (2.16)$$

Equations (2.16), (2.8), (2.9) together with the Bethe-Salpeter wave function equation which determines X as a function of g , μ and M_B provide the bootstrap conditions. The modifications of these equations for the case of particles with spin will be given in Section 5.

Since the wave function satisfies a homogeneous equation, it is clear that the above procedure would be ambiguous without a suitable normalization condition on X . Such a condition was first derived by Mandelstam ¹⁵⁾ under restrictive assumptions and later proved in generality by Cutkosky and Leon ¹⁶⁾. This condition reads

$$\int d^4 p \bar{X}^\alpha(P, k) \left[\frac{\partial}{\partial p_\mu} G_0^{-1}(P, k) \right] X^\beta(P, k) - \int d^4 p \int d^4 q \bar{X}^\alpha(P, k) \left[\frac{\partial}{\partial p_\mu} I(P, k, q) \right] X^\beta(P, q) = 2i p_\mu \delta_{\alpha\beta} \quad (2.17)$$

α and β label possible degeneracies of the system. It is to be understood that a trace is implied on the left-hand side for the case of spin $\frac{1}{2}$ constituents.

We note first of all that if the normalization integral converges, so that no cut-off parameter is required, then we may have a well-defined bootstrap. On the other hand, the introduction of a cut-off parameter to ensure a finite norm would be analogous to the use of a subtraction constant in the N/D equations. In this latter case the bootstrap is ambiguous since there are fewer equations than unknowns. In all applications of the bootstrap mechanism that we make in this paper, the normalization integrals will be convergent and no cut-off parameters have to be introduced. However, even if the normalization integral is convergent, Eq. (2.7) need not always lead to a discrete eigenvalue spectrum for g^2 , in which case our bootstrap equations are again like the once-subtracted N/D equations in that there is an adjustable parameter - the value of g itself. Even with this extra freedom, there is not always a bootstrap solution as we shall see.

Secondly, we note that the bootstrap condition, Eq. (2.16), is meaningful only if the wave function contains first order poles on the mass shell, for in this case the vertex function has no poles on the shell and the effective coupling constant, f , is well defined. There exist solutions to the Bethe-Salpeter equation, however, in which higher order poles arise. An example is the scalar-scalar case with massless particle exchange. Furlan and Mahoux¹⁷⁾ have given arguments to suggest that the occurrence of these higher order poles is a reflection of the infinite range of the massless particle exchange force, and Nakanishi¹⁸⁾ has shown that the solutions to the scalar-scalar problem with massive particle exchange would not exhibit such higher order poles. Furthermore, the latter author has shown that in the limit as the exchanged particle mass goes to zero, the absence of higher order poles persists. Since these results show that the analytic structure of the scalar-scalar solutions with zero and non-zero mass is quite different, our statement that it is reasonable to approximate the fact that the mesons are much lighter than the constituent particles by taking the exchanged mesons to have zero mass might appear to be invalidated. However, since no higher order poles appear in the fermion-

antifermion case to which we shall apply our bootstrap in Section 5, it is not unreasonable to believe that the essential features of our solution might persist in a more realistic treatment of the meson mass. An intuitive understanding of this situation can be gained from a consideration of an analogous situation in potential theory. It can be shown ¹⁹⁾ that the Bethe-Salpeter equation for the fermion-antifermion system interacting via meson exchange is analogous to the Schroedinger equation with a marginally singular $1/r^2$ potential, while the equation for a pair of mesons interacting via scalar meson exchange is analogous to the Schroedinger equation with a non-singular $1/r$ potential. This difference, which is due simply to the less singular high-energy behaviour of the meson propagators, has the following consequence. Since the range of the wave function in a non-singular potential is to some extent determined by the range of the potential, the boson-boson wave function "feels" the infinite range of the $1/r$ potential and hence one expects a significant difference between the wave function in a screened and unscreened potential. On the other hand, for a marginally singular potential, the range of the wave function is determined by the binding energy rather than the potential range, if the latter is sufficiently great, and hence, one expects little difference between the wave function in an unscreened and a slightly screened potential. Therefore, the fermion-antifermion Bethe-Salpeter wave function might be expected to be insensitive to the approximation of taking the exchanged particle mass to be zero, whereas this is not expected to be true for the boson-boson wave function.

It is instructive to mention that in a theory with crossing symmetry the interaction kernel would correspond not to the exchange of the bound state in the sense of a single particle exchange force, but rather to the exchange of the entire Green's function. This is illustrated in Fig. 4. The entire wave function would then be bootstrapped rather than just the coupling constant which is simply the residue of the pole in the wave function for particular values of the momentum variables. The solution of the non-linear and very difficult integral equation which would result is, of course, the central problem in any realistic bootstrap, but is beyond reach now.

3. MULTICHANNEL GENERALIZATION

The situation described by Eqs. (2.7), (2.16), (2.17) pertains to a single meson bootstrapped in a single constituent channel. In this Section the bootstrap conditions are generalized to include the situation in which multiplets of particles are bootstrapped as the bound states of multiplets of constituent particles. Let us denote the bound states by Greek indices and the constituent particles by Latin indices. Then the generalization of the Bethe-Salpeter wave function equation [Eqs. (2.7)] is given by

$$X_{\alpha,ij}(P,F) = \sum_{\beta,l,m} \lambda_{\alpha}(P^2) (G_0(P,F))_{ij} \int d^4k I_{\beta,il;\beta,jm}(P,F,k) X_{\beta,lm}(P,k) \quad (3.1)$$

where $\lambda_{\alpha}(P^2 = M_{\alpha}^2) = 1$. We will assume now that $M_{\alpha}^2 = M_B^2$ for all α and will set $P^2 = M_B^2$. Then it can be shown that the generalized normalization condition becomes

$$\sum_{ij} \int d^4k \bar{X}_{\alpha,ij}(P,F) \frac{\partial}{\partial P_{\mu}} (G_0(P,F))_{ij} X_{\beta,ij}(P,F) - \sum_{\alpha,il;\beta,jm} \int d^4k \int d^4q \bar{X}_{\alpha,il}(P,F) \frac{\partial}{\partial P_{\mu}} [I_{\beta,il;\beta,jm}(P,F,q)] X_{\beta,jm}(P,q) = 2iP_{\mu} \delta_{\alpha\beta} \quad (3.2)$$

In the above equations, $(G_0)_{ij}$ refers to the product of the i^{th} and j^{th} particle propagators, $X_{\alpha,ij}$ refers to the wave function of the α^{th} bound state in the i, j channel (see Fig. 5) and the meaning of $I_{il;jm}$ is explained diagrammatically by Fig. 6. The derivation of Eq. (2.2) follows along essentially the same lines as the derivation given by Cutkosky and Leon for the one-channel case, and will not be given here. We will be interested in fermion-anti-fermion bound states and hence there is no $I_{\beta,im;\beta,jl}$ term in Eq. (3.1).

Suppose, now, that all the bound state masses are degenerate and also, all the constituent particle masses are degenerate. Referring to Eq. (2.16) we write

$$\chi_{\alpha, i j}(P, p) = f_{\alpha i j} \chi'(P, p) \quad (3.3)$$

where

$$i(2\pi)^4 G_0^{-1}(P, p) \chi'(P, p) \Big|_{\substack{P_0 = m_1 \\ p_0 = m_2}} = 1 \quad (3.4)$$

and the function χ' is, by our degeneracy assumption, independent of all particle labels. We also express the interaction kernel in the form

$$I_{\alpha, i j; \alpha, k m}(P, p, q) = g_{\alpha i j} g_{\alpha k m} I'(P, p, q) \quad (3.5)$$

where I' is given in ladder approximation by

$$I'(P, p, q) = \frac{i}{(2\pi)^4} \frac{1}{(p-q)^2 - \mu^2} \quad (3.6)$$

and again, I is independent of particle indices. Substitution of Eqs. (3.4), (3.5) into Eqs. (3.1), (3.2) yields

$$G_0^{-1}(P) \chi'(P, p) = C_1 \int d^4 q I'(P, p, q) \chi'(P, q) \quad (3.7)$$

and

$$C_2 \int d^4 p \bar{\chi}' \frac{\partial [G_0^{-1}]}{\partial P_\mu} \chi' - C_3 \int d^4 p \int d^4 q \bar{\chi}' \frac{\partial [I']}{\partial P_\mu} \chi' = 2i P_\mu \quad (3.8)$$

where

$$C_1 = \sum_{\beta, l, m} g_{\beta l e} g_{\beta l m} f_{\alpha l m} / f_{\alpha i j} \quad (3.9)$$

$$g_{\alpha \beta} C_2 = \sum_{i, j} f_{\alpha i j} f_{\beta i j}, \quad g_{\alpha \beta} C_3 = \sum_{\gamma, i, l, e, m} f_{\alpha i j} g_{\gamma i l} g_{\gamma l m} f_{\beta l m}.$$

In ladder approximation, Eq. (3.8) simplifies since the term proportional to C_3 vanishes. For the one-channel case, we have as the bootstrap condition that

$$C_1 = C_2 \quad (3.10)$$

for this will imply Eq. (2.9). However, let us look more generally at Eq. (3.9). It can be shown that if the particles transform as members of irreducible representations of a semi-simple Lie group, then Eq. (3.9) is always valid for some C_1 , C_2 and C_3 which are independent of the particles indices. Furthermore, for the coupling between two fundamental representations and the adjoint representation or for the antisymmetric coupling between three adjoint representations of any Lie group SU_n , the relation

$$2C_1 = C_2 \quad (3.11)$$

is valid. For example, in SU_3 this states that the F type $8 \times 8 \rightarrow 8$ and the $3 \times \bar{3} \rightarrow 8$ coupling constants satisfy Eq. (3.11).

The values of C_1 and C_2 are determined solely from dynamics. Their ratio will determine whether there is a single particle, single channel bootstrap [Eq. (3.10)], a multiplet bootstrap of the type indicated in Eq. (3.11), or a bootstrap with more general group theoretic properties (e.g., higher representations or F and D couplings).

4. J, P AND C ANALYSIS OF THE BOUND STATE
WAVE FUNCTIONS

Before proceeding to some applications of the bootstrap dynamics we must determine the spin, parity and charge parity properties of the Bethe-Salpeter wave functions. For a fermion-antifermion system, the most general wave function (which transforms like the direct product of two spinors) can be written ^{*})

$$\chi = \begin{pmatrix} A_1 + A_4 + (\vec{B}_1 + \vec{B}_4) \cdot \vec{\sigma} & A_2 - A_3 + (\vec{B}_2 - \vec{B}_3) \cdot \vec{\sigma} \\ A_1 + A_3 + (\vec{B}_2 + \vec{B}_3) \cdot \vec{\sigma} & A_1 - A_4 + (\vec{B}_1 - \vec{B}_4) \cdot \vec{\sigma} \end{pmatrix} \quad (4.1)$$

where $A_1, \dots, B_4$ are arbitrary functions of p and P . We will assume ²⁰⁾ the validity of the Wick rotation ⁸⁾ to the Euclidean metric, and therefore, following Gourdin ¹²⁾, it is most convenient to expand the functions A_i and B_i in four-dimensional spherical and vector spherical harmonics. Thus we write

$$A_{i, \dots, 4} = \sum_{n, J, m} Y_{n, J, m}(\psi, \theta, \varphi) a_{i, \dots, 4}^{(n, J, m)} \\ \vec{B}_{i, \dots, 4} = \sum_{n, J, L, m} Y_{n, J, L, m}(\psi, \theta, \varphi) b_{i, \dots, 4}^{(n, J, L, m)} \quad (4.2)$$

where the functions a and b are arbitrary functions of p^2 . The $Y_{n, J, L, m}$ are given in terms of appropriate Clebsch-Gordan coefficients by

^{*}) In our metric and notation : $k_\mu k^\mu = k_0^2 - \vec{k}^2 = k^2$, $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$
 $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$, $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma_5 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_{\mu\nu} = \frac{i}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$

$$Y_{n,J,L,m}(\psi, \theta, \varphi) = \sum_{m'=-1}^1 C_{L,1}(J, m; m-m', m') Y_{n,L,m-m'}(\psi, \theta, \varphi) \vec{\chi}_{m'} \quad (4.3)$$

ψ , θ and φ are angles in a Euclidean, four-dimensional space defined by the equations

$$\begin{aligned} p_1 &= |p| \sin \psi \sin \theta \cos \varphi = |p| \frac{Y_{111} + Y_{11-1}}{2} \\ p_2 &= |p| \sin \psi \sin \theta \sin \varphi = |p| \frac{Y_{111} - Y_{11-1}}{2i} \\ p_3 &= |p| \sin \psi \cos \theta = |p| Y_{110} \\ p_4 &= |p| \cos \psi = |p| Y_{100} \end{aligned} \quad (4.4)$$

and the $Y_{n,J,m}$ are given in terms of Gegenbauer polynomials and spherical harmonics by the definition

$$Y_{n,J,m} = \sin^J \psi C_{n-J}^{J+1}(\cos \psi) P_J^m(\cos \theta) e^{im\varphi} \quad (4.5)$$

The function $\vec{\chi}_{m'}$ is a fixed spin-one vector with a polarization index m' .

The parity and charge parity of the wave functions can be easily determined ¹³⁾ from the reduction formula that relates the Bethe-Salpeter wave function to the three point function. Consider the case of a scalar or pseudoscalar bound state first. We introduce an effective bound state field operator $B(x)$ defined such that in the asymptotic limit, the positive frequency part of $B(P)$, operating on the vacuum creates the bound state with momentum P . Thus

$$B(x) = \frac{1}{(2\pi)^3} \int \frac{d^3P}{\sqrt{2\omega_P}} [b^+(P) e^{iP \cdot x} + b(P) e^{-iP \cdot x}] \quad (4.6)$$

and

$$b^+(P) |0\rangle = |B, P\rangle \quad (4.7)$$

Then using

$$b^+(P) = i \int d^3x e^{iPx} \vec{\partial}_x B(x) \quad (4.8)$$

we obtain for the fermion-antifermion wave function

$$\begin{aligned} X_p(x_1, x_2) &\equiv \langle 0 | T \{ \psi(x_1) \bar{\psi}(x_2) \} | B \rangle \\ &= \langle 0 | T \{ \psi(x_1) \bar{\psi}(x_2) b^+(P) \} | 0 \rangle \\ &= \lim_{x_0 \rightarrow -\infty} i \int d^3x e^{iPx} \vec{\partial}_x \langle 0 | T \{ \psi(x_1) \bar{\psi}(x_2) B(x) \} | 0 \rangle \\ &= -i \int d^4x e^{iPx} (\not{\partial}_x - M_B^2) \langle 0 | T \{ \psi(x_1) \bar{\psi}(x_2) B(x) \} | 0 \rangle \end{aligned} \quad (4.9)$$

where ψ and $\bar{\psi}$ are the Heisenberg field operators for the spin $\frac{1}{2}$ constituent particles. Now, the transformation properties of ψ and B under parity and charge parity are known. In terms of the parity operator, P , we have from Eq. (4.9)

$$\begin{aligned} X_p(x_1, x_2) &= -i \int d^4x (\not{\partial}_x - M_B^2) \langle 0 | T \{ P P^{-1} \psi(x_1) P P^{-1} x \\ &\quad \times \bar{\psi}(x_2) P P^{-1} B(x) P P^{-1} \} | 0 \rangle \\ &= -i \int d^4x (\not{\partial}_x - M_B^2) \langle 0 | T \{ \gamma_0 \psi(\bar{x}_1) \gamma_0 \bar{\psi}(\bar{x}_2) \gamma_0 \bar{B}(\bar{x}) \} | 0 \rangle \\ &= \pm \gamma_0 X_{\bar{P}}(\bar{x}_1, \bar{x}_2) \gamma_0 \end{aligned} \quad (4.10)$$

where $\bar{x} = (-\vec{x}, x_0)$, $\bar{B}(x) = \pm B(x)$ and the two signs refer to bound states with even and odd parity. Similarly, in terms of the charge parity operator, C , defined by

$$C^{-1} \gamma_\mu^T C = -\gamma_\mu \quad (4.11)$$

we have, using

$$C^{-1} \psi(x) C = C \bar{\psi}^T(x) \quad (4.12)$$

the result

$$\chi_p(x_1, x_2) = \pm C^{-1} \chi_p^T(x_2, x_1) C. \quad (4.13)$$

The two signs refer to bound states with even and odd charge parity.

For spin-one bound states, using

$$B_p(x) = \sum_r \int \frac{d^3p}{\sqrt{2\omega_p}} [e^{i p x} \epsilon_p^r(p) a_r^+(p) + e^{-i p x} \epsilon_p^r(p) a_r(p)] \quad (4.14)$$

we obtain

$$\chi_p^r(x_1, x_2) = \sum_p \epsilon_p^r(p) \int d^4x (\square_x - M_B^2) K_0 |T\{\psi(x_1) \bar{\psi}(x_2) B^r(x)\}|0\rangle. \quad (4.15)$$

Hence the parity and charge parity analysis goes through as before. In Table 1 we list the spin, parity and charge parity associated with the wave functions given in Eqs. (4.1), (4.2).

It will be more convenient later to express the wave functions in terms of the four vectors P_μ , p_μ , and 3 unit polarization vectors ϵ_μ^r . Thus, for example, the functions $\vec{\gamma} \cdot \vec{\chi}_m$ and Y_{1Jm} can be rewritten as $\gamma^\mu \epsilon_\mu^r$ and $p^\mu \epsilon_\mu^r / |p|$ respectively where a given value of r refers to appropriate linear combinations of J and m as determined from Eq. (4.4). In Table 2 we have listed the spin, parity and charge parity for a few wave functions formed in this manner.

5. APPLICATIONS

In this Section a few applications of the bootstrap mechanism will be given. The models are necessarily unrealistic, but serve to illustrate the ideas discussed above. First we examine a pseudoscalar meson bound state in the fermion-antifermion channel, assuming the force is provided by the exchange of a massless pseudoscalar meson and taking the centre-of-mass momentum to be zero in order to arrive at an equation that can be exactly solved. The resulting Bethe-Salpeter equation is equivalent to that treated by Goldstein⁶⁾. We shall find a continuum of solutions in the coupling constant (which is indicative of an unrealistic bound state problem) and our object will be to see if the bootstrap equations select out of this continuum a discrete point.

For the fermion-antifermion system the Bethe-Salpeter equation reads

$$\begin{aligned} S_F^{-1}(P_1) \chi(P_1, P_2) S_F^{-1}(P_2) &\equiv \left(\not{P}_1 + \not{P} - m \right) \chi(P, P) \left(-\not{P}_2 + \not{P} - m \right) = \\ &= - \int d^4 q \, I(P, q) \chi(P, q) \end{aligned} \quad (5.1)$$

The function $I \chi$ will depend on the type of coupling assumed for the interaction of the exchanged particle and the fermion current.

Let us now consider a pseudoscalar meson bound state with a wave function given by (see Table 2)

$$\chi_{P.S.}(P, P) = \gamma_5 \varphi(P^2) \quad (5.2)$$

The interaction kernel that appears in Eq. (5.1) can be determined from the bootstrap hypothesis. From Eq. (2.14), we can obtain the effective vertex function for the interaction of the pseudoscalar meson with the fermion current from the relation

$$\Gamma_{P.S.}^{-1}(P) \equiv S_F^{-1} \chi_{P.S.}(P) S_F^{-1} = (m^2 - P^2) \gamma_5 \varphi(P^2) \quad (5.3)$$

for $P = 0$. Hence only γ_5 coupling arises and Eq. (5.1) becomes

$$(p^2 - 1) \varphi(p^2) = - \frac{\lambda_{p.s.}}{\pi^2} \int \frac{d^4 q \varphi(q^2)}{(q-p)^2} \quad (5.4)$$

for $P = 0$. Here, $\lambda_{p.s.} = g_{p.s.}^2 / (4\pi)^2$ and we choose units such that $m^2 = 1$. Equation (5.3) has been approximated by its on-shell value.

This equation is most easily solved by making the Wick rotation $p_0 \rightarrow -ip_0 \equiv p_4$ of the timelike component of the momentum variables and then converting the four dimensional integral equation that results into a differential equation. We refer the reader to Refs. 7), 8) for the details of this procedure. We write the differential equation that one obtains in this way for the case of an arbitrary Y_{nlm} multiplying the function φ given in Eq. (5.2), since this generalization will be useful later. The pseudoscalar meson bound state corresponds to the particular value $n = 0$.

$$s(s-1) \varphi''(s) + (4s-2) \varphi'(s) + \left[2 + \lambda_{p.s.} - \frac{n(n+2)}{4} + \frac{n(n+2)}{4s} \right] \varphi(s) = 0. \quad (5.5)$$

Here the prime refers to differentiation with respect to the variable $s = -(p_4^2 + \vec{p}^2)$. This equation is subject to the following boundary conditions found by integrating Eq. (5.5) to obtain Eq. (5.4) and setting the surface terms that result from the integration by parts equal to zero.

$$\left(\frac{n}{2} + 2 \right) s^{1-\frac{n}{2}} \varphi(s) + s^{2-\frac{n}{2}} \varphi'(s) \Big|_{s \rightarrow 0} \rightarrow 0 \quad (5.6)$$

$$\left(\frac{n}{2} \right) s^{1+\frac{n}{2}} \varphi(s) - s^{2+\frac{n}{2}} \varphi'(s) \Big|_{s \rightarrow \infty} \rightarrow 0$$

The solution to Eqs. (5.5), (5.6) with $n = 0$ is the hypergeometric function

$$\varphi(s) = c F \left(\frac{3 + \sqrt{1-4\lambda_{p.s.}}}{2}, \frac{3 - \sqrt{1-4\lambda_{p.s.}}}{2}; 2; s \right) \quad (5.7)$$

where c is a constant to be determined from the normalization condition.

We note that this solution contains a simple pole at $s = 1$ [see Eq. (A.11)]. Thus the coupling constant, f , is well defined in terms of the residue at that pole by the generalization of Eq. (2.16) to the case of spin $\frac{1}{2}$ constituents and pseudoscalar bound states :

$$i f_{\pi s} = \lim_{s \rightarrow 1} (2\pi)^4 / 5_F(p) \chi_{\lambda s} (p) \Sigma_F(p) / \lambda_{\pi s}^2 = (2\pi)^4 / \chi_s(s-1) \psi(s) / s=1^2 \quad (5.8)$$

Secondly, the solution behaves for large s like

$$s^{\frac{3 - \sqrt{1 - 4\lambda_{\text{p.s.}}}}{2}}$$

and therefore the normalization integral is convergent for $\lambda_{\text{p.s.}} > 0$. The normalization of this solution can be carried out by making use of either an integral representation or the power series expansion for the hypergeometric functions. The details are given in the Appendix.

The results are exhibited in Fig. 7. The solid line and the dashed line represent the coupling constants $f_{\text{p.s.}}^2$ and $g_{\text{p.s.}}^2$ respectively and their values are plotted along the ordinate. $\lambda_{\text{p.s.}}$ is plotted along the abscissa. The equation for $g_{\text{p.s.}}^2$ is simply $g_{\text{p.s.}}^2 = 16\pi^2 \lambda_{\text{p.s.}}$, and the values of $f_{\text{p.s.}}$ are obtained from Eqs. (5.8), (2.17). The bootstrap described by Eq. (3.10) in which a single particle is bootstrapped in a single constituent channel would be obtained if the two curves intersected while the situation described by Eq. (3.11) would be obtained if, for some value of $\lambda_{\text{p.s.}}$, the solid line were twice the height of the dashed line. The slope of the curve $f_{\text{p.s.}}^2(\lambda_{\text{p.s.}})$ evaluated at $\lambda_{\text{p.s.}} = 0$ is exactly equal to $32\pi^2$. Therefore, referring to Fig. 7, bootstraps of the first kind [Eq. (3.10)] can only occur for $g_{\text{p.s.}}^2 > 32\pi^2 \lambda_{\text{p.s.}}$ and those of the second kind Eq. (3.11) can only occur for $g_{\text{p.s.}}^2 > 16\pi^2 \lambda_{\text{p.s.}}$. In fact neither situation is obtained and we conclude that there is no value of $\lambda_{\text{p.s.}}$ for which a bootstrap is possible.

The Bethe-Salpeter equation for bound states with higher spin is complicated by the fact that the commutation properties of the γ matrices give rise to coupled integral equations for the amplitudes corresponding to the various possible forms for the wave function. For example, consider a vector meson bound state, and, as a first approximation, take $P = 0$. Then the most general wave function can be written

$$X_V^r(p) = \psi_0(p^2) \not{e}^r + \psi_1(p^2) \not{p} \cdot e^r + \psi_2(p^2) \not{p} p \cdot e^r \quad (5.9)$$

Substitution of this function into Eq. (5.1) gives for the left-hand side

$$\begin{aligned} S_F^{-1}(p) X_V^r(p) S_F^{-1}(p) = \\ = \psi_0(p^2) [(m^2 - p^2) \not{e}^r + 2 \not{p} p \cdot e^r - 2m \not{p} \cdot e^r] + \\ + \psi_1(p^2) [(p^2 + m^2) \not{p} \cdot e^r - 2m \not{p} p \cdot e^r] + \\ + \psi_2(p^2) [(p^2 + m^2) \not{p} p \cdot e^r - 2m p^2 p \cdot e^r] \end{aligned} \quad (5.10)$$

and hence, from Eq. (2.14) we see that the off-shell interaction kernel would contain three kinds of couplings corresponding to the vertices

γ_μ , p_μ , $\not{p} p_\mu$. As stated in Section 2, however, we approximate the vertex function by the on-shell value which is easily obtained from the identity

$$\bar{u}(p) \not{p} u(p) = \frac{p \cdot a}{2m} \bar{u}(p) u(p) \quad (5.11)$$

for $P = 0$. Even still, the resulting integral equations are extremely complicated and no solutions have been obtained.

A further possible approximation is to keep only those terms in the bound state equation which are proportional to one of the three possible terms $\not{\epsilon}^r$, $p \cdot \epsilon^r$, $\not{p} p \cdot \epsilon^r$ (21). This easily can be shown to be equivalent to placing the other two terms on the mass shell and using Eq. (5.11) to incorporate them into the first term. Hence the approximation is similar to that discussed above in our treatment of the interaction kernel and amounts to sacrificing some more of the off-shell properties of the Bethe-Salpeter equation in order to obtain a solution. For example, writing $\chi_v^r(p) = \psi_0(p^2) \not{\epsilon}^r$ and dropping terms in $S_F^{-1} \chi_v^r S_F^{-1}$ which involve $p \cdot \epsilon^r$ and $\not{p} p \cdot \epsilon^r$, we obtain (22) the bound state equation

$$(p^2 - 1) \psi_0(p^2) \not{\epsilon}^r = - \frac{2i g_v^2}{16\pi^4} \int d^4 q \frac{\psi_0(q^2) \not{\epsilon}^r}{(q-p)^2}. \quad (5.12)$$

Equation (5.12) is practically equivalent to Eq. (5.4) with the difference of a factor of 2 on the right-hand side of Eq. (5.12). Hence the bootstrap equation for the coupling constant is similar to that described graphically in Fig. 7 except that the slope of the line $g_v^2(\lambda_v)$ would be $\frac{1}{2}$ the slope of the line $g_{p.s.}^2(\lambda_{p.s.})$ indicated in the figure and thus there would be no bootstrap solution.

Had we chosen

$$\chi_v^r(p) = \psi_1(p^2) \hat{p} \cdot \epsilon^r \quad (5.13)$$

however, then the bound state equation, in the approximation discussed above, would read

$$(p^2 - 1) \psi_1(p^2) \hat{p} \cdot \epsilon^r = - \frac{4i g_v^2}{16\pi^4} \int d^4 q \frac{\psi_1(q^2) \hat{q} \cdot \epsilon^r}{(q-p)^2}. \quad (5.14)$$

This is just a special case of Eq. (5.5) with $n = 1$. The equation with arbitrary n is considerably more complicated than the equation with $n = 0$. We shall see, however, that if we impose the condition

$$\frac{g_v^2}{16\pi^2} \equiv \lambda_v = n \quad (5.15)$$

then the equation simplifies considerably and for $n = 1$ yields a bootstrap solution. The change of variables

$$s = \frac{1+t}{t} \quad (5.16)$$

$$\psi_n(s) = \left(\frac{t}{1+t}\right)^{\frac{n+2}{2}} h_n(t)$$

allows us, using Eq. (5.15), to rewrite Eq. (5.5) in the simplified form

$$t(1+t)h_n''(t) + nh_n'(t) = 0 \quad (5.17)$$

where $h'(t)$ means dh/dt . This equation is easily solved and in terms of our original variables the solution is

$$\psi_n(s) = a_1(s)^{-\frac{(n+2)}{2}} + a_2 s^{-\frac{(n+2)}{2}} \int_1^{\frac{1}{1-s}} \left(\frac{t-1}{t}\right)^n dt' \quad (5.18)$$

The boundary conditions [Eq. (5.6)] are only satisfied for $n = 1$ which is just the n value required for vector mesons with the wave function given in Eq. (5.13). We then obtain the solution

$$\psi_1(s) = a \left(\frac{1}{\sqrt{s}(s-1)} - \frac{\ln(1s-1)}{s^{3/2}} \right) \quad (5.19)$$

where a is a constant to be determined by the normalization condition. Note that this solution has a simple pole at $s = 1$ which corresponds to a finite vertex function and a well-defined bootstrap. The normalization of Eqs. (5.13), (5.19) is straightforward. Using

$$\int |Y_{lm}(\varphi, \theta, \varphi)|^2 d\Omega_4 = \frac{2\pi^2}{4} \quad (5.20)$$

where $d\Omega_4$ is the four-dimensional angular element, we obtain from the normalization condition (Eq. (2.17) with S_F replacing Δ_F)

$$\begin{aligned} \text{tr} \left\{ |a|^2 \int d^4p \left(\frac{1}{\sqrt{s}(s-1)} - \frac{\ln(s-1)}{s^{3/2}} \right)^2 |Y_{lm}(\varphi, \theta, \varphi)|^2 \right. \\ \left. \times \frac{\partial}{\partial p_\nu} \left[\left(\frac{p}{2} + k - m \right) \left(-\frac{p}{2} + k - m \right) \right] \right\} = 2i P_\nu \end{aligned} \quad (5.21)$$

the result

$$|a|^2 = 2/\pi^2. \quad (5.22)$$

Then using the following generalization of Eq. (5.8) to the vector meson wave function [Eq. (5.13)]

$$\begin{aligned} \sum_r f_\nu^2 \epsilon^r \cdot \hat{p} \epsilon^r \cdot \hat{p} &= \sum_r (2\pi)^4 |a|^2 (s-1)^2 \times \\ &\times \left[\frac{1}{\sqrt{s}(s-1)} - \frac{\ln(s-1)}{s^{3/2}} \right]^2 \epsilon^r \cdot \hat{p} \epsilon^r \cdot \hat{p} \Big|_{s=1} \\ &= \sum_r (2\pi)^4 |a|^2 \epsilon^r \cdot \hat{p} \epsilon^r \cdot \hat{p} \end{aligned} \quad (5.23)$$

we obtain the relation

$$f_\nu^2 = (2\pi)^4 |a|^2 = 32\pi^2 \quad (5.24)$$

However, from Eq. (5.15), we have

$$g^2_{\nu} = 16\pi^2 \lambda_{\nu} = 16\pi^2 \quad (5.25)$$

Hence we have an exact bootstrap of the type described by Eq. (3.11). For instance, if the constituents and the bound states form representations of the group SU_3 , then we have an exact bootstrap only if the constituents are triplets and the bound states are members of an octet or if the constituents and bound states are members of an octet and interact via pure F type coupling.

The value $\lambda_{\nu} = n = 1$, for which the vector meson bootstraps, is just the branch point in the coupling constant plane of the wave function. In the pseudoscalar meson problem, the branch point was given by $\lambda_{p.s.} = \frac{1}{4}$, and at this value the pseudoscalar meson very nearly bootstraps in the sense that $f^2_{p.s.}(\lambda_{p.s.})$ achieves a minimum value (see Fig. 7). It is encouraging that this should be the case because the branch point also corresponds to a value of λ , for arbitrary n , at which the wave function most nearly satisfies a high-energy condition ¹⁵⁾ which is required in order to allow the usual discarding of the surface terms at $|p| \rightarrow \infty$ in the proof of the orthogonality of the wave functions. This condition reads

$$\lim_{|p| \rightarrow \infty} p^3 \chi(p) \rightarrow 0 \quad (5.26)$$

whereas at the branch point, the solution to Eq. (5.5) satisfy

$$\lim_{|p| \rightarrow \infty} p^{3-\epsilon} \chi(p) \rightarrow 0 \quad (5.27)$$

for arbitrary $\epsilon > 0$. The failure of the exact solutions to satisfy Eq. (5.26) is perhaps due to the ladder graph approximation, but it may not be too serious since the normalization integrals converge for all cases we treat here. The above choice of the branch point value for λ_v can at most be taken as a heuristic device for obtaining an eigenvalue condition, and is by no means intended as a substitute for a rigorous understanding of the difficulties associated with the spinor-spinor Bethe-Salpeter equation.

Finally, we wish to discuss the approximations that are necessary in order to obtain solutions for bound states with mass. If we substitute Eq. (5.2) into Eq. (5.1) and keep $P \neq 0$ we obtain for the left-hand side

$$\bar{S}_F^{-1} X_{PS} S_F^{-1} = \left[\left(m^2 + \frac{P^2}{4} - \beta^2 \right) \gamma_5 - m \not{P} \gamma_5 + \frac{1}{2} (\not{K} \not{P} - \not{P} \not{K}) \gamma_5 \right] \psi(P^2) \quad (5.28)$$

It is clear that there is no term on the right-hand side of Eq. (5.1) that is proportional to $\not{P} \gamma_5$ while such a term appears in Eq. (5.28). Hence, Eq. (5.1) cannot have any solutions for $P \neq 0$. It is necessary, therefore, to write a more general wave function of the form

$$X_{PS} = \psi_1(P^2) \gamma_5 + \psi_2(P^2) \not{P} \gamma_5 + \psi_3(P^2) (\not{K} \not{P} - \not{P} \not{K}) \gamma_5 \quad (5.29)$$

Substitution of Eq. (5.29) into Eq. (5.1) leads to coupled equations for the amplitudes ψ_1 , ψ_2 and ψ_3 which are extremely difficult to solve, while, if all terms proportional to M_B/m are neglected, we are

back with Eq. (5.4) whose solutions we have examined. Thus the approximation of setting $P = 0$ can be restated more physically as the approximation of ignoring the presence of gradient and tensor coupling of the mesons which arise from the commutation properties of the γ matrices. One could hope that the presence of higher order kernels or fermion self-energy corrections might cancel the difficulties resulting from these terms. Or, perhaps the bootstrap mechanism we have discussed is a useful dynamical method for obtaining the leading term in an expansion of the wave function in the parameter M_B/m but that it is demanding too much to require all the terms in χ to be self-generating.

If we consider only the term in Eq. (5.28) proportional to γ_5 , then we obtain Eq. (5.4) with the replacement $m^2 \rightarrow m^2 - M_B^2/4$ on the left-hand side. Hence the bootstrap equation for the dimensionless coupling constant is unchanged by the presence of the mass term, and the wave function becomes only trivially modified. A similar situation arises for the vector meson bound states if the condition $P \cdot \epsilon^r = 0$ is imposed.

CONCLUSION

In Section 5 we have described a possible application of our bootstrap mechanism ; the many approximations we made rendered this application of limited practicality. We have given arguments to suggest that our bootstrap dynamics might describe a relativistic quark model of the mesons. The most severe approximation that was made was the restriction to ladder graphs. This approximation is almost certainly unjustified and perhaps is the cause of certain of the other difficulties such as the appearance of the gradient and tensor coupled terms in the wave functions which, as we saw, do not bootstrap. It would be of interest to try and estimate the effect of higher order interaction kernels on the bootstrap solutions.

In our applications we treated the meson mass bootstrap equation in a crude way, while the bootstrap equation for the coupling constant was solved exactly. In a more realistic application, the treatment of the meson mass would have to be improved upon, perhaps using an appropriate generalization of the ingenious but complicated representations for the solutions of a Bethe-Salpeter equation with arbitrary mass parameters given by Wanders ⁵⁾ and Nakanishi ⁵⁾. In spite of the shortcomings mentioned above, the results of Section 5 indicate that it may be useful to formulate the bootstrap idea in a field theoretic context.

We note that we have not obtained any bootstraps of the type described by Eq. (3.10) (bootstraps of a single bound states in a single constituent channel). The Bethe-Salpeter equation in ladder approximation and the normalization condition seem to show a tendency to "give out more than is put in" ; that is, to yield values for f that are greater than the input coupling constant g . This can be seen, for example, in Fig. 7. This is just the situation favoured by the bootstraps involving certain higher dimensional representations of particles. It is interesting to speculate that a similar phenomenon might arise when symmetry breaking effects are introduced and would lead to a preferred direction for the symmetry breaking when self-consistency requirements of the type we have used here are imposed.

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A P P E N D I X

In this Appendix the normalization integrals used in Section 5 are explicitly carried out. Consider, first, Eq. (4.7) at the branch point $\lambda_{p.s.} = \frac{1}{4}$. Then from Eq. (2.17) we have

$$\begin{aligned} I &= |c|^2 + r \int d^4 p |F(\frac{3}{2}, \frac{3}{2}; 2; p^2)|^2 \gamma_5 \times \\ &\times \left[\left(\frac{p}{2} + p - m \right) \frac{\partial}{\partial p_\mu} \left(-\frac{p}{2} + p - m \right) + \left\{ \frac{\partial}{\partial p_\mu} \left(\frac{p}{2} + p - m \right) \right\} \left(-\frac{p}{2} + p - m \right) \right] \gamma_5 \quad (A.1) \\ &= -|c|^2 P_\mu \int d^4 p |F(\frac{3}{2}, \frac{3}{2}; 2; p^2)|^2 = 2i P_\mu. \end{aligned}$$

The p_0 axis is now Wick rotated and the four dimensional solid angle integration performed to yield

$$I = i |c|^2 \pi^2 P_\mu \int_0^\infty ds \, s |F(\frac{3}{2}, \frac{3}{2}; 2; s)|^2 \quad (A.2)$$

This expression is most easily evaluated by making use of the following integral representation for the hypergeometric function ²³⁾

$$F(a, b; c; s) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \frac{t^{b-1} (1-t)^{c-b-1}}{(1-ts)^a} \quad (A.3)$$

Hence we have

$$\begin{aligned} I &= i |c|^2 \pi^2 P_\mu \left[\frac{\Gamma(2)}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})} \right]^2 \times \\ &\times \int_0^\infty ds \, s \int_0^1 dt \int_0^1 dt' \frac{(tt')^{\frac{1}{2}} (1-t)^{-\frac{1}{2}} (1-t')^{-\frac{1}{2}}}{(1-ts)^{3/2} (1-t's)^{3/2}} \quad (A.4) \end{aligned}$$

The validity of the interchange of the order of integration is assured by uniform convergence. The result, after evaluating the s integral is

$$I = 2\epsilon/c^2 \pi^2 P_\nu \left[\frac{\Gamma(2)}{\Gamma(3/2)\Gamma(1/2)} \right]^2 J \quad (\text{A.5})$$

where

$$J = \int_0^1 dt \int_0^1 dt' \left\{ \frac{1}{(1-t)^{1/2} (1-t')^{1/2} (t^{1/2} + t'^{1/2})^2} \right\} \quad (\text{A.6})$$

The t' integration can be done by making the substitution

$$t' = (y - t^{1/2})^2 \quad (\text{A.7})$$

and using a standard table of integrals. Then we arrive at

$$J = \int_0^1 dt' \left[(1-t)^{-2} \log \left(\frac{1+(1-t)^{1/2}}{t^{1/2}} \right) - (1-t)^{-3} \right]. \quad (\text{A.8})$$

This integral involves two terms each of which diverges at the upper limit but which is finite due to a cancellation of divergences. The final result is

$$I = 8\epsilon/c^2 P_\nu \pi^2 \left[\frac{\Gamma(2)}{\Gamma(3/2)\Gamma(1/2)} \right]^2 = 2\epsilon P_\nu \quad (\text{A.9})$$

To determine the coupling constant $f_{p.s.}$ we require the value of the function

$$S_F^{-1}(p) \chi(p) S_F^{-1}(p) = (s-1) \chi(s) = c \chi_s (s-1) F(3/2, 3/2; 2; s) \quad (\text{A.10})$$

for $s = 1$. Using the identity ²³⁾

$$F(a, b; c, s) = (1-s)^{c-a-b} F(c-a, c-b; c, s) \quad (\text{A.11})$$

and the following relation ²³⁾ which is valid for $\text{Re}(c-a-b) > 0$

$$F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad (\text{A.12})$$

we obtain from Eqs. (5.8), (A.9)

$$f_{p.s.}^2 = (2\pi)^4 / S_F^{-1}(b) \times S_F^{-1}(b) \Big|_{b^2=1}^2 = 16\pi^2 \quad (\text{A.13})$$

as given in Fig. 7.

For other values of $\lambda_{p.s.}$ the integrations are more difficult. An approximate value for the normalization integral can be obtained from the identity ²³⁾

$$F(a, b; c; s) = (1-s)^{-a} F(a, c-b; c; s/s-1). \quad (\text{A.14})$$

For $-\infty \leq s \leq 0$, the range of the variable $t = s/s-1$ is $1 \geq t \geq 0$ in which range the power series expansion ²³⁾ for F

$$F(a, b; c; t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{t^n}{n!} \quad (\text{A.15})$$

is well defined except possibly on the unit circle. However, for $s = \infty$ or $t = 1$ the integrand is rapidly decreasing and a few terms of the series are generally sufficient to obtain a reasonable approximation.

For example, for the case $\lambda_{p.s.} = \frac{1}{4}$, the value obtained from the first 3 terms of the series expansion gives the result $f_{p.s.}^2 \approx 14\pi^2$ which is nearly identical to Eq. (A.9). As another example we treat the limiting case $\lambda_{p.s.} \rightarrow \infty$. We let

$$\sqrt{1-4\lambda_{p.s.}} = i/\beta \quad (\text{A.16})$$

and use Eq. (A.15) to obtain

$$\begin{aligned}
I &= i/c|^2 \pi^2 P_\mu \left| \frac{\Gamma(2)}{\Gamma(\frac{3+i\beta}{2}) \Gamma(\frac{1+i\beta}{2})} \right|^2 X \\
&\times \int_0^{-\infty} ds \, s \left| \sum_{n=0}^{\infty} \frac{\Gamma(\frac{3+i\beta}{2}+n) \Gamma(\frac{1+i\beta}{2}+n) s^n}{(1-s)^{\frac{3+i\beta}{2}} \Gamma(2+n) (s-1)^n n!} \right|^2 \\
&= 2i P_\mu
\end{aligned} \tag{A.17}$$

keeping just the first term in the sum, we get,

$$I = i/c|^2 \pi^2 P_\mu \int_0^{-\infty} \frac{ds \, s}{(1-s)^3} = 2i P_\mu. \tag{A.18}$$

Now, from Eqs. (A.10), (A.11), (A.12)

$$f_{p.s.}^2 = (2\pi)^4 / c|^2 \left| \frac{\Gamma(\frac{3+i\beta}{2}) \Gamma(\frac{3-i\beta}{2})}{\Gamma(2)} \right|^{-2} \tag{A.19}$$

The relation

$$\Gamma\left(\frac{1}{2} + i\eta\right) \Gamma\left(\frac{1}{2} - i\eta\right) = \frac{\pi}{\cosh \pi \eta} \tag{A.20}$$

then gives

$$f_{p.s.}^2 \propto \cosh^2 \pi \beta. \tag{A.21}$$

The inclusion of more terms in Eq. (A.17) simply alters the coefficient in front of this expression. Hence, for large, increasing $\lambda_{p.s.}$, $f_{p.s.}$ is exponentially increasing.

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TABLE CAPTIONS

Table 1 Spin, parity and charge parity for the fermion-antifermion wave function given in Eqs. (4.1), (4.2). The a_i and b_i are arbitrary functions of p^2 .

Table 2 Spin, parity and charge parity for some fermion-antifermion wave functions formed from the four vectors p_μ , P_μ and ϵ_μ^r ($P^\mu \cdot \epsilon_\mu^r = 0$). The wave functions, χ , listed in column 1 are understood to be multiplied by arbitrary functions of p^2 .

χ	SPIN	PARITY	CHARGE PARITY
$a_1(n, J, m)$	J	$(-1)^J$	$(-1)^n$
$a_2(n, J, m)$	J	$(-1)^{J+1}$	$(-1)^n$
$a_3(n, J, m)$	J	$(-1)^{J+1}$	$(-1)^n$
$a_4(n, J, m)$	J	$(-1)^J$	$(-1)^{n+1}$
$b_1(n, J, L, m)$	J	$(-1)^L$	$(-1)^{n+1}$
$b_2(n, J, L, m)$	J	$(-1)^{L+1}$	$(-1)^{n+1}$
$b_3(n, J, L, m)$	J	$(-1)^{L+1}$	$(-1)^{n+1}$
$b_4(n, J, L, m)$	J	$(-1)^L$	$(-1)^n$

T A B L E 1

χ	SPIN	PARITY	CHARGE PARITY
1	0	+	+
γ_5	0	-	+
$\gamma_5 \gamma^\mu p_\mu$	0	-	+
$\gamma^\mu \epsilon_\mu^r$	1	-	-
$p^\mu \epsilon_\mu^r$	1	-	-
$\sigma^{\mu\nu} p_\nu \epsilon_\mu^r$	1	-	-
$\gamma_5 \gamma^\mu \epsilon_\mu^r$	1	+	+
$\gamma_5 p^\mu \epsilon_\mu^r$	1	+	-

T A B L E 2

FIGURE CAPTIONS

- Figure 1 Diagrammatic representation of the Green's function equation [Eq. (2.1)] for elastic scattering. The shaded rectangle represents the interaction kernel and the shaded circle gives the contribution of all iterations of this kernel.
- Figure 2 Ladder approximation to the interaction kernel.
- Figure 3 Effective T matrix element at the bound state pole.
- Figure 4 Approximation to the interaction kernel in a crossing symmetric bootstrap. The shaded circle represents the exchange of the entire Green's function at the bound state pole.
- Figure 5 The multichannel wave function $\chi_{\alpha,ij}$. α labels the bound state and i, j label the constituent particles.
- Figure 6 Multichannel interaction kernel in ladder approximation.
- Figure 7 Bootstrap equation for the coupling constant for 0^{--} states with $P = 0$. Here, $g_{p.s.}^2$ is the input coupling constant and $f_{p.s.}^2$ is the output coupling as calculated from Eqs. (2.16), (2.17). A single channel bootstrap is obtained if $g_{p.s.}^2 = f_{p.s.}^2$ while a multichannel bootstrap of the type described by Eq. (3.11) is obtained if $g_{p.s.}^2 = \frac{1}{2}f_{p.s.}^2$.

FIG.1

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6

FIG. 7