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ON A PROBLEM OF MAXIMUM IN DISPERSION THEORY

L. Łukaszuk ^{*)}

CERN - Geneva

A B S T R A C T

We present a technique for finding maximal bounds on strengths of interaction when the general inequality of the type arising from dispersion relations is given. It is shown that for quite a large class of functionals involved in such inequalities the solution exists and is unique.

^{*)} On leave of absence from Institute for Nuclear Research,
Warsaw, Poland.

INTRODUCTION

In the problem of finding bounds on strength of interaction one gets conditions of the type ^{1),2),3)}:

$$\oint [F] = \int_1^{\infty} g(x, |F(x)|) dx = A \leq M$$

$g(x, |F(x)|)$ and $F(x)$ - the latter has physical interpretation, e.g., as form factor ²⁾ or scattering amplitude for $t \geq 0$ ³⁾ - satisfy the following conditions (comp. Ref. ²⁾):

- a) $F(x)$ is an analytic function of x except for a cut from 1 to ∞ . Also $F^*(x) = F(x^*)$;
- b) $F(x)$ is bounded at ∞ by some power of $x^{1/2}$;
- c) $F(x)$ is non-zero on the cut except at a finite number of discrete points;
- d) $g(x, |F|)$ is positive;
- e) $\oint [F]$ exists.

Two types of problem occur:

- 1) M is given and one wants to get the maximum of $|F|$ at some point outside the cut $x = 1 \rightarrow x = \infty$. Here we shall restrict ourselves to the problem of finding the maximum of $|F(x=c)|$, $c < 1$ and real;
- 2) M is itself a function of $F(c)$ and one wants to find, by self-consistency, an absolute bound of $F(c)$. Here we shall restrict ourselves to the case $M = F(c)$. We shall deal with these problems in the first and second part of the paper respectively.

2.

The functions $g(x, |F|)$ for which our technique gives the unique solution have to satisfy the following conditions:

f) when $|F|$ is treated as independent variable, both $g(x, |F|)$ and $(\partial g(x, |F|))/(\partial |F|)$ are continuous and positive functions of x and $|F|$ for $x > 1$ and $|F| \geq 0$. Moreover, $(g(x, |F|))/|F|$ and $(\partial g(x, |F|))/(\partial |F|)$ are increasing with $|F|$ for $x > 1$:

g) there exist functions $F(x)$ such that

$$\sqrt{x-1} (x-c) |F| \partial g(x, |F|) / \partial |F| = B > 0 \Big|_{x \rightarrow 1}$$

and

$$\sqrt{x-1} (x-c) |F| \partial g(x, |F|) / \partial |F| = D > 0 \Big|_{x \rightarrow \infty}$$

We demand that they satisfy

$$(x-1)^\beta < |F(x)| < (x-1)^\alpha \quad \text{for } x \rightarrow 1$$

and

$$x^{\beta'} < |F(x)| < x^{\alpha'} \quad \text{for } x \rightarrow \infty$$

where $B, D, \alpha, \beta, \alpha', \beta'$ are arbitrary constants; $c < 1$.

I. By the change of variables (comp., e.g., Ref. 2))

$$Z = -(\bar{t} - i)/(t + i) \text{ with } t = \tan \frac{\Theta}{2} = \left[\frac{x-1}{1-c} \right]^{1/2} \text{ (notice } x=c \text{ implies } z=0)$$

one maps the cut x plane onto the unit disc $|z| \leq 1$. Next, denoting $F(z = e^{i\Theta})$ by $F(\Theta)$ one gets a condition

$$\oint [F] = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\Theta, |F(\Theta)|) d\Theta = A \leq M \quad (1)$$

where f is expressed in a simple way by $g(x, |F|)$ and satisfies condition d). $\bar{f}(\Theta, |F|)$ will be involved now in equalities $|F| \frac{\partial}{\partial |F|} f(\Theta, |F|) \big|_{\Theta \rightarrow 0} = B$ and $|F| \frac{\partial}{\partial |F|} f(\Theta, |F|) \big|_{\Theta \rightarrow \pi} = D$ in condition g). With properties a) - e) the problem of minimum for $\oint [F]$ - which is an essential step in finding bounds - has been solved for the following $\oint [F]$:

$$\oint [F] = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\Theta) |F(\Theta)|^n d\Theta, \quad (2)$$

n being an arbitrary positive constant. The solution for $n = 2$ - found by Szegő⁴⁾ - was generalized to the case of several functions F_i by Meimann⁵⁾. A simpler proof with the additional cut of $F(z)$ from $z = \beta$ ($0 < \beta < 1$) to $z = 1$, is given by Drell et al.²⁾. The last method may be easily applied to any n . One gets for the minimum of (2)

$$\phi_{\min} = |F_0|^n \exp \left\{ -\frac{n}{2\pi i} \int_0^1 \frac{\operatorname{disc} \ln F(z)}{z} dz \right\} \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln p(\theta) d\theta \right\}; \quad F_0 = F(z=0)$$

However, in the case of the $\pi - \pi$ scattering one is forced ⁶⁾ to put a more complicated form of $f(\theta, |F|)$ in (1), namely $f_F \rightarrow \infty \sqrt{|F|} \exp \{ \alpha(\theta) \sqrt{|F|} \}$. Then Meimann's technique does not seem to be convenient. This may be already seen in the comparatively simple case of

$$\Phi[F] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{ p_1(\theta) |F|^2 + p_2(\theta) |F|^4 \} d\theta \quad p_1, p_2 > 0 \quad (3)$$

Using the notation of Ref. ³⁾ one could write:

$$\Phi[F] = \Phi_1[F] + \Phi_2[F]$$

$$\Phi_1[F] = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_1(\theta) |F(\theta)|^2 d\theta = \sum_n F_n^{(1)} F_n^{(1)}$$

$$\Phi_2[F] = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_2(\theta) |F(\theta)|^4 d\theta = \sum_n F_n^{(2)} F_n^{(2)}$$

$F_n^{(1)}, F_n^{(2)}$ are coefficients of expansion $F(\theta)$ and $(F)^2$ in polynomials with weight $p_1(\theta)$ and $p_2(\theta)$ respectively. $\{ F_n^{(2)} \}$ are now dependent on $\{ F_n^{(1)} \}$ and the dependence between them is by no means expressible in a simple way. The existence of these relations makes the procedure of minimizing $\{ F_n \}$ rather unhandy. It is still possible to treat $\{ F_n^{(1)} \}$ and $\{ F_n^{(2)} \}$ as if they were mutually independent but we do not use then some important information about $\Phi[F]$ in our considerations. For the case of the functional (3) it may be for example explicitly shown that in such an approach there exist

no $F_{\text{extr}}(\theta)$ which minimizes simultaneously both ϕ_1 and ϕ_2 unless $p_2(\theta)$ is proportional to $(p_1(\theta))^2$ ⁷⁾. It generally means that we get too low a value for ϕ_{min} . We want to propose a technique which was found for the functional (3) by A. Martin ⁷⁾, and which is efficient for more general forms of $f(\theta, |F|)$. (To illustrate the essential features of our method, we apply it in an appendix explicitly to a simple particular case of the functional ϕ .)

Let us consider a functional of the type (1) with conditions a) - g) fulfilled. We weaken condition a) allowing - as in Ref. ²⁾ - for an additional cut from $z = \beta$ to $z = 1$ ($0 < \beta < 1$).

Let us now assume that we are given some arbitrary positive $\lambda(\theta)$ and look for $F_{\text{extr}}(\theta)$ which minimizes a functional

$$\Psi[F] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{ f(\theta, |F(\theta)|) - \lambda(\theta) |F(\theta)| \} d\theta \quad (4a)$$

We get a condition

$$\left. \frac{\partial \Psi[F]}{\partial |F|} \right|_{|F|=F_{\text{extr}}(\theta)} = \left. \frac{\partial f(\theta, |F|)}{\partial |F|} \right|_{|F|=F_{\text{extr}}} - \lambda = 0 \quad (4b)$$

There may exist only one $F_{\text{extr}}(\theta)$ satisfying (4b) for given $\lambda(\theta)$ because of condition f). Combining (4a,b) with (1) we have

$$A \equiv \phi[F] \equiv \Psi[F] + \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda |F| d\theta \geq \Psi[F_{\text{extr}}] + \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda |F(\theta)| d\theta$$

which gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda(\theta) |F(\theta)| d\theta \leq A - \Psi[F_{\text{extr}}] \quad (5a)$$

where the equality sign is achieved for $|F(\theta)| = F_{\text{extr}}(\theta)$. The minimum of the left-hand side may be found by applying the theorem of the geometric and arithmetic means (comp. Refs. 2), 4)). One gets then

$$\frac{|F_0|}{r_1^2 \dots r_n^2} \exp\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \lambda(\theta) d\theta\right] \cdot \exp\left\{-\frac{1}{2\pi i} \int_{\beta}^{\gamma} \frac{d \ln F(z)}{z} dz\right\} \leq \mu \quad (5b)$$

$$\mu = A - \Psi[F_{\text{extr}}]$$

$\bar{r}_i < 1$ are moduli of zeros of $F(z = \rho e^{i\theta})$ inside the circle $|z| < 1$. The equality sign applies if $|F(\theta)| = F_{\text{extr}}(\theta)$ and $F_{\text{extr}}(\theta) \cdot \lambda(\theta) = \text{const.}$ We have still freedom in choosing the function $\lambda(\theta)$. Let us take that $\lambda_{\text{extr}}(\theta)$ which brings inequality (5b) closest to equality. It means that we look for the minimum of the functional

$$\chi[\lambda] = \frac{(A - \Psi[F_{\text{extr}}]) r_1^2 \dots r_n^2}{|F_0| \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \lambda(\theta) d\theta\right\} \exp\left\{-\frac{1}{2\pi i} \int_{\beta}^{\gamma} \frac{d \ln F(z)}{z} dz\right\}}$$

This is equivalent to minimizing against $\ln \chi[\lambda]$. We are led then to:

$$\frac{1}{\lambda} \Big|_{\lambda=\lambda_{\text{extr}}} = \frac{1}{\mu} \left[-\frac{\delta \Psi[F_{\text{extr}}]}{\delta \lambda} \right] \Big|_{\lambda=\lambda_{\text{extr}}} = \frac{1}{\mu} \left[-\frac{\partial \ln |F|}{\partial |F|} \Big|_{|F|=F_{\text{extr}}} \cdot \frac{\partial F_{\text{extr}}}{\partial \lambda} + \lambda \frac{\partial F_{\text{extr}}}{\partial \lambda} + F_{\text{extr}} \right] \Big|_{\lambda=\lambda_{\text{extr}}}$$

The first two terms of the r.h.s. cancel in virtue of Eq. (4b) and we get

$$\frac{\mu}{\lambda_{\text{extr}}(\theta)} = F_{\text{extr}}(\theta) \quad (6a)$$

with

$$\mu = A - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta, F_{\text{extr}}) d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda_{\text{extr}}(\theta) F_{\text{extr}} d\theta$$

Because of (6a) we may change the above condition for μ to:

$$A = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta, F_{\text{extr}}(\theta)) d\theta \quad (6b)$$

Making use of (4b) one may rewrite (6a,b) in the following way:

$$\lambda_{\text{extr}} F_{\text{extr}} = F_{\text{extr}} \frac{\partial f(\theta, F_{\text{extr}})}{\partial F_{\text{extr}}} = \mu = \text{const} \quad (7a)$$

$$A = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta, F_{\text{extr}}(\theta)) d\theta \quad (7b)$$

Equation (7a) fixes F_{extr} for given μ . Equation (7b) fixes μ . Our inequality (5b) becomes, after inserting λ from (7a)

$$\frac{1}{r_1^2 \dots r_n^2} |F_0| \exp \left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln F_{\text{extr}}(\theta) d\theta \right\} \exp \left\{ -\frac{1}{2\pi i} \int_{\beta}^1 \frac{d \ln F(z)}{z} dz \right\} \leq 1 \quad (7c')$$

For any μ the r.h.s. integral in (7b) exists, because $|F| \frac{\partial f}{\partial |F|} > f$ (from the fact that $\frac{1}{|F|} f$ is an increasing function of $|F|$); also the integral $\int \ln F_{\text{extr}}(\theta) d\theta$ in (7c') exists [comp. condition (g)].

Next, it is evident that for $|F| = F_{\text{extr}}$ the equality sign in (7c') holds. Because, on the other hand, $F_{\text{extr}}(\theta, A)$ satisfies Eq. (1) [comp. (7b)] and may be expressed - in virtue of condition g) - as the modulus of an analytic function satisfying conditions a) - e), it is impossible to make the r.h.s. of (7c') smaller than unity.

From Jensen's ⁸⁾ theorem one has $(1/r_1^2 \dots r_n^2) \geq 1$. Further, if for $\beta < \gamma < 1$ $\text{Im } Fz = (y+i\epsilon) > 0$ we may write instead of (7c')

$$|F_0| \exp \left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln F_{\text{extr}}(\theta) d\theta \right\} \leq 1/\beta \quad (7c)$$

Now let us come to the problems of uniqueness and existence of $F_{\text{extr}}(\theta)$. As regards the first, it is easy to prove the following statement:

if for a given A there exist $F_{\text{extr}}(\theta)$ and μ , satisfying (7a,b) then they are unambiguously determined by these equations.

When μ is increased, $F_{\text{extr}}(\theta)$ gets larger for any θ because of Eq. (7a) and condition f). It means that $F_{\text{extr}}(\theta)$ is uniquely determined by μ . Next, increased $F_{\text{extr}}(\theta)$ makes the value of A larger [comp. (7b) and f)] which means that there is only one μ , $F_{\text{extr}}(\theta)$ for given A , which proves our statement.

Equation (7a) has always a solution; the danger at points $\theta = 0, \pi$ ($x = 1, \infty$) is avoided because of condition g).

Next, we want to point out the following inequality resulting from Eq. (1) and condition f):

$$A = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\vartheta, |F|) d\vartheta > \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\vartheta, 0) d\vartheta$$

It results from (7a) that the above condition is equivalent to:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\vartheta, F_{\text{extr}}) d\vartheta < A \quad \text{for } \mu = 0 \quad (F_{\text{extr}} = 0 \quad \text{for } \mu = 0)$$

Now

$$I = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\vartheta, F_{\text{extr}}) d\vartheta$$

is an increasing function of μ . Making μ sufficiently large it is always possible to reach the value A for I .

We have proved then that there always exists one and only one solution of our equations for $F_{\text{extr}}(\vartheta)$. What we do know about A is $A \leq M$, M being a given constant. However, because $F_{\text{extr}}(\vartheta)$ increases with A [it results from Eqs. (7a,b) and condition f)], the maximal value of F_0 is obtained when one puts the maximal value of A in (7c), i.e., $A = M$. This means that we have only to solve (7a,b,c) for $A = M$ and the obtained F_0 will be a maximal one satisfying Eq. (1).

II. In cases of physical interest one is not given the value of M but usually an inequality of the type $A \leq M = F_0$. This leads to the same equations as (7a,b,c) with A replaced by F_0 :

$$F_{\text{extr}}(\theta) \cdot \frac{\partial f(\theta, F_{\text{extr}})}{\partial F_{\text{extr}}} = \mu(F_0) = \text{const} \quad (8a)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta, F_{\text{extr}}(\theta)) d\theta = F_0 \quad (8b)$$

$$F_0 \exp \left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln F_{\text{extr}}(\theta) d\theta \right\} \leq \frac{1}{\beta} \quad (8c)$$

The condition for F_0 is contained now implicitly in Eqs. (8a,b,c) and it is difficult to see what values of F_0 will be forbidden.

We shall prove now that Eqs. (8a,b,c) give a maximum for F_0 . In the first we want to show that the l.h.s. of inequality (8c) increases when $F_{\text{extr}}(\theta)$ increases. To this end we have to show that the functional derivative of the l.h.s. with respect to $F_{\text{extr}}(\theta)$ is positive:

$$\exp \left\{ -\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln F_{\text{extr}} d\theta \right\} \left[\frac{\delta F_0}{\delta F_{\text{extr}}(\theta)} + F_0 \frac{\delta}{\delta F_{\text{extr}}(\theta)} \left(-\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln F_{\text{extr}}(\theta) d\theta \right) \right] \geq 0$$

This, after using (8b) and dropping the "exp" term, is equivalent to:

$$\frac{\partial f(\theta, F_{\text{extr}})}{\partial F_{\text{extr}}} - F_0 \cdot \frac{1}{F_{\text{extr}}} \geq 0$$

With the help of (8a) it gives:

$$\frac{1}{F_{\text{extr}}} \left\{ \mu(F_0) - F_0 \right\} \geq 0$$

Now, $\mu(F_0) - F_0 = - \Psi[F_{\text{extr}}] \overline{\text{comp. (4a), (8b)}}$ and is larger than zero because $\frac{1}{|F|} \frac{\partial f}{\partial |F|}$ increases with $|F|$ [condition f)]. This proves that the l.h.s. of (8c) is increasing with $F_{\text{extr}}(\theta)$. But $F_{\text{extr}}(\theta)$ itself increases (for any θ) when F_0 is made larger [as seen from Eqs. (8a,b)]. So we proved that really the l.h.s. of (8c) is growing with F_0 , which simultaneously gives our statement. We also want to make this remark that the obtained maximum will be larger than zero if

$$\lim_{|F| \rightarrow 0} \frac{\partial f(\theta, |F|)}{\partial |F|} = 0$$

In the case of finite limit, unless

$$\beta \cdot \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \frac{\partial f(\theta, |F|)}{\partial |F|} d\theta \right\}_{|F| \rightarrow 0} < 1$$

we obtain $F_0 = 0$ or no F_0 satisfying (8a,b,c).

In cases of physical interest, where $f(\theta, |F|) \sim |F|^2$ for small $|F|$, the above mentioned condition is of course fulfilled and always $F_0 > 0$.

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A P P E N D I X

As an illustration let us consider a functional of the form:

$$\phi[F] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{ p_1(\theta)|F| + p_2(\theta)|F|^2 \} d\theta = A \quad (\text{A.1})$$

Assume that $F(z)$ has no cut apart from the integration region. What we want to obtain is a maximum for $F(z=0) = F_0$.

The first step is the replacement of $\phi[F]$ by a linear functional in F by means of minimizing the functional:

$$\psi[F] = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \{ p_1(\theta)|F| + p_2(\theta)|F|^2 - \lambda(\theta)|F| \} d\theta \quad (\text{A.2})$$

The extremum condition

$$\left. \frac{\delta \psi[F]}{\delta |F|} \right|_{|F| = F_{\text{extr}}} = 0$$

leads to

$$F_{\text{extr}}(\theta) = \frac{\lambda(\theta) - p_1(\theta)}{2 p_2(\theta)} \quad (\text{A.3})$$

Now, from $\Psi[F] \geq \Psi[F_{\text{extr}}]$ we have:

$$A = \Phi[F] \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda(\theta) |F| d\theta + \Psi[F_{\text{extr}}]$$

which, after using (A.3), leads to

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda(\theta) |F| d\theta \leq A + \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{(1-p_1)^2}{4 p_2} = \mu$$

Applying to the l.h.s. the theorem of the geometric and arithmetic means we come to

$$F_0 \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \lambda(\theta) d\theta\right\} \leq \mu \quad (\text{A.4})$$

Now we choose a $\lambda_{\text{extr}}(\theta)$ which makes the two sides of the inequality as close each to the other as possible. This means, one has to find the minimum of a functional

$$\chi[\lambda] = \frac{\mu}{F_0 \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \lambda d\theta\right\}}$$

or, which is equivalent, the minimum of

$$\ln \chi[\lambda] = \ln \mu - \ln F_0 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \lambda d\theta$$

with respect to $\lambda(\theta)$.

and from (A.4) using (A.3) and (A.7):

$$F_0 \exp \left[-\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln F_{\text{ext}} d\vartheta \right] \leq 1 \quad (\text{A.7''})$$

Equations (A.7), (A.7') and (A.7'') are just those which one could obtain directly from (7a,b,c) applied to functional (A.1) ($\beta = 1$).

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