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A BOUNDARY CONDITION MODEL FOR THE

 P_{33} PION-NUCLEON STATE^{*)}

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A B S T R A C T

A singular boundary condition model with potential tail is developed for the P_{33} pion-nucleon resonance. The logarithmic derivative F is given a pole form suggested by theoretical considerations. Both Yukawa and square well tails are treated. It was found that for boundary radii $r_0 = 0.50 \mu^{-1}$ or $0.35 \mu^{-1}$ an attractive Yukawa tail of strength -2μ and range $0.5 \mu^{-1}$ gives an excellent fit to the P_{33} scattering data correcting violations of the Wigner condition, $dF/dE \leq 0$, previously seen at low energies in zero potential tail models. Only repulsive square well tails give good fits, and then the improvement seems to depend on the peculiar behaviour of relativistic corrections. Below $r_0 = 0.20 \mu^{-1}$ the single pole model merges into a constant F model with $F = -1.95$. Attractive tails agree with theoretical studies of the long range pion nucleon force.

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I. INTRODUCTION

There have been a number of papers applying the boundary condition model (BCM) to various strong interactions in recent years, especially to the nucleon-nucleon and pion-nucleon interaction. The idea behind this model is relatively simple. In the inner region of the strong interaction, the energy of the incoming particle is expected to be spread over so many virtual states that the wave function within this region is almost independent of the incoming energy. Therefore, if the inner region is defined by a boundary radius r_0 , the wave function inside r_0 is completely determined, as far as its effect on the outside is concerned, by the logarithmic derivative F of the radial wave function $\Psi(r) = r^{-1}u(r)$ at r_0 . The behaviour resulting from constraining F to be independent of energy or to have some other simple form, theoretically surmised from the requirements of unitarity and causality, is then examined. The parameter r_0 can be chosen as the radius inside of which the interaction is no longer simple, e.g. becomes non-local. It is further possible to consider a simple potential tail outside r_0 , as given by the usual potential theory, with the complex short range interaction being completely defined by F at r_0 .

One of the chief advantages of the BCM is that it is able to fit experimental data over a large range of energy with just a few parameters. For example, Goldberg and Lomon¹⁾ have obtained fits up to about 700 MeV pion lab energy (ϵ_π) for the pion-nucleon phase shifts α_{11} ($T = \frac{1}{2}$, $J = \frac{1}{2}$) and α_{31} ($T = \frac{3}{2}$, $J = \frac{1}{2}$). For $T = \frac{3}{2}$ they obtain the simplest result (constant F , no potential tail)

$$k r_0 \cot \alpha_{31} = \frac{F + 1 + k r_0 \tan k r_0}{1 - (F + 1)(k r_0)^{-1} \tan k r_0}$$

where $F = 0.28$ and $r_0 = 0.45 \mu^{-1}$ (μ = pion mass in natural units).

2.

A second paper²⁾ gives fits to the total cross-section and the D_{13} and D_{33} amplitudes over the range $\epsilon_{\pi} = 400-700$ MeV.

In a general review paper by Feshbach and Lomon³⁾ discussing the theoretical support and applications of the BCM, it was shown that the logarithmic derivative should have a pole form when there is an intermediate single particle (composite) state. Choosing the P_{33} ($L = 1$, $J = \frac{3}{2}$, $T = \frac{3}{2}$) pion-nucleon resonance as a well defined composite state for which there is abundant experimental data, Jacobs^{4,5)} tested this hypothesis and found it to be qualitatively true - the logarithmic derivative fitted a hyperbola fairly well over the range $\epsilon_{\pi} = 0-700$ MeV except at low energies where it appeared to violate the Wigner condition $dF/dE \leq 0$ required by causality. He obtained his best fits for $r_0 = 0.45\mu^{-1}$ and $0.55\mu^{-1}$. Figure 1 shows a typical fit for $r_0 = 0.50\mu^{-1}$. If the apparent Wigner violation is examined closely, it will be seen that a non-violating curve might just barely be fitted using the upper ends of the error brackets. However, such a curve generated by a single pole in F could not fit the higher energy points. As Jacobs used no potential tail, it was hoped that such an addition might correct this condition and give a better fit. It is the job of this paper to add a potential tail and see what conclusions can be drawn.

As a historical note the BCM has been previously applied to the P_{33} resonance by Weisskopf⁶⁾. By using an energy dependent form for F , he is able to closely reproduce the effective range results.

Boundary condition models with non-zero potential tails (sometimes called the modified boundary condition model) have been applied to the nucleon-nucleon interaction^{7,8,9)}. The first two references treat the proton-proton interaction with field theoretic tails OPEP and TPEP and obtain satisfactory fits up to $\epsilon_p = 310$ MeV. Lomon and Nauenberg⁹⁾, however, take advantage of the fact that an exponential tail is solvable in S states and obtain fits for the 1S_0 nucleon-nucleon state up to $\epsilon_p = 310$ MeV.

This paper treats the solvable square well tail and the theoretically more significant Yukawa tail which must be handled numerically. The next section considers the relation of F to the S -matrix and the reasons for expecting the logarithmic derivative to contain poles for composite states. The third section describes the method of calculation of F from the scattering data and the fourth section displays and discusses the results.

II. THE STRUCTURE OF F AND OF THE SCATTERING MATRIX

Since the logarithmic derivative completely defines the interaction inside r_0 , it must be related to the S-matrix which completely describes the scattering. In the zero potential tail case S depends only on F; with a potential tail solution of the radial Schrödinger equation integrated in from infinity, the Jost functions $f(k, r)$, also enter the picture. Feshbach and Lomon³⁾ construct the S-matrix from F

$$S_L(k) = (-1)^L \frac{(F_L + 1) f_L(k, r_0) - r_0 f_L'(k, r_0)}{(F_L + 1) f_L(-k, r_0) - r_0 f_L'(-k, r_0)} \quad (1)$$

A bound state situation is manifested as a pole in the S matrix in the upper half k-plane, which should be on the imaginary axis if the corresponding energy is to be real and negative. It is possible to derive the Wigner condition on F

$$\frac{dF}{dE} \leq 0$$

by requiring that these poles be on the imaginary axis for arbitrary potential tails³⁾. The Wigner condition was originally derived from classical causality; the incoming wave proceeds with at most infinite velocity through the core region $r \leq r_0$ ¹⁰⁾ (this derivation is also in reference 3). As mentioned in the introduction, Jacobs' curves of F for the P_{33} resonance appeared to violate this condition at low energy (see Fig. 1).

A resonance situation is manifested by a pole in the S-matrix at $k = k_r - i\Gamma$. By unitarity there must also be a pole at $k = -k_r - i\Gamma$ and zeros at $k = \pm k_r + i\Gamma$. This situation can be studied more easily in the case of zero potential tail and $L = 1$ for a P resonance. With

no potential tail the Jost functions are simply related to Hankel functions of the second kind and the S-matrix becomes

$$S_1 = \frac{(F+2)(1+ikr_0) - (kr_0)^2}{(F+2)(1-ikr_0) - (kr_0)^2} e^{-2ikr_0} \quad (2)$$

Using the resonance condition $S = -1$, F passes slightly above -2 at the resonance energy agreeing with Fig. 1. As will be discussed below -2 is the value below which a constant F implies a bound state for $L = 1$.

From the relation of F to the S-matrix, it is expected that F will also have poles for composite states. Feshbach and Lomon³⁾ were able to demonstrate this explicitly by considering two channels in the BCM. In the limit that the second channel simulates a single particle state, they showed that the logarithmic derivative for the first channel has a pole form.

It is also possible for some constant values of F to generate a single resonance in states with $L > 0$. For instance, a pole in S_1 is given by the zero of the denominator of (2).

$$k_p r_0 = -\frac{1}{2}(F+2) \pm \frac{1}{2} \sqrt{-(F+2)^2 + 4(F+2)}$$

k_p is purely imaginary positive for $(F+2) < 0$; this is the condition for a bound state:

$$F < -2 \quad \text{implies bound state for } L = 1.$$

If F is a constant slightly greater than -2 , say $F + 2 = \epsilon$, then

$$k_p r_0 = -i\epsilon/2 \pm \sqrt{\epsilon}$$

and k_p is slightly above threshold with a small negative imaginary part, i.e. in the proper position for a resonance. The boundary radius must be small in this situation or the resonance would only appear at a very low energy. The situation of constant F producing a resonance is peculiar to $L > 0$, for the angular momentum supplies the only barrier to penetration.

The resonances generated by the boundary condition as above are called "internal resonances". It is also possible for a strongly attractive tail to bind a resonance which would then be called an "external resonance". This paper considers a model in which the resonance is generated by the boundary condition and the long range forces of the potential tail only modify the resonance background.

The wave function has a zero at the position of the pole of F . Given the energy W_r and the half-width $\Gamma/2$ of the resonance, the phase shift near the resonance is

$$\delta(w) = \delta_0 + \tan^{-1} \frac{1}{2} \Gamma / (W_r - w)$$

where δ_0 is the slowly varying "background" phase shift. So the pole could be located by solving the transcendental equation

$$u(W_r, \delta(W_r)) \Big|_{r_0} = 0 \quad (3)$$

where u is the reduced radial wave function. In the case $L = 1$ the equation is

$$\tan(k_p r_0 + \delta(W_r)) - k_p r_0 = 0$$

III. THE INTERACTION AND CALCULATIONAL METHOD

For the P_{33} phase shift data, necessary as input to calculate the logarithmic derivative, a recent analysis of Roper¹¹⁾ of all non-zero pion-nucleon phase shifts over the range 0-700 MeV pion lab energy was used. This analysis employs an energy dependent method in which resonances for which there is the best experimental evidence are assumed to exist and the phase shift solutions are constrained to produce these resonances. Thus, in the energy range 0-350 MeV, of the non-negligible amplitudes, only the P_{33} was assumed to resonate and in the range 0-700 MeV the following choices were tried: (a) D_{13} , (b) P_{13} , (c) $D_{13} + P_{13}$, (d) no resonance. Roper obtained his best fit for case (a) and it was therefore chosen for the basic data of this problem. A list of the phase shift as a function of lab and c.m. energy is given in Table I.

To obtain the best estimates for the error in α_{33} , the Fermi I solutions of other workers (a list of these is given in reference 12) were compared with Roper's. It is encouraging that the average error brackets given by the workers is in approximate agreement with the deviation of their results from Roper's (and therefore among themselves). However, they deviate from Roper's smooth energy dependence by much less than these errors. The errors are shown in Table II along with the number of pieces of information used in obtaining the estimates. The error bracket $\pm 7.5^\circ$ above 600 MeV is the uncertainty introduced by Roper's four choices of the proper resonating amplitude.

A similar error analysis was done on the pion-lab energy. These errors, which introduce an error in W , the total c.m. energy, of ± 2 MeV at threshold to ± 7 MeV at the high end are too small to be shown in the figures.

Using natural units; $\hbar = c = 1$, and M = mass of nucleon, m = mass of pion, μ = reduced mass, E_{π} = the lab energy of a pion colliding with a stationary nucleon, then the total c.m. energy is given by

8.

$$W^2 = (\mu + M)^2 + 2 \epsilon_\pi M \quad (4)$$

and the c.m. wave number is given by

$$k = \frac{M}{W} \left[\epsilon_\pi (\epsilon_\pi + 2\mu) \right]^{\frac{1}{2}} \quad (5)$$

Since the maximum pion energy is 700 MeV, several times the pion rest mass and comparable with the nucleon rest mass, relativistic kinematics should be used. There is, however, some question about relativistic corrections to the potential or to the form of the Schrödinger equation. It is clear that the energies involved are high enough to warrant relativistic forms, but such corrections are ambiguous. As will be seen later on, they lead to an awkward, non-analytic, dependence of the effective wave number on W in the physical region. This behaviour also appears in K in the Schrödinger equation, but only below threshold. Although an interaction, V , always exists for which the Schrödinger equation generates the correct physical amplitude, there is still the question of which equation leads to the simplest structure of V and better extends the amplitude beyond the physical region. In view of these uncertainties it was thought best to try and compare both cases. In the first case the non-relativistic radial equation

$$\frac{d^2 u_L(r)}{dr^2} + \left[k^2 - 2mV(r) - L(L+1)r^{-2} \right] u_L(r) = 0 \quad (6)$$

was used but with the wave number k handled relativistically as Eq. (5) above. For the second case a Klein-Gordon-like approach was adopted in which a local wave number was computed from

$$(\chi^2 + \mu^2)^{\frac{1}{2}} + (\chi^2 + M^2)^{\frac{1}{2}} = W - V(r) \quad (7)$$

and substituted into

$$\frac{d^2 u_L}{dr^2} + \left[\chi^2 - L(L+1)r^{-2} \right] u_L(r) = 0 \quad (6')$$

(An ambiguity in the relativistic correction is apparent already:

Should the centrifugal potential have been included in the V of (7)?)

Equation (7) may be solved for χ^2

$$\chi^2 = \frac{1}{4} \left[(W-V)^2 - 2(\mu^2 + M^2) + (W-V)^{-2} (M^2 - \mu^2)^2 \right] \quad (8)$$

In the non-relativistic limit χ^2 reduces properly to

$$\chi^2 \xrightarrow{N.R.} k^2 - 2mV(r) \quad (9)$$

and W reduces to

$$W \xrightarrow{N.R.} \mu + M + \frac{k^2}{2m}$$

$u_1(r)$ should have the asymptotic form

$$u_1(r) \xrightarrow{r \rightarrow \infty} k^{-1} \sin(kr - \frac{1}{2}\pi + \alpha_{33}) \quad (10)$$

The dimensionless logarithmic derivative is defined as

$$F \equiv r_0 [\psi(r_0)]^{-1} \left[\frac{d\psi}{dr} \right]_{r_0} = r_0 [u(r_0)] \left[\frac{du}{dr} \right]_{r_0} - 1 \quad (11)$$

Of the three types of potential tails studied, an explicit solution for F can be given for the zero and square well tail cases¹²⁾. However, the Yukawa tail must be handled numerically so all calculations were performed by a computer with several zero and square well tails also done by hand to check the programme. An IBM 7044 computer integrated the radial equation

$$u'' + \left[\chi^2 - \frac{2}{r} \right] u = 0$$

with χ^2 given by (9) or (8) for the non-relativistic or Klein-Gordon-like cases respectively and

$$V(r) = 0 \quad \text{for zero potential tail}$$

or

$$\left. \begin{array}{ll} V \times 120 \text{ MeV} & r \leq A\mu^{-1} \\ 0 & r > A\mu^{-1} \end{array} \right\} \text{square well tail}$$

or

$$V\mu e^{-A\mu r}/\mu r \quad \text{Yukawa tail} \quad (12)$$

The integration ran from $r = 5\mu^{-1}$ with the initial condition (10) to r_0 in steps of $\Delta r = 0.01\mu^{-1}$. The calculation was done for each set

of (r_0, A, V) for twenty eight values of ϵ_{π} in steps of 25 MeV, and the twenty-eight values of F , Eq. (11), were computed.

To fit the logarithmic derivative, consider a hyperbola of the form

$$H(W) = \frac{B}{W - E_0} + C$$

The standard method for finding the best hyperbola is to minimize the sum

$$S = \sum_{i=1}^{28} [\Delta F(W_i)]^2 [F(W_i) - H(W_i)]^2 \quad (13)$$

where $\Delta F(W_i)$ is the experimental root mean square deviation of F . However, in view of the fact that it takes as much machine time to compute $\Delta F(W_i)$ from $\Delta \alpha_{33} \text{ exp}$ as to compute $F(W_i)$ itself, an analytic form for $\Delta F(W_i)$ is useful. A main feature of ΔF is that it becomes very large near the pole (Fig. 1). Consider

$$\Delta F(W_i) \sim \frac{1}{W_i - E_0}$$

or

$$S' = \sum_{i=1}^{28} \left\{ [F(W_i) - H(W_i)] [W_i - E_0] \right\}^2 \quad (14)$$

This form has a special mathematical advantage; namely, the sum S' can now be carried out and the Eqs. for B , C , and E_0 of the minimum be solved explicitly¹²⁾, thus avoiding the lengthy process of trying many solutions to find the minimum. Therefore, it was decided to minimize S' to find the best hyperbola and afterwards compute S for several representative fits.

IV. RESULTS AND DISCUSSION

The three lowest order diagrams contributing to pion-nucleon scattering are the Feynman graphs for:

- a) a single particle intermediate state, which must be an N^* for the $T = \frac{3}{2}$ $J = \frac{3}{2}$ state,
- b) single baryon exchange, either N or N^* , and
- c) the exchange of two pions or a di-pion.

(One pion exchange is forbidden by G-parity.) Of these three (a) gives a zero range force, (b) both a short range force and a long range force of range $\sim (M\mu)^{-\frac{1}{2}} \sim 0.4\mu^{-1}$, and (c) a long range force of range $\sim 0.5\mu^{-1}$. The short range forces are expected to be covered by the boundary condition, so the potential tail comes from the long range part of (b) and from (c). On the basis of these considerations a range of $0.5\mu^{-1}$ was chosen for the Yukawa potential tails. An "equivalent" square well tail would have a somewhat longer range $\sim 0.7\mu^{-1}$.

Three values of r_0 were chosen to be studied first to see how the fits depend on this parameter - $r_0 = 0.35, 0.50$, and $0.75\mu^{-1}$. Jacobs obtained his best fit for $r_0 = 0.45\mu^{-1}$ agreeing with the value used by Goldberg and Lomon¹⁾ in their pion-nucleon fits.

A complete set of fits was obtained for the three values of r_0 for Yukawa and square well tails, the strength of the potential being varied over five steps from strongly attractive to strongly repulsive. The parameters of these fits are displayed in Tables III - VI for non-relativistic and Klein-Gordon-like types of tails. Figure 2 shows the trend of the quality of fit S' (Eq. (14)) for the different types of potential tails as the potential strength is varied.

These four tables yield the following observations:

1. For $r_0 = 0.35\mu^{-1}$ and $0.50\mu^{-1}$ excellent fits are obtained for strongly attractive Yukawa tails in both the non-relativistic and Klein-Gordon-like cases. By contrast good fits are obtained for strongly repulsive square well tails and then only in the Klein-Gordon-like case. This point will be discussed below. Sample good fits for $r_0 = 0.50\mu^{-1}$ are in Figs. 3-5. Notice that any Wigner violation is completely corrected by the addition of a potential tail. The low energy points no longer tend downward but are smoothly fitted by the hyperbola.

Figure 3 shows a fit for an attractive Yukawa tail of strength -2μ and range $0.5\mu^{-1}$, $r_0 = 0.5\mu^{-1}$. Error brackets as computed from the errors in α_{33} (Table II) are shown. For this tail S (Eq. (13)) was computed to be 1.3. In contrast to the zero potential case (Fig. 1) $S = 496$. S would normally be approximately equal to the number of points fitted, viz. 28, less the number of parameters, viz. 4. The improbably low value of S obtained here indicates a conservative choice of $\Delta\alpha_{33}$. It is also due in part to the fact that Roper's curve is a smooth fit to less than 28 experimental energies. At any rate the attractive tail gives an excellent fit to the data.

2. The Klein-Gordon-like case exaggerates the variation in the trends of the parameters; that is, it makes the potential appear as if it were stronger. Also in the Klein-Gordon-like case a repulsive tail (square well or Yukawa) gives a better fit. To see why this is so, examine more closely the Eqs. (7) and (9) for the local wave number.

So χ^2 of Eq. (9) is analytic, but that of Eq. (7) has branch points at $-\mu^2$ and $-M^2$. If V is repulsive enough ($V \geq \mu$), the branch point $-\mu^2$ will be above threshold. As W increases from its threshold value $M + \mu$, $\chi^2(K - G)$ decreases until $\chi^2 = -\mu^2$ and then increases, whereas $\chi^2(n-r)$ increases uniformly from threshold. This decrease of $\chi^2(K - G)$ at low energies makes the repulsive tail appear to be attractive; hence, good fits are obtained. Also the branch point caused

the tail to have a more rapidly changing effect and thus the parameters show greater variation for Klein-Gordon-like tails.

3. To see how sharply the quality of the fit depends on the strength of the potential, computations were done for an $r_0 = 0.50\mu^{-1}$ non-relativistic Yukawa tail with V varied in steps of 0.25μ from -3.00μ to -1.00μ . These results are shown in Table III and sketched in Fig. 2 (a). The quality of fit does indeed depend sharply on the strength of the potential; for this case $V = -1.9\mu \pm 0.4\mu$ for a good fit.

4. Which values of r_0 give a good fit? Although the dF/dE of an $r_0 = 0.75\mu^{-1}$, zero potential tail BCM satisfies the Wigner condition, it is poorly fitted by the pole because the two branches have different curvatures (see figure in Ref. 5). Furthermore neither attractive nor repulsive tails significantly improve the fit (see Fig. 2 (a)). Therefore the upper limit for r_0 is between $r_0 = 0.50\mu^{-1}$ and $r_0 = 0.75\mu^{-1}$.

To get a lower limit on r_0 , programmes were run for non-relativistic Yukawa tails with $r_0 = 0.30\mu^{-1}$ and $0.20\mu^{-1}$. The results are displayed in Table III. It is seen that at $r_0 = 0.20\mu^{-1}$, F becomes nearly a constant at $F = -1.95$ (Fig. 6). However, $\Delta F(W_i)$ is very small so that S is an order of magnitude larger than the best fit. The slight Wigner violations are again corrected by an attractive tail although the addition of a tail tends to give a poorer fit to a constant F . The fact that under certain conditions the F of a resonating state can be represented by a constant was discussed in the second section. Because of the increase in S , and consistent with the belief that an interaction contained entirely in $r_0 = 0.20\mu^{-1}$ is unlikely theoretically, it is concluded that the lower limit for r_0 is around $0.30\mu^{-1}$.

5. We conclude with a brief reference to theoretical studies of the long range forces in the P_{33} pion-nucleon state. Referring to the first order diagrams, the long range part of diagram (b) is thought to dominate diagram (c) for a low energy P-wave interaction^{13),14)}. Hamilton et al.¹⁴⁾ find that this diagram contributes a strongly attractive long

range force. They also analyze diagram (c) in terms of a $T = 0$ π - π interaction and again obtain an attractive long range force. There is therefore good agreement between the model for the P_{33} state developed in this paper and the above theoretical results.

V. CONCLUSIONS

The P_{33} phase shift up to 700 MeV pion laboratory energy has been interpreted in terms of a BCM with a potential tail and relativistic kinematics. The data is fitted very well and the five parameters of the model determined sufficiently closely to permit some theoretical inferences. The best fitting values of the boundary radius, r_0 , are in the theoretically indicated range, and the best values of the potential strength have the theoretical sign and a reasonable strength. The range of the potential tail was taken from theoretical indications and not varied.

The statistically better fits all require a pole term in the $F(W)$ function, and are very well fitted by this simple form. The pole in F drives the resonance; the angular momentum barrier is unimportant at $r_0 = 0.5 \text{ } \mu^{-1}$ and the potential is of the wrong sign to sharpen a resonance. Therefore this is not an example of an "external resonance" due to a barrier penetration effect. Also the structure of F shows no sign of a cut appearing a few hundred MeV above the resonance, as would be the case if the resonance was due to the coupling to another two (or more) body channel. Such a cut is expected to start at the $\pi + N^*$ threshold (400 MeV), but in this channel the threshold behaviour would be that of a P state and the effect may be weak up to 700 MeV.

We conclude on this basis that the P_{33} state is an example of an internal resonance; i.e. that it is a short range effect due to the existence of a "compound state" of the system.

The comparatively simple amplitude of this BCM could be used as an input into partial wave dispersion relations, or used with crossing symmetry as a restriction on related channels. The fitted potential tail is a measure of the nearer part of the unphysical cut.

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Table IP₃₃ Pion-Nucleon Phase Shift as a Function of Energy

E_{π} (MeV)	W (MeV)	δ_{33} (Degrees)
0	1078.2	0
25	1099.4	2
50	1120.5	6
75	1141.2	12
100	1161.6	21
125	1181.7	34
150	1201.3	52
175	1220.7	74
200	1239.8	94
225	1258.6	109
250	1277.1	122
275	1295.3	130
300	1313.3	134
325	1331.1	139
350	1348.6	143
375	1365.9	144
400	1382.9	146
425	1399.8	148
450	1416.5	150
475	1432.9	151.5
500	1449.2	153
525	1465.3	153.5
550	1481.3	154.5
575	1497.0	155
600	1512.6	156
625	1528.0	156.5
650	1543.3	157
675	1558.4	157.5
700	1573.4	158

Table IIErrors in α_{33} and ϵ_{π}

Energy range (MeV)	$\Delta\alpha_{33}^{\text{exp}}$ (degree)	Bits	$\Delta\epsilon_{\pi}^{\text{exp}}$ (MeV)	Bits
0 - 50	2.2	7	1	4
50 - 100	3.0	3	1	2
100 - 200	6.1	14	2	11
200 - 300	6.5	11	3	9
300 - 400	7.6	10	4	13
400 - 500	9.0	6	5	10
500 - 600	11.0	4	5	6
600 - 700	12.5	4	7.5	0

Table III

Non-Relativistic Yukawa Tails ($A = 2.00$ for all fits)

μr_0	V	B(GeV)	C	E_0 (MeV)	S'
0.75	-2.00	0.178	0.1844	1348	0.0987
0.75	-1.00	0.173	-0.0084	1345	0.0554
0.75	0.0	0.169	-0.1736	1342	0.0354
0.75	1.00	0.166	-0.3153	1338	0.0264
0.75	2.00	0.161	-0.4420	1334	0.0227
0.50	-3.00	0.495	0.7842	1500	0.1035
0.50	-2.50	0.468	0.5671	1490	0.0247
0.50	-2.25	0.450	0.4610	1486	0.0126
0.50	-2.00	0.432	0.3537	1481	0.0051
0.50	-1.75	0.415	0.2534	1477	0.0052
0.50	-1.50	0.401	0.1641	1473	0.0084
0.50	-1.00	0.378	0.0169	1465	0.0216
0.50	0.0	0.330	-0.2273	1446	0.0543
0.50	1.00	0.289	-0.4712	1432	0.0610
0.50	2.00	0.256	-0.6754	1418	0.0703
0.35	-2.00	0.534	-0.2506	1659	0.0027
0.35	-1.00	0.362	-0.7441	1611	0.0199
0.35	0.0	0.330	-0.9146	1587	0.0446
0.35	1.00	0.334	-0.9409	1555	0.0846
0.35	2.00	0.258	-1.160	1511	0.0963
0.30	-2.00	0.411	-0.6755	1712	0.0018
0.30	-1.00	0.216	-1.190	1632	0.0099
0.30	0.0	0.173	-1.409	1603	0.0176
0.30	1.00	0.183	-1.483	1588	0.0303
0.20	-2.00	0.096	-1.482	1718	0.0003
0.20	-1.00	0.025	-1.759	1587	0.0012
0.20	0.0	0.010	-1.897	1562	0.0017
0.20	1.00	0.011	-1.993	1563	0.0023
0.20	2.00	0.022	-2.055	1571	0.0038

Table IVKlein-Gordon-like Yukawa Tails ($A = 2.00$ for all fits)

λr_0	V	B(GeV)	C	E_0 (MeV)	S'
0.75	-2.00	0.2259	0.4632	1364	0.1326
0.75	-1.00	0.1969	0.1547	1349	0.0754
0.75	0.0	0.1699	-0.1736	1342	0.0354
0.75	1.00	0.1487	-0.4364	1333	0.0239
0.75	2.00	0.1288	-0.6757	1327	0.0170
0.50	-2.00	0.7068	1.120	1545	0.0086
0.50	-1.00	0.4823	0.3141	1494	0.0330
0.50	0.0	0.3300	-0.2273	1446	0.0543
0.50	1.00	0.2262	-0.7000	1413	0.0370
0.50	2.00	0.1606	-1.049	1389	0.0219
0.35	-2.00	1.125	0.8574	1860	0.0076
0.35	-1.00	0.3642	-0.6962	1643	0.0243
0.35	0.0	0.3300	-0.9146	1587	0.0446
0.35	1.00	0.2149	-1.179	1492	0.0399
0.35	2.00	0.1085	-1.487	1427	0.0143

Table VNon-Relativistic Square Well Tails (Range = $0.7\mu^{-1}$)

μr_0	V	B(GeV)	C	E_0 (MeV)	S'
0.50	-2.00	0.3918	0.2460	1451	0.0708
0.50	-1.00	0.3591	0.0063	1449	0.0631
0.50	0.0	0.3300	-0.2273	1446	0.0543
0.50	1.00	0.3052	-0.4451	1445	0.0463
0.50	2.00	0.2843	-0.6449	1442	0.0399
0.35	-2.00	0.4269	-0.5193	1517	0.0460
0.35	-1.00	0.3886	-0.7462	1601	0.0430
0.35	0.0	0.3300	-0.9146	1587	0.0446
0.35	1.00	0.3187	-1.010	1572	0.0547
0.35	2.00	0.2598	-1.224	1551	0.0488

Table VIKlein-Gordon-Like Square Well Tails (Range = $0.7\mu^{-1}$)

μr_0	V	B(GeV)	C	E_0 (MeV)	S'
0.50	-2.00	0.5384	0.9763	1467	0.1601
0.50	-1.00	0.4231	0.3229	1454	0.1108
0.50	0.0	0.3300	-0.2287	1446	0.0543
0.50	1.00	0.2710	-0.6090	1442	0.0186
0.50	2.00	0.2317	-0.8226	1438	0.0035
0.35	-2.00	0.3978	-0.3594	1630	0.1207
0.35	-1.00	0.3757	-0.6836	1617	0.0675
0.35	0.0	0.3300	-0.9146	1587	0.0446
0.35	1.00	0.2297	-1.220	1540	0.0191
0.35	2.00	0.1517	-1.463	1510	0.0025

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FIGURE CAPTIONS

- Fig. 1: $r_0 = 0.50\mu^{-1}$, zero potential tail
 $B = 0.3300$ GeV, $C = 0.2273$, $E_0 = 1446$ MeV, $S' = 0.0543$,
 $S = 496.4$
- Fig. 2: Trends of the parameter S' , the quality of fit, for the various types of tails as a function of potential strength, -2μ to $+2\mu$ 0 $r_0 = 0.35\mu^{-1}$, - - - Δ - - -
 $r_0 = 0.50\mu^{-1}$, — $\square$ — $r_0 = 0.75\mu^{-1}$.
 a) Non-relativistic Yukawa tails.
 Note that for $r_0 = 0.50\mu^{-1}$ the curve is extended to $V = -3\mu$
 b) Klein-Gordon-like Yukawa tails
 c) Non-relativistic square well tails
 d) Klein-Gordon-like square well tails
- Fig. 3: $r_0 = 0.50\mu^{-1}$, non-relativistic Yukawa tail, range $0.5\mu^{-1}$, strength -2μ ,
 $B = 0.4323$, $C = 0.3523$, $E_0 = 1481$, $S' = 0.0051$, $S = 1.306$.
- Fig. 4: $r_0 = 0.50\mu^{-1}$, Klein-Gordon-like Yukawa tail, range $0.5\mu^{-1}$, strength -2μ ,
 $B = 0.7068$, $C = 1.120$, $E_0 = 1545$, $S' = 0.0086$.
- Fig. 5: $r_0 = 0.50\mu^{-1}$, Klein-Gordon-like square well tail, range $0.7\mu^{-1}$, strength $+2\mu$,
 $B = 0.2317$, $C = -0.8226$, $E_0 = 1438$, $S' = 0.0035$.
- Fig. 6: $r_0 = 0.20\mu^{-1}$, zero potential tail.

Fig. 2

Fig. 2

Fig. 4

Fig. 5

Fig. 6