EXTENSION OF THE AXIOMATIC ANALYTICITY DOMAIN OF
SCATTERING AMPLITUDES BY UNITARITY - II

A. Martin
CERN - Geneva

ABSTRACT

This paper, the second of a series on the subject, is entirely devoted to the pion-pion scattering amplitude. The main results are :

1) The scattering amplitude can be continued till the border-line of the double spectral function, except for $8\mu^2 < s < 32\mu^2$. Hence the nearest singularities are really induced by two-particle unitarity. Another consequence is that the only possible static potential describing low energy pion-pion scattering is a Yukawa superposition. The domain of validity of fixed transfer dispersion relations is slightly enlarged, as compared to I.

2) A partial analytic completion is carried out by various methods. As a result one finds a very large domain of analyticity for the fixed angle amplitude and the partial wave amplitudes; this domain extends from $s = -28\mu^2$ to $s = +\infty$. However, only in the interval $-28\mu^2 < \text{Re } s < 78\mu^2$ the extension in $\text{Im } s$ is appreciable ($|\text{Im } s|_{\text{max}} = 70\mu^2$).

3) The result of Bros, Epstein and Glaser on the validity of fixed, negative t quasi-dispersion relations is extended to any t inside the parabola with focus $t = 0$ and summit $t = \mu^2$.

I. INTRODUCTION

In a previous paper denoted hereafter by I ¹⁾, we have shown that the use of unitarity makes it possible to extend considerably the analyticity domain of scattering amplitudes deduced from axiomatic field theory (more precisely the Wightman axioms). The central result was that while the linear programme of field theory leads to the result that dispersion relations hold on a segment $-t_0 < t \leq 0$, the additional information that due to unitarity the absorptive part of the scattering amplitude has positivity properties allows to prove the validity of dispersion relations in a region $|t| < R$, including in particular $t = +R$. In the special case of pion-pion scattering, more careful use of the existing results of field theory, of positivity and unitarity, and crossing symmetry allows to make some further progress and in particular to prove dispersion relation when t is inside a certain domain $\mathcal{D}$ containing the real segment $t = -28 \mu^2$ to $t = 4 \mu^2$. Then one gets, using only crossing symmetry and unitarity, an analyticity domain which, for not too high energies contains part of the Mandelstam cuts, and also an analyticity domain for the partial wave amplitude.

What we want to do here is to make more full use of the unitarity condition and to exploit the possibilities of analytic completion. In Section II we prove that both at low energies ($4 \mu^2 < s < 16 \mu^2$) and at high energies ($s > 32 \mu^2$) the absorptive part of the scattering amplitude is analytic at least up to the border of the Mandelstam double spectral function. The low energy result was already obtained in I. The high energy result is not obvious from crossing symmetry as it would be if Mandelstam representation would hold. It is only by repeated use of unitarity that it can be obtained. As a consequence, the ellipse of analyticity of the absorptive part is enlarged. Therefore the intersection of all elliptic discs for energies running from $s = 4 \mu^2$ to $+\infty$ is enlarged and a somewhat bigger domain of validity of dispersion relations is obtained.

Section III begins by a new presentation of Mandelstam's method of analytic completion ²⁾, using analytic hypersurfaces of the type $|s-s_0||t-t_0| = C$. It is then possible, using the results of Section I, to prove that the scattering amplitude $F(s,t)$ can be continued till the border of the boundary curve of Mandelstam's double spectral function for $4\mu^2 < s < 8\mu^2$ and $32\mu^2 < s < \infty$. As a consequence, the analyticity domain of the absorptive part for $4\mu^2 < s < 8\mu^2$ is further enlarged. Another consequence is that as $s - 4\mu^2$ approaches 0 by positive values, the analyticity domain in t tends to cover the whole complex t plane (minus cuts) so that we can conclude that if low energy pion-pion scattering can be described by a static potential, this static potential has to be a Yukawa superposition. This, to our knowledge, is the only good argument in favour of this family of potentials (previously people used to say: we want Yukawa superpositions to have Mandelstam representation, or since we have Mandelstam representation we necessarily have Yukawa superpositions).

Section IV deals again with analytic completion, using partly Mandelstam's method ²⁾ and partly the tube theorem applied to the regions $\text{Im } s, \text{Im } t > 0$ and permutations. This allows a rather impressive enlargement of the analyticity domain of partial wave amplitudes. However, the domain we obtain is not the final one. Using information on the t -dependence of the analyticity domain for fixed negative s ³⁾ one proves that the partial wave amplitudes can be continued up to $s = +\infty$. However, the thickness of the domain in $\text{Im } s$ shrinks very rapidly as $\text{Re } s$ increases, and I happen to know that Bros and Glaser are in the process of getting new results on the analyticity domain for arbitrary negative transfers. These results will allow to calculate a bigger domain for partial wave amplitudes.

Finally, in Section V, we present a qualitative result which is the direct consequence of the work of Bros, Epstein and Glaser ³⁾ on the analyticity for real negative transfers. It is shown that the scattering amplitude is analytic in a cut plane minus a finite region whenever t is inside a parabola with focus $t = 0$ and summit $t = \mu^2$.

Section VI contains concluding remarks concerning the question of the validity of Mandelstam representation. It is the personal opinion of the author that the chances that Mandelstam representation is right are very high, especially if one accepts to add to all what has been obtained here the results which survive from the attempts at proving Mandelstam representation in perturbation theory.

We recommend to the reader who is uninterested in mathematical details to look first at the figures.

II. ANALYTICITY OF THE ABSORPTIVE PARTS AND CONSEQUENCES ON THE FIXED TRANSFER DISPERSION RELATIONS

In I we have seen that in the elastic pion-pion region $4\mu^2 < s < 16\mu^2$ (this holds for all charge states of the pion-pion system) the scattering amplitude $F(s, t, u)$ is analytic in a region which certainly contains the ellipse with foci

$$t = 0 \quad t = 4 - s$$

and with extremities

$$t = 4 \quad t = -s$$

By using unitarity in this region (summing over the various admissible charge states of the intermediate two-pion state), it is easy to see that the absorptive part $A_s(s, t)$ is analytic inside the ellipse with foci

$$t = 0 \quad t = 4 - s$$

and extremities

$$t = 16 + \frac{64}{s-4} \quad t = 4 - s - 16 - \frac{64}{s-4} \quad (1)$$

Equations (1) are just the equation of the border-line of Mandelstam's double spectral function in $4 < s < 16$.

4.

Now for $16 < s < 64$ we only have at our disposal the result of Lehmann, which is that the extremities of the ellipse are

$$t = \frac{256}{s} \quad t = 4 - s - \frac{256}{s} \quad (2)$$

and for $s > 64$ the extremities, as a consequence of I, are

$$t = 4 \quad \text{and} \quad t = -s \quad (3)$$

However, if the Mandelstam representation was true, we could use the fact that if around some point s_0, t_0 of the real s, t plane the absorptive part $A_s(s, t)$ is analytic in t then the double spectral function vanishes and hence the absorptive part in the t channel $A_t(s, t)$ is analytic in s in the same region. The existing information concerns $4 < s < 16$ and $4 < t < \infty$. Then by crossing symmetry, we could enlarge the analyticity domain of $A_s(s, t)$ for $s > 16$.

What we want to do here is to find a substitute to prove a similar result without knowing in advance the validity of Mandelstam representation. The main tool, as will appear soon, is again the positivity properties of $A_s(s, t)$ and its derivatives with respect to t .

To exploit crossing symmetry we write dispersion relations in two ways, for s and t inside the triangle $s < 4, t < 4, u < 4$, restricting ourselves for the moment to the $\pi^0 \pi^0 \rightarrow \pi^0 \pi^0$ scattering amplitude (the true one, not the one of neutral pseudoscalar theory) :

$$\begin{aligned} F(s, t, u) &= \frac{1}{\pi} \int_4^\infty \frac{A(x, t)}{x - s} dx + \frac{1}{\pi} \int_4^\infty \frac{A(x, t)}{x - u} dx \\ &= \frac{1}{\pi} \int_4^\infty \frac{A(x, s)}{x - t} dx + \frac{1}{\pi} \int_4^\infty \frac{A(x, s)}{x - u} dx \end{aligned} \quad (4)$$

We have omitted subtractions, but this is legitimate, as we shall consider now the object $(d/ds)^n(d/dt)^n F(s,t)$ (taking s and t as independent variables). That such an expression can be obtained by differentiating under the integral can be justified along the same lines as in I. We therefore get :

$$\begin{aligned}
 & n! \frac{1}{\pi} \int_4^{\infty} \frac{\left(\frac{d}{dt}\right)^n A(x,t)}{(x-s)^{n+1}} dx \\
 & + \left(\frac{d}{ds}\right)^n \left(\frac{d}{dt}\right)^n \frac{1}{\pi} \int_4^{\infty} \frac{A(x,t) - A(x,s)}{x-u} dx \\
 & = n! \frac{1}{\pi} \int_4^{\infty} \frac{\left(\frac{d}{ds}\right)^n A(x,s)}{(x-t)^{n+1}} dx
 \end{aligned} \tag{5}$$

Now we shall take $0 < s < 4$ and $t = 0$ and from the existing information on the left-hand side try to get an upper bound on the right-hand side integral, which, due to the positivity of the derivatives of $A(x,s)$ for $0 < s < 4$ is positive. First we shall have to get rid of the left-hand cut contributions. We shall prove the following, for $t = 0$, $0 \leq s < 4$

$$\phi(x,s,t) = (-)^n \left(\frac{d}{ds}\right)^n \left(\frac{d}{dt}\right)^n \frac{A(x,t) - A(x,s)}{x-u} \tag{6}$$

is negative. We have

$$\phi(x,s,0) = n! \left[\left(\frac{d}{dt}\right)^n \frac{A(x,t)}{(x-4+s+t)^{n+1}} \right]_{t=0} - \left(\frac{d}{ds}\right)^n \frac{A(x,s)}{(x-4+s)^{n+1}}$$

6.

Now, by crossing symmetry

$$A(x, t) = A(x, 4-x-t)$$

putting $4-x-t = \sigma$, we transform ϕ into

$$\begin{aligned} \phi &= -n! \left[\left(\frac{d}{d\sigma} \right)^n \frac{A(x, \sigma)}{(\sigma-s)^{n+1}} \right]_{\sigma=4-x} + \left(\frac{d}{ds} \right)^n \frac{A(x, s)}{(x-4+s)^{n+1}} \\ &= -\frac{(n!)^2}{2\pi i} \left[\int_{\gamma_1} \frac{A(x, \zeta) d\zeta}{(\zeta-s)^{n+1} (\zeta+x-4)^{n+1}} + \int_{\gamma_2} \frac{A(x, \zeta) d\zeta}{(\zeta-s)^{n+1} (\zeta+x-4)^{n+1}} \right] \end{aligned}$$

where γ_1 surrounds $\zeta = 4-x$ and γ_2 surrounds $\zeta = s$.

This expression can be replaced by a unique contour integral containing $\zeta = 4-x$ and $\zeta = s$

$$\phi = -\frac{(n!)^2}{2\pi i} \int_{\Gamma} \frac{A(x, \zeta) d\zeta}{(\zeta-s)^{n+1} (\zeta+x-4)^{n+1}} \quad (7)$$

The contour Γ lies entirely inside the ellipse of analyticity of $A(x, \zeta)$. Now

$$A(x, \zeta) = \sum c_e P_e \left(1 + \frac{2\zeta}{x-4} \right)$$

where, from unitarity, $c_e(x)$ is positive.

To prove that ϕ is negative it is therefore sufficient to prove that

$$\frac{n!}{2\pi i} \int_{\Gamma} \frac{P_e \left(1 + \frac{2\zeta}{x-4} \right) d\zeta}{(\zeta-s)^{n+1} (\zeta+x-4)^{n+1}} \quad (8)$$

is positive.

We notice first of all that (8) is zero for $\ell \leq 2n$ because then the contour could be deformed to infinity. We can use the following expansion for P_ℓ (ℓ even only occurs due to crossing)

$$P_\ell(y) = \text{const} \prod (y^2 - y_i^2) = \text{const} \prod (y^2 - 1 + 1 - y_i^2) = \sum C_{p\ell} (y^2 - 1)^p$$

where, clearly, the $C_{p\ell}$ are positive since $|y_i| < 1$. Hence it is enough to prove that an individual term of the expansion of P_ℓ gives a positive contribution to (8). Apart from factors depending on x an individual term will be

$$\frac{n!}{2\pi i} \int \frac{(x-4+\zeta)^p \zeta^p}{(\zeta-s)^{n+1} (\zeta+x-4)^{n+1}} d\zeta \quad (9)$$

Two cases occur :

$\alpha)$ if $p \leq n$ this integral is zero:

$\beta)$ if $p \geq n+1$ the integrand has no singularity at $\zeta = 4-x$ and (9) is just

$$\left(\frac{d}{d\zeta}\right)^n \left[(x-4+\zeta)^{p-n-1} \zeta^p \right]_{\zeta=s} \quad (10)$$

which, since $x > 4$ and $s > 0$, is obviously a sum of positive terms.

Going backwards we conclude

$$\frac{1}{\pi} \int_4^\infty \frac{\left(\frac{d}{ds}\right)^n A(x,s)}{x^{n+1}} dx < \frac{1}{\pi} \int_4^\infty \frac{\left(\frac{d}{dt}\right)^n A(x,0)}{(x-s)^{n+1}} dx \quad (11)$$

for $0 \leq s < 4$ for n even, and the converse for n odd.

8.

Now we shall majorize the right-hand side integral in (11) using the existing information (1), (2), (3) on the analyticity domain of $A(x, t)$. We know that if $A(x, t)$ is analytic in a circle $R(x)$ contained in the ellipses, we have

$$\left(\frac{d}{dt}\right)^n A(x, 0) < \frac{n!}{[R(x)]^n} A(x, R(x))$$

We have, from (1), (2) and (3), restricting ourselves to even n :

$$\begin{aligned} & \frac{1}{\pi} \int_4^{\infty} \frac{\left(\frac{d}{ds}\right)^n A(x, s)}{x^{n+1}} dx < \\ & \frac{n!}{\pi} \int_4^{16} \frac{A(x, 16 + \frac{64}{x-4})}{(x-s) \left[(x-s) \left(16 + \frac{64}{x-4}\right)\right]^n} dx \\ & + \frac{n!}{\pi} \int_{16}^{204} \frac{A(x, \frac{256}{x})}{(x-s) \left[(x-s) \frac{256}{x}\right]^n} dx \\ & + \frac{n!}{\pi} \int_{204}^{\infty} \frac{A(x, 2)}{(x-s)^3 4 \left[(x-s) 2\right]^{n-2}} dx \\ & < \frac{n!}{\left[16(2 + \sqrt{4-s})^2\right]^n} \frac{1}{\pi} \int_4^{16} \frac{A(x, 16 + \frac{64}{x-4})}{x-s} dx \\ & + \frac{n!}{\left[16(16-s)\right]^n} \frac{1}{\pi} \int_{16}^{204} \frac{A(x, \frac{256}{x})}{x-s} dx \\ & + \frac{n!}{4x(400)^{n-2}} \frac{1}{\pi} \int_{204}^{\infty} \frac{A(x, 2)}{(x-s)^3} dx \end{aligned} \tag{12}$$

Notice that according to a theorem by Jin and the author, the last integral is convergent. It is easy to see that for

$$0 \leq s \leq 4$$

$$400 > 16(16-s) \geq 16[2+\sqrt{4-s}]^2$$

Hence we get finally :

$$\int_4^{\infty} \frac{\left(\frac{d}{ds}\right)^n A(x, s_0)}{x^n} dx < C \frac{n!}{(16(2+\sqrt{4-s_0}))^{n-2}} \quad (13)$$

for $0 \leq s_0 < 4$, n even.

One could argue that in (12) we should have majorized the derivatives by values of A taken inside the domain. This can be done by putting in some epsilons but alter as little as one wishes the result (13). From (13) we conclude that

$$\int_{x_1}^{x_2} \phi(x) \left(\frac{d}{ds}\right)^n A(x, s_0) dx < C \frac{(x_2)^n n! \sup(\phi(x))}{[16[2+\sqrt{4-s_0}]^2]^{n-2}}$$

n even, where $\phi_{x_1, x_2}(x)$ is a continuous positive function with compact support x_1, x_2 . Hence if we define

$$A(\phi_{x_1, x_2}, s) = \int_{x_1, x_2} \phi(x) A(x, s) dx$$

we have for even n :

$$\left(\frac{d}{ds}\right)^n A(\phi_{x_1, x_2}, s_0) \leq C \frac{\sup \phi(x) x_2^n n!}{[16(2+\sqrt{4-s_0})^2]^{n-2}}$$

The odd derivatives can be majorized as follows :

$$\left(\frac{d}{ds}\right)^{n-1} A(\phi_{x_1, x_2}, s_0) < s_0 \left(\frac{d}{ds}\right)^n A(\phi_{x_1, x_2}, s_0) + \left(\frac{d}{ds}\right)^{n-1} A(\phi_{x_1, x_2}, 0)$$

since for $0 \leq s \leq s_0$, $(d/ds)^n A(\phi_{x_1, x_2}, s_0)$ is a positive increasing function. Now we have

$$\left(\frac{d}{ds}\right)^n A(x, 0) < n! \frac{A(x, \frac{256}{x}) x^n}{(256)^n}$$

and hence

$$\left(\frac{d}{ds}\right)^{n-1} A(\phi_{x_1, x_2}, s_0) < x_2^{n-1} (n-1)! \left[\frac{C s_0 \sup(\phi) x_2^n}{[16(2+\sqrt{4-s_0})]^2} + \frac{\int_{x_1}^{x_2} \phi(x) A(x, \frac{256}{x}) dx}{(256)^{n-1}} \right].$$

Since $256 > 16[2+\sqrt{4-s_0}]^2$, we get, using the fact that $A(x, \frac{256}{x})$ is a measure, for both odd and even n :

$$\left(\frac{d}{ds}\right)^n A(\phi_{x_1, x_2}, s_0) < \frac{x_2^n n! (n+1)}{[16(2+\sqrt{4-s_0})]^2} n-2 C_{x_1, x_2} \sup(\phi(x))$$

where C_{x_1, x_2} is bounded for x_1 and x_2 finite.

Hence $A(x, s)$ is a measure in x analytic in s in :

$$|s - s_0| x < 16 [2 + \sqrt{4 - s_0}]^2 \quad (14)$$

By crossing, $A_s(s, t)$ is analytic in t in

$$|t - t_0| \leq 16 [2 + \sqrt{4 - t_0}]^2 \quad (15)$$

for $0 \leq t_0 < 4$.

However, the positivity properties of $A_s(s, t)$ allow to make the stronger statement : $A_s(s, t)$ is analytic in

$$|t| < t_0 + \frac{16}{s} [2 + \sqrt{4 - t_0}]^2 \quad (16)$$

for any $0 \leq t_0 < 4$.

The optimum t_0 turns out to be $t_0 = 0$ for $s < 32$, which gives a domain $|t| < \frac{256}{s}$, which is nothing but the Lehmann result $t_0 = 4 - (32/(s-16))^2$ for $s > 32$, which gives

$$|t| < 4 + \frac{64}{s-16} \quad (17)$$

i.e., exactly the Mandelstam equation for the boundary of the double spectral function. The region (17) can be, of course, replaced by an ellipse with foci $t = 0$, $t = 4 - s$ and right extremity $t = 4 + (64/(s-16))$. Figure 1 represents the real section of the analyticity domain of the absorptive part in the s channel.

So far this proof is only valid for the $\pi^0 \pi^0 \rightarrow \pi^0 \pi^0$ amplitude. However, it is clear that if $A_s^{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0}(s, t)$ is analytic in a certain ellipse in the t plane, $A_s^{I=0}(s, t)$ and $A_s^{I=2}(s, t)$ will be analytic in the same ellipse since from

$$\mathcal{M}_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0} = \frac{1}{3} \mathcal{M}_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0}^{I=0} + \frac{2}{3} \mathcal{M}_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0}^{I=2}$$

we have

$$\begin{aligned} 0 &\leq \mathcal{M}_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0}^{I=0} \leq 3 \mathcal{M}_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0} \\ 0 &\leq \mathcal{M}_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0}^{I=2} \leq \frac{3}{2} \mathcal{M}_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0} \end{aligned}$$

It means that the same result holds for $\pi^+\pi^+ \rightarrow \pi^+\pi^+$ and $\pi^+\pi^- \rightarrow \pi^0\pi^0$. To get the $\pi^+\pi^- \rightarrow \pi^+\pi^-$ analyticity domain for the absorptive part, one can start from

$$\begin{aligned} \frac{1}{\pi} \int \frac{A^{\pi^+\pi^- \rightarrow \pi^+\pi^-}(x, t)}{x-s} dx + \frac{1}{\pi} \int \frac{A^{\pi^+\pi^+ \rightarrow \pi^+\pi^+}(x, t) - A^{\pi^+\pi^+ \rightarrow \pi^+\pi^+}(x, s)}{x-u} dx \\ = \frac{1}{\pi} \int \frac{A^{\pi^+\pi^- \rightarrow \pi^+\pi^-}(x, s)}{x-t} dx \end{aligned} \quad (18)$$

and carry through the whole argument. This is possible because of the symmetries of the integrand in the integral over the u cut.

A first, obvious, consequence of the enlargement of the ellipses of analyticity of the absorptive part is that the intersection of all elliptic discs of analyticity of $A_s(s, t)$, for $4 \leq s < \infty$ will become bigger. This intersection $\mathcal{D}'$ now replaces the domain $\mathcal{D}$ of validity of fixed t dispersion relations. What may be considered of some interest is that part of the boundary of $\mathcal{D}'$ is given by $A_s(\infty, t)$, i.e., it is an arc of parabola of focus $t = 0$ and extremity $t = 4\mu^2$. The domain has been computed on the CERN 3400 CDC by Mr. W. Klein and is represented on Fig. 2.

III. ANALYTICITY OF THE AMPLITUDE IN THE NEIGHBOURHOOD OF THE DOUBLE SPECTRAL FUNCTION BOUNDARY BY THE MANDELSTAM METHOD

We want to exploit the information obtained in Section II to prove that the scattering amplitude itself (and not only the absorptive parts) can be continued through a complex path till the line $t = 16 + (64/(s-4))$ for $4 \leq s \leq 8$ and correspondingly the line $t = 4 + (64/(s-16))$ for $s > 32$, and, in between, to the line $t = 256/s$ for $8 \leq s \leq 32$. So, apart from the region $8 \leq s \leq 32$, we can continue the amplitude till the border of the double spectral function. First we shall consider the $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ case.

To prove this result, we shall use a technique due to Mandelstam ²⁾ which we want to present here in a somewhat different way. Consider an analytic hypersurface $|s-\lambda||t-\lambda| < C$, where λ has been chosen inside the domain $\mathcal{D}'$ of validity of fixed transfer dispersion relations, for instance $-28 < \lambda < 4$. C is fixed by the requirement that the fixed $s > 4\mu^2$, $t > 4\mu^2$, $u > 4\mu^2$ sections of the hypersurface are inside the corresponding ellipses for the absorptive parts. Let us define :

$$\begin{aligned} \phi(s, t, u) = & \frac{s^2}{\pi} \int_4^\infty \frac{A(x, \lambda + \frac{(s-\lambda)(t-\lambda)}{x-\lambda})}{x^2(x-s)} dx \\ & + \frac{t^2}{\pi} \int_4^\infty \frac{A(x, \lambda + \frac{(s-\lambda)(t-\lambda)}{x-\lambda})}{x^2(x-t)} dx \\ & + \frac{u^2}{2\pi} \int_4^\infty \frac{A(x, t_1(x, s, t)) + A(x, t_2(x, s, t))}{x^2(x-u)} dx \end{aligned} \quad (19)$$

where t_1 and t_2 are the two solutions of the equation

$$(4-x-t(x)-\lambda)(t(x)-\lambda) = (s-\lambda)(t-\lambda) \quad (20)$$

First we notice that if we have

$$\begin{aligned} |s-\lambda||t-\lambda| & < C \\ A(x, \lambda + \frac{(s-\lambda)(t-\lambda)}{x-\lambda}) \end{aligned}$$

is analytic, for fixed x in s and t . Indeed if we call

$$t' = \lambda + \frac{(s-\lambda)(t-\lambda)}{x-\lambda}$$

we have

$$|t'-\lambda||x-\lambda| = |s-\lambda||t-\lambda| < C$$

Similarly, $A(x, t_1(x, s, t))$ and $A(x, t_2(x, s, t))$ are, by Eq. (20), separately analytic in s and t for $|s-\lambda| |t-\lambda| < C$ apart from the branch point appearing in the explicit solution of Eq. (20). However, the sum $A(x, t_1) + A(x, t_2)$ is a symmetric function of t_1 and t_2 and has, therefore no branch point.

Finally, the discontinuity of $\emptyset$ across the s cut is just $A_s(s, t, u)$, across the t cut $A_t(s, t, u)$, across the u cut $\frac{1}{2}[A_u(s, t, u) + A_u(t, s, u)]$ which, by crossing symmetry of the $\pi^0 \pi^0$ amplitude, is just $A_u(s, t, u)$.

The convergence of the integrals in $\emptyset$ is guaranteed by the fact that asymptotically the arguments appearing in the absorptive parts tend to values which are inside the analyticity domain.

Now consider the difference $F(s, t, u) - \emptyset(s, t, u)$. In this difference the $s > 4\mu^2$, $t > 4\mu^2$, $u > 4\mu^2$ cuts have disappeared. Therefore this difference is certainly analytic in

$$[|s-\lambda| < \epsilon \cup |t-\lambda| < \epsilon] \cap |s-\lambda| |t-\lambda| < C \quad (21)$$

where ϵ is taken small enough so that $|s-\lambda| = \epsilon$ is inside the domain $\mathcal{D}'$. If we use new variables $\bar{s} = \log(s-\lambda)$, $\bar{t} = \log(t-\lambda)$ the domain (21) becomes

$$[\operatorname{Re} \bar{t} < \log \epsilon \cup \operatorname{Re} \bar{s} < \log \epsilon] \cap [\operatorname{Re} \bar{t} + \operatorname{Re} \bar{s} < \log C] \quad (22)$$

This is a tube, the holomorphy envelope of which should have a convex base just given by

$$\operatorname{Re} \bar{t} + \operatorname{Re} \bar{s} < \log C \quad (23)$$

Hence $\phi(s,t,u)-F(s,t,u)$ is analytic in $|s-\lambda||t-\lambda| < C$ and since $\phi(s,t,u)$ is analytic in $|s-\lambda||t-\lambda| < C$ minus the cuts, we conclude that $F(s,t,u)$ is analytic in

$$|s-\lambda||t-\lambda| < C \text{ minus the cuts} \quad (24)$$

$$s \text{ real} > 4, \quad t \text{ real} > 4, \quad u \text{ real} > 4.$$

This simplified derivation of the Mandelstam rigorous domain is possible, because, contrary to what was the case in Mandelstam's work, we started from an initial domain which was not "flat" in one variable.

Now we shall take $0 < \lambda < 4$ and adjust C so that the curve $(s-\lambda)(t-\lambda) = C$ is tangent to the curve made of the three arcs

$$\left. \begin{aligned} (s-4)(t-16) &= 64 \\ 4 < s < 8 \\ st &= 256 \\ 8 < s < 32 \\ (s-16)(t-4) &= 64 \\ 32 < s < \infty \end{aligned} \right\} \quad (25)$$

If such a curve is obtained, it is obvious from Section II that $|t-\lambda| < (C_\lambda/s-\lambda)$ is inside the analyticity domain of A_s for $s > 4$, and that $|s-\lambda| < (C_\lambda/t-\lambda)$ is inside the analyticity domain of A_t for $t > 4$. The only thing which is not completely obvious is that $|s-\lambda||t-\lambda| < C_\lambda$ for $u \text{ real} > 4$ is inside the analyticity domain of A_u .

However, it is easy to see that the real sections of $|s-\lambda||t-\lambda| < C_\lambda$ are inside the analyticity domain of A_u . Indeed, the section is given by $|u+t-4+\lambda||t-\lambda| < C_\lambda$. For $\text{Re } t > 4-2\lambda$ this domain is contained in $|t-\lambda||u-\lambda| < C_\lambda$. Similarly, for $\text{Re } s > 4-2\lambda$ this domain is contained in $|s-\lambda||u-\lambda| < C_\lambda$.

We remark now that the ellipse of analyticity of A_u , with foci at $s = 0$ and $t = 0$ contains the ellipse with foci at $t = \lambda > 0$ and $s = \lambda > 0$ which has the same extremities. Along this ellipse the sum $|s - \lambda| + |t - \lambda|$ is constant. Along this ellipse the product $|s - \lambda||t - \lambda|$ is always larger than its value at the extremities of the major axis. Therefore since the extremities of our domain are in between the extremities of the ellipse of analyticity of A_u , the domain is entirely contained in A_u .

Now it is a trivial matter (at least for a Frenchman who studied before Bourbakism penetrated undergraduate courses) to find the family of domains tangent to (25). It is

$$\left| s - 16 \frac{16 - (s_0 - 4)^2}{64 - (s_0 - 4)^2} \right| \left| t - 16 \frac{16 - (s_0 - 4)^2}{64 - (s_0 - 4)^2} \right| < 64 \left[\frac{3 s_0^2}{64 - (s_0 - 4)^2} + 4 \left(\frac{16 - (s_0 - 4)^2}{64 - (s_0 - 4)^2} \right)^2 \right] \quad (26)$$

where s_0 varies from $s_0 = 4$ to $s_0 = 8$. For $s_0 = 8$ the domain reduces to $|st| < 256$. It is in contact with (25) for $8 < s < 32$. For $s_0 = 4$ it is $|(s-4)(t-4)| \leq 64$ which has a triple intersection at $s = 4$ and $s = \infty$ with (25). The points of contact with (25) are given by

$$\begin{cases} s = s_0, & t = 16 + \frac{64}{s_0 - 4} \\ s = 16 + \frac{64}{s_0 - 4}, & t = s_0 \end{cases}.$$

It is easy to check that

$$\lambda = 16 \frac{16 - (s_0 - 4)^2}{64 - (s_0 - 4)^2}$$

varies between 0 and 4.

Therefore one can analytically continue the scattering amplitude to any point of the boundary of the double spectral function, $s_0 t_0$, for $4 < s_0 \leq 8$ and $32 \leq s_0 < \infty$. Figure 3 represents the points of the real s, t plane where the amplitude can be continued through a complex path. It is not clear whether there is a basic obstacle to continue to the boundary for $8 \leq s_0 \leq 16$ and for $64/3 \leq s_0 \leq 32$. It is clear, however, that in this approach, where only positivity and elastic unitarity is used, there is no hope to reach points inside the corner $16 < s < 64/3$, $16 < t < 64/3$. Even in the framework of Mandelstam representation, there is no good argument to prove that the double spectral function vanishes in this region which is neither in the s elastic strip nor in the t elastic strip. Now since we could construct a Mandelstam representation with a positive double spectral function decreasing fast enough at infinity which would satisfy all the requirements of positivity and crossing and be different from zero in this corner, it is clear that we cannot remove it ^{*)}.

The existence of the domains (26) has an important implication on the low energy behaviour of the scattering amplitude. Consider the domains

$$|s-4||t-4| < 64 \quad |s-4||u-4| < 64 \quad (27)$$

These are two circles whose radius tends to infinity as s approaches 4 by positive values. Therefore it is a good very low energy approximation to consider that the $\pi^0 \pi^0 \rightarrow \pi^0 \pi^0$ scattering amplitude is analytic in a cut plane in the $\cos\theta$ or t variable. There is clearly a continuous transition between $s = 4 + \epsilon$ where we have this very large region of analyticity and $s = 4 - \epsilon$ where we have, according to I, the full cut plane. If a static potential description of pion-pion scattering at very low energies is permissible (as it seems to be the case for nucleon-nucleon scattering) the only acceptable potentials will be those which give analyticity in the cut t plane. These are, and only are

*) I am indebted to Dr. Cornwall who in a different context drew my attention to this point.

Yukawa superpositions plus potentials decreasing faster than any exponential. However, the description should be just as good for s slightly less than $4\mu^2$, where - we know - we have a finite number of subtractions in t . This excludes the potentials decreasing faster than any exponential, and leaves only Yukawa superpositions. To our knowledge, this is the first relatively good argument in favour of Yukawa superpositions. Previously Yukawa superpositions were preferred only for two reasons :

- i) model calculations of potentials in the past gave rise to potentials of this family (but the theoretical value of these potentials is not clear):
- ii) they give Mandelstam representation.

Now we can recycle the information obtained on the analyticity domain of the amplitude and use elastic unitarity to enlarge further the analyticity domain of the absorptive part in the s channel for $4 < s < 16$, as was done in I. Let us just indicate that for $4 < s < 8$, $A_s(s, t)$ is analytic in a region of the complex t plane which contains the real segment

$$-t_M + 4 - s < t < t_M$$

minus the real cut

$$16 + \frac{64}{s-4} < t < t_M \quad (28)$$

and the symmetric cut, with

$$t_M = \frac{2}{(s-4)^2} (s-4+8)^3 - 2(s-4) - 12$$

This is an enormous region. For instance for $s = 5$, $t_M = 1444$, while the cut of $A_s(s, t)$ begins at $t = 80$, and the cut of $F(s, t)$ begins at $t = 4$.

These results have been obtained for the $\pi^0\pi^0 \rightarrow \pi^0\pi^0$ amplitude. It is easy to see, however, that what matters really, once one knows the analyticity domain of the various absorptive parts, is the symmetry of the scattering amplitude in the u channel. Therefore if we take the $\pi^+\pi^+$ amplitude in the u channel (corresponding to $\pi^+\pi^- \rightarrow \pi^+\pi^-$ in the s and t channels) we get the analyticity domains (26) (but not the permuted domains). Similarly, if we take the $\pi^0\pi^0 \rightarrow \pi^+\pi^-$ amplitude in the u channel (corresponding to $\pi^+\pi^0 \rightarrow \pi^+\pi^0$ in the s channel and $\pi^-\pi^0 \rightarrow \pi^-\pi^0$) we get the analyticity domain (26). We therefore get the domain (26) for three independent amplitudes and hence for any charge state of the two-pion system. Hence we can now say that the pion-pion amplitude, irrespective of the charges, is analytic in (26) and its permutations over s , t and u .

IV. FURTHER ANALYTIC CONTINUATION AND APPLICATION TO THE DOMAIN OF ANALYTICITY OF PARTIAL WAVES

In this Section three techniques will be used. One is again the Mandelstam method with domain $|u-\lambda||t-\lambda| \leq C$ where λ is no longer restricted to $\lambda > 0$. This technique has the advantage of eliminating the cuts and of allowing to get new real points of analyticity. The second will exploit the fact that when t is inside $D' \cap \text{Im } t > 0$, $F(s, t, u)$ is analytic in $\text{Im } u > 0$ and conversely, exchanging u and t . It will provide points of analyticity in the complex plane but no new real points. The last technique exploits the results of Glaser, Epstein and Bros³⁾ which are that given $t_0 < 0$ arbitrarily large in magnitude and given $s_0 > -t_0 + 4$ then for $t_0 \leq t \leq 0$ the scattering amplitude is analytic in s in $\emptyset_{s_0 t_0} \text{Arg}(s-s_0) > 0$.

IV.1 The Mandelstam method has been applied as follows : one requires that the real sections of $|u-\lambda||t-\lambda| < C$ lie inside the ellipses :

$$1) \quad 4 < s < 16$$

$$\text{foci } t = 0, \quad t = 4-s \quad \text{right extremity } t = 16 + \frac{64}{s-4}$$

$$2) \quad 16 < s < 32$$

$$\text{foci } t = 0, \quad t = 4-s \quad \text{right extremity } t = \frac{256}{s} \quad (29)$$

$$3) \quad 32 < s < \infty$$

$$\text{foci } t = 0, \quad t = 4-s \quad \text{right extremity } t = 4 + \frac{64}{s-16}$$

and similar requirements in the t and u channels. In the t and u channels, as we said in the last Section, the sections of the domain are circles. One determines the largest C by requiring :

- a) that the real extremities of the circle lie inside the ellipse;
- b) that the largest circle is smaller than the circle with the same centre tangent in complex points at the ellipse (here one must make sure that these points are not "complex" in the geometrical sense).

In the s channel the sections are Cassini ovals. One must impose the same requirements plus the additional requirement that the Cassini oval does not touch the extremities of the minor axis of the ellipse. Fortunately, for real λ all this can be done algebraically, by means of second degree equations, discriminants, etc. The results for $-26 < \lambda < 0$ are given in the Table. It would, of course, be very desirable to repeat this for any complex λ inside the region $\mathcal{D}'$, but we are then faced with a complex algebraic problem, and this probably does not help to get new real points of analyticity.

Now how can one use these results to get information on the analyticity domain of partial waves ? The simplest thing to do is to look at the analyticity properties of the 90° scattering amplitude. At $\cos \theta_s = 0$ we have $t = u = 4-s/2$. Hence, if we have a domain $|t-\lambda||u-\lambda| < C$ we find for the 90° scattering amplitude a domain

$$|s - 4 + 2\lambda| < 2\sqrt{C_2} \quad (30)$$

Now, for a given s inside (30) consider the straight line in the t plane connecting $t = \lambda$, $t = 4 - s - \lambda$ (i.e., $u = \lambda$). Any point on this complex path is parametrized by

$$\begin{aligned} t &= \alpha \lambda + (1 - \alpha)(\lambda + 4 - s) \\ u &= \alpha(-\lambda + 4 - s) + (1 - \alpha)\lambda \end{aligned} \quad (31)$$

So we have along this path

$$|t - \lambda|/|u - \lambda| = \alpha(1 - \alpha)|4 - s - 2\lambda|^2 \leq |4 - s - \lambda|^2$$

So that if s is inside the domain (30), the whole segment defined by (31) is inside the analyticity domain $|t - \lambda||u - \lambda| < C_2$. On the other hand, the segments $t = 0$, $t = \lambda + i\varepsilon$ (irrespective of s) and $u = 0$, $u = \lambda + i\varepsilon$ (irrespective of t) are inside the analyticity domain since they are inside the domain $\mathcal{D}'$. Therefore, whenever (30) is satisfied, we can connect by a complex path the point $t = 0$ and $u = 0$. The partial wave amplitudes are defined by

$$\frac{1}{s - 4} \int_{t=4-s}^0 P_e \left(1 + \frac{2t}{s-4} \right) F(s, t, u) dt$$

Starting from s real, we can deform the integration path as s becomes complex and use the complex path we have just obtained. Therefore the partial wave amplitudes are analytic in (30), and more generally in the union of all domains (30) obtained from the Table. This is a rather big domain which extends on the real axis from $s = -28$ to $s = -78$. It is somewhat disturbing, if one has in mind further analytic completion, that the complex path which we use to define partial waves is not the straight line connecting $t = 0$ and $u = 0$. The condition for this is

$|\lambda| |4-s-\lambda| < C_\lambda$. So if $|\frac{4-s}{2}-\lambda|^2 = C_\lambda$, a sufficient condition is $2|\lambda|\sqrt{C_\lambda} + \lambda^2 < C_\lambda$. It can be checked, from the Table, that this is fulfilled for $-6 \leq \lambda \leq 0$. For λ large negative the straight line connecting $t=0$ and $u=0$ is no longer entirely inside the domain. However, one can check by looking at sections of the domain (which, when $|\frac{s-4}{2} + \lambda|^2 = C_\lambda$ are Bernoulli lemniscates) the following :

- 1) for $\lambda = -20$, $|\text{Arg}(s-44)| < \frac{3\pi}{8}$ the segment connecting $t=0$ and $s=0$ is always either inside $|u-\lambda| |t-\lambda| < C_\lambda$, or such that t or u are in $\mathcal{D}'$;
- 2) for $\lambda = -22$, $|\text{Arg}(s-48)| < \frac{\pi}{4}$ the same happens.

After the following analytic completion is made, it will appear that this is really all what we need.

IV.2 Now we want to exploit the information that whenever t is in $\mathcal{D}'$, $F(s,t,u)$ is analytic in a cut plane in u and conversely. The difficulty is that one of the cuts is moving. However, if we restrict ourselves to t in $(\mathcal{D}' \cap \text{Im } t > 0)$ we can safely say that $F(s,t,u)$ is analytic when t is in this region and $\text{Im } u > 0$, and conversely. For simplicity, we can insert in $(\mathcal{D}'_t \cap \text{Im } t > 0)$ an arc of circle

$$\alpha < \text{Arg} \left(\frac{t-B}{t-A} \right) < \pi \quad (32)$$

and similarly in $(\mathcal{D}'_u \cap \text{Im } u > 0)$:

$$\alpha < \text{Arg} \left(\frac{u-B}{u-A} \right) < \pi \quad (33)$$

the domain becomes :

$$1) \alpha < \text{Arg} \left(\frac{t-B}{t-A} \right) < \pi, \quad 0 < \text{Arg} \left(\frac{u-B}{u-A} \right) < \pi$$

$$2) 0 < \text{Arg} \left(\frac{t-B}{t-A} \right) < \pi \quad \alpha < \text{Arg} \left(\frac{u-B}{u-A} \right) < \pi$$

If we use as variables $t' = \log\left(\frac{t-B}{t-A}\right)$ and $u' = \log\left(\frac{u-B}{u-A}\right)$ we see that this domain is a tube whose base should be convex. We therefore get as analytic completion the domain satisfying the simultaneous conditions

$$\begin{aligned}\alpha &< \operatorname{Arg}\left(\frac{t-B}{t-A}\right) + \operatorname{Arg}\left(\frac{u-B}{u-A}\right) \\ \operatorname{Arg}\left(\frac{t-B}{t-A}\right) &< \pi \\ \operatorname{Arg}\left(\frac{u-B}{u-A}\right) &< \pi\end{aligned}\tag{34}$$

Figure 4 illustrates the result. Here again the simplest thing to do is to find an analyticity domain for the scattering amplitude for $\cos\theta_s = 0$, i.e., $t = u = 4-s/2$, which gives as expressed in t

$$\begin{aligned}\frac{\alpha}{2} &< \operatorname{Arg}\left(\frac{t-B}{t-A}\right) < \pi \\ \text{or} \quad \frac{\alpha}{2} &< \operatorname{Arg}\left(\frac{4-s-2B}{4-s-2A}\right) < \pi\end{aligned}\tag{35}$$

i.e., that portion of a certain circle which lies in $\operatorname{Im}s < 0$ (in $\operatorname{Im}s > 0$ we find of course a symmetric region). The gain, as compared to what we get by making the brutal union of (32) and (33) is that α is replaced by $\alpha/2$, i.e., we fill a much bigger region of the s plane. Now arises again the question whether :

- 1) for any s in (33) can we connect $t = 0$ and $u = 0$ by a complex path to define partial wave amplitude ?
- 2) can this path be a straight line ?

It can easily be checked by geometric methods that if $t = 0$ lies between A and B , i.e., $A \leq 0 \leq B$, both conditions are fulfilled, i.e., that when s is in the domain (35), $F(s, \cos\theta_s)$ is analytic in the neighbourhood of $-1 \leq \cos\theta_s \leq +1$ and hence, $f_e(s)$, the partial wave amplitude is analytic in the same domain.

If $A < B < 0$ the first condition is still fulfilled because we can go from $t = 0$ to $t = B$ inside $\mathcal{D}'$, from $t = B$ to $t = 4-s-B$ and from $t = 4-s-B$ to $t = 4-s$. The second condition is not fulfilled. However, in the only case where we had to consider this situation, one found that due to the complex extension of $\mathcal{D}'$ the straight line $t = 0 \rightarrow t = 4-s$ was still inside the domain.

It turns out that the best results are obtained by filling $\mathcal{D}'$ with three circles. Of course, it would be better to perform a mapping which would bring $t = 4$ to $t = \infty$ on a straight line, the border of $\mathcal{D}'$ in $\text{Im } t > 0$ on a circle and the line $t = -\infty$, $t = -28$ on a straight line and then only carry the analytic completion. We believe that this would not bring a very serious improvement. The new domain we have obtained for $F(s, \cos\theta_s)$ and also for $f_e(s)$ now extends up to $\text{Im } s = 70 \mu^2$ in the complex direction. When one makes the union of this domain with the one obtained by the Mandelstam method, one sees that, taking the best of the two information, it is always true that $F(s, \cos\theta_s)$ for $-1 \leq \cos\theta_s \leq 1$ is analytic when s is in the union of the two domains. This has some importance, because as we know from I that along the physical cut $s \text{ real} > 4$ the scattering amplitude is analytic in a certain ellipse with foci at $\cos\theta = \pm 1$ we can, in a further analytic completion, extend the analyticity domain of $F(s, \cos\theta_s)$ in $\cos\theta$ for any s inside the domain. This will be done later.

IV.3 The final source of information on the domain of partial wave amplitudes comes from the work of Bros, Epstein and Glaser ³⁾. It is somehow complementary to what we have done up to now, because as it will appear, this provides a domain which extends to infinity in the direction $s \rightarrow +\infty$. Bros, Epstein and Glaser have shown the following: the fixed negative t_0 scattering amplitude $F(s, t_0)$ is analytic in

$$\phi_{s_0 t_0} > \text{Arg}(s - s_0) > 0 \quad -\phi_{s_0 t_0} < \text{Arg}(s - s_0) < 0$$

$$\cos \phi_{s_0 t_0} = 1 - 2 \left(\frac{32}{36 - t_0} \right)^{\frac{\pi}{\theta_{s_0 t_0}}} \quad (36)$$

with $\cos \theta_{s_0 t_0} = \frac{36 - t_0}{s_0}$

s_0 can be chosen real arbitrary > 0 , provided $(36 - t_0)/s_0 < 1$.

More generally, it is easy to see that $F(s, t)$ for $t_0 < t < 0$ is analytic in $\varnothing_{s_0 t_0} > \text{Arg}(s - s_0) > 0$.

Now from I we know that for any arbitrarily large s real $F(s, t)$ is analytic in a certain ellipse in the t which is deduced by unitarity from the ellipse of analyticity of the absorptive part, which for $s > 32$ (the only case we shall consider here) has foci at $t = 0$, $t = 4 - s$, and right extremity at $t = 4 + \frac{64}{s-4}$, i.e.,

$$\cos \theta_M = 1 + \frac{2}{s-4} \left[4 + \frac{64}{s-4} \right]$$

The corresponding ellipse for the amplitude, which results from $|\text{Im} f_e| > |f_e|^2$ has a semi-major axis measured in $\cos \theta$ by $(\cos \theta_M + 1/2)^{\frac{1}{2}}$. If one translates this in the t variable, one finds that the right extremity of the ellipse of analyticity for $F(s, t)$ is always to the right of $t = 1$. Hence $F(s, t)$ is certainly analytic in the ellipse with foci $t = 0$, $t = 4 - s$, extremity $t = 1$. As s increases the ellipse gets bigger and bigger. So we can just say that for $s > s_0$ (the restriction $s_0 > 32$ can be removed, with the help of the excellent low energy information we have) $F(s, t)$ is analytic inside the ellipse with foci $t = 0$, $t = 4 - s_0$, extremity $t = 1$. But since from (36) we are forced anyway to take $s_0 > 36 - t_0$ we have $t_0 > 4 - s_0$.

It will be sufficient to use the fact that $f(s, t)$ is analytic inside the ellipse with foci $t = 0$, $t = t_0$ and right extremity $t = 1$, for s real s_0 . Since on the other hand we have analyticity for $t_0 \leq t$ real ≤ 0 in $|\text{Arg}(s - s_0)| < \varnothing_{s_0 t_0}$. It is extremely easy to perform the analytic completion. Let $z = -((t - t_0/2)/(t_0/2))$, $y = z + \sqrt{z^2 - 1}$ (where the cut is taken from -1 to $+1$). The analyticity domain for s real becomes

$$1 < |y| < 1 + \frac{2}{|t_0|} + \sqrt{\left(1 + \frac{2}{|t_0|}\right)^2 - 1}$$

and for $0 < \text{Arg}(s-s_0) < \phi_{s_0 t_0}$ reduces to the circle $|y| = 1$.

If we use as variables $i \log(y)$ and $\log(s-s_0)$, we see this domain is a tube and can be extended to the convex base of the tube. Hence we obtain the union of the domains

$$\begin{cases} 0 < \text{Arg}(s-s_0) < \phi_{s_0 t_0} (1-\alpha) \\ 1 \leq |y| < \left[1 + \frac{2}{|t_0|} + \sqrt{\left(1 + \frac{2}{|t_0|}\right)^2 - 1} \right]^\alpha \end{cases} \quad (37)$$

so for

$$0 < \text{Arg}(s-s_0) < \phi_{s_0 t_0} (1-\alpha)$$

the analyticity domain in t is an ellipse with foci $t = t_0$, $t = 0$ and extremity

$$t = \frac{|t_0|}{2} \left[\left(1 + \frac{2}{|t_0|} + \sqrt{\left(1 + \frac{2}{|t_0|}\right)^2 - 1} \right)^\alpha + \left(1 + \frac{2}{|t_0|} - \sqrt{\left(1 + \frac{2}{|t_0|}\right)^2 - 1} \right)^\alpha - 2 \right] \quad (38)$$

First let us look at the asymptotic form of the domain for large $|t_0|$. Then it turns out that the optimum choices are

$$s_0 = 2|t_0| \left[1 - \frac{1}{\log|t_0|} \right]$$

for which
$$\phi_{s_0 t_0} \simeq \frac{(64)^2 \exp(-\frac{4}{\pi})}{|t_0| \log |t_0|}$$

and $\alpha(t_0)$ going to zero less fast than $|t_0|^{-\frac{1}{2}}$ for $|t_0| \rightarrow \infty$.
Then, for

$$\text{Res} = 2|t_0| + 4$$

$$0 < \text{Im } s \leq [1 - \alpha(t_0)] \frac{(64)^2 \exp(-\frac{4}{\pi})}{|t_0| \log |t_0|}$$

$F(s, t)$ is analytic inside an ellipse with foci $t = 0$, $t = -|t_0|$ and extremity $t \simeq 2(\alpha(t_0))^2$. This ellipse contains the rectangle $-|t_0| < \text{Re } t < 0$, $|\text{Im } t| < 2(\alpha(t_0))^2$.

If we take

$$s_1 = 2|t_0| + 4 + i [1 - \alpha(t_0)] \frac{(64)^2 \exp(-\frac{4}{\pi})}{|t_0| \log |t_0|}$$

and if $\alpha(t_0)$ decreases less fast than $|t_0|^{-\frac{1}{2}}$ we see that the segment connecting $t = 0$ to $t = (4 - s_1)/2$ is inside the ellipse of analyticity in the t plane. By $t \rightleftharpoons u$ crossing symmetry, the line $t = 0$, $t = 4 - s_1$ is inside the analyticity domain. Hence, re-expressing t_0 in terms of s_1 we see that for large $\text{Re } s$ the fixed angle amplitude $F(s, \cos \theta)$, $(-1 < \cos \theta < 1)$, and the partial wave amplitudes are analytic for

$$0 < \text{Im } s < \frac{2 \times (64)^2 \exp(-\frac{4}{\pi})}{\text{Res} \log(\text{Res})} \simeq \frac{2300}{\text{Res} \log(\text{Re } s)} \quad (39)$$

Unfortunately, this formula is only valid for $\text{Re } s > 2000 \mu^2$. If one computes, adjusting each time s_0 at best, the domain for smaller values of $\text{Re } s$, one finds a very small domain which is rather disappointing :

for $\text{Re } s = 68$ the domain shrinks to zero.
 $\text{Re } s = 200$ gives $0 < \text{Im } s < 0.4$
 $\text{Re } s = 600$ gives $0 < \text{Im } s < 0.35$
 $\text{Re } s = 1000$ gives $0 < \text{Im } s < 0.27$

We think that these poor results are due to the very drastic majorizations used in the derivation of equations (36). The unpleasant feature of Eq. (36) is that as t_0 approaches -28 they still give a very small domain while we know that then we have the full cut plane. It would not be astonishing, for instance, to be able to replace in (36)

$$\cos^2 \theta_{s_0 t_0} = \frac{36 - t_0}{s_0} \quad \text{by} \quad \cos^2 \theta_{s_0 t_0} = \frac{-28 - t_0}{s_0}$$

So, at least we have obtained a method of analytic continuation of partial waves for $s \rightarrow +\infty$. As soon as better results are available on fixed negative t analyticity properties, we shall repeat our calculations.

We have presented in Fig. 5 the analyticity domain for $F(s, \cos \theta)$ and for $f_\ell(s)$ obtained by the first two methods obtained in this Section. The points obtained from the third method are too close to the real positive axis to be seen.

V. ANALYTICITY DOMAIN FOR t INSIDE THE PARABOLA OF FOCUS $t=0$ AND EXTREMITY $t=\mu^2$

We have already indicated in the last part of the last Section that Bros, Epstein and Glaser have shown that for arbitrary negative transfer the scattering amplitude $F(s, t)$ is analytic in some angle $0 < \text{Arg}(s-s_0) < \phi_{s_0 t_0}$, for $0 > t > t_0$.

In fact, they have obtained more than that : for any negative t the scattering amplitude is analytic in $\text{Im} s > 0$ and $\text{Im} s < 0$ minus a finite region of the s complex plane. We shall say that $F(s, t)$ satisfies a "quasi" dispersion relation because $F(s, t)$ can be represented as a Cauchy integral over a contour consisting of the physical cuts (from $s = 4-t$ to $+\infty$ and $s = -\infty$ to $s = 0$) and a contour in the complex plane avoiding the complex singularities from $s = 0$ to $s = 4-t$.

We want to show here that this result is not only valid for t real < 0 but also for any t inside the parabola with focus $t = 0$ and extremity $t = \mu^2$, which of course contains in particular the line t real < 0 .

The argument is as follows : for $t_0 < t < 0$, $F(s, t)$ is analytic in $\text{Im} s > 0$ minus some finite region which is the union of the regions of possible singularities for fixed t , $t_0 < t < 0$. This region being finite, we can choose s_0 real such that $F(s, t)$ is analytic in $0 < \text{Arg}(s-s_0) < \psi(s_0, t_0)$, and also $-\psi(s_0, t_0) < \text{Arg}(s-s_0) < 0$, where, by taking s_0 large enough, one can make $\psi(s_0, t_0)$ as close as one wishes to π . For $s_0 > s_{\min}(t_0, \varepsilon)$ let us say that $\psi(s_0, t_0) > \pi(1-\varepsilon)$. Hence, applying the same technique as in Section IV, we find that $F(s, t)$ is analytic in

$$s: \quad 0 < |\text{Arg}(s-s_0)| < \alpha \pi(1-\varepsilon) = \chi$$

$$t: \quad \text{ellipse } E(\alpha, t_0) \text{ with foci } t=0 \quad t=t_0$$

and extremity :

(40)

$$\frac{|t_0|}{2} \left[\left(1 + \frac{2}{|t_0|} + \sqrt{\left(1 + \frac{2}{|t_0|} \right)^2 - 1} \right)^{1-\alpha} + \left(1 + \frac{2}{|t_0|} - \sqrt{\left(1 + \frac{2}{|t_0|} \right)^2 - 1} \right)^{1-\alpha} - 2 \right]$$

We can choose

$$\alpha > \frac{1}{2(1-\varepsilon)}$$

so that χ is larger than $\pi/2$. In this case, using $s \rightleftharpoons u$ crossing symmetry, we see that the domains $0 < \text{Arg}(s-s_0) < \chi$, and $-\chi < \text{Arg}(u-s_0) < 0$ for fixed t overlap, and similarly $-\chi < \text{Arg}(s-s_0) < 0$ and $0 < \text{Arg}(u-s_0) < \chi$ overlap. So that it is again true that the scattering amplitude is analytic in a twice cut plane minus a finite region. If we consider the intersection of the domains obtained when t varies in (40) we find, taking into account the motion of the left-hand cut, an analyticity domain with singularities inside $\text{Re } s < s_0$, $|\text{Im } s| < C(t_0, s)$ plus the cut $s \text{ real} > s_0$.

Choosing now $s_1 \gg s_0$ we can then interpolate again between $E(\alpha, t_0)$ and $E(1, t_0)$ and then iterate the procedure. The final result is that, given any t inside the ellipse $E(1, t_0)$, $F(s, t)$ is analytic in a cut plane minus a finite region. Now letting $t_0 \rightarrow -\infty$ we see that the ellipse $E(1, t_0)$ fills the parabola with focus $t = 0$ and summit $t = 1$, which is what we wanted to establish (Fig. 6). Of course, this result is qualitative in the sense that we do not know the size of the region in the s plane where singularities occur. This ignorance is entirely due to our ignorance, or rather lack of sufficiently accurate information, in the case of negative real t . However, we think it has some importance, since it prevents the analyticity domain from being flat, like the central result of I, and it makes it very difficult to construct counter examples to Mandelstam representation.

VI. CONCLUDING REMARKS

As the reader may have noticed, if he has had the courage to read thoroughly this paper, this article contains mainly transpiration and very little inspiration. One may question whether it was worth to take so much pain to get what is mainly a quantitative improvement of the

results presented in I and one could wonder whether one should not wait for a new idea to make a big jump forward and get at once Mandelstam representation which is the final goal. The answer to this is that the technique of the termite is sometimes efficient in the end.

Let us summarize the results obtained. First we have been able to continue the absorptive parts and the scattering amplitude till the border of the Mandelstam double spectral function (except for $8 < s < 32$). This means that one can prove from axiomatic field theory that the two particle singularities induced by elastic unitarity are really the nearest singularities to the physical region. This is what everybody is ready to believe, but now it is proved. As by-product, we get that the fixed energy sections of the amplitude, irrespective of the energy, contain part of the cuts $t > 4$, $u > 4$. Another important consequence is that if low energy pion-pion scattering admits a potential description, this potential is a Yukawa superposition.

Then we have carried out a partial analytic completion which could certainly be pushed further since :

- 1) we have never used s, t, u crossing symmetry at the same time;
- 2) we get a union of domains which certainly gives a non-natural domain.

However, one could get out of it a very large domain of analyticity for partial waves which practically extends from $s = -28$ to $s = 78$ (with $\text{Im } s_{\text{max}} = 70$) and theoretically goes to $s = +\infty$ (the thickness in $\text{Im } s$, for $78 < \text{Re } s < \infty$ never exceeds 0.40). We think that this result has some practical importance since it shows that in the range $0 < s < (1 \text{ GeV})^2$ the complex singularities are rather far away and this, if one could extend a little bit further to the left of $-28 \mu^2$ would be enough as a basis for all existing approximate calculations of the pion-pion partial waves. However, while there is considerable hope to improve the situation for $\text{Re } s \rightarrow +\infty$, it does not seem so easy to extend the domain to the left of $\text{Re } s = -28 \mu^2$.

Finally, we have shown that "quasi" dispersion relations hold not only for t negative arbitrarily large, but also for t inside a parabola containing the negative t axis. Notice that if one could get rid of the complex singularities at finite distance in the s plane this would be enough to prove Mandelstam representation, by using crossing symmetry.

Now if one collects all this, one sees that counter-examples to Mandelstam representation, consistent with all what we know at present, are very difficult to construct. If you take only part of the information you may succeed. For instance, starting only from the existence of dispersion relations for t in $\mathcal{D}$ and the requirement of full s, t, u crossing symmetry, Bros and Glaser⁵⁾ have been able to construct a function which does not satisfy Mandelstam representation. However, this function has singularities in the region $t = -\lambda + i\varepsilon$, $u = -\lambda + i\varepsilon$ which is the physical region of the s channel, where, we know, we have analyticity, and a fortiori, in the region obtained in Section V.

Another question is what happens if you accept the fact that the scattering amplitude is also analytic in the domain obtained from perturbation theory. To my knowledge, the only safe domain in perturbation theory is the union of the points⁶⁾

$$\begin{aligned} s &= s_0 + \lambda z & t &= t_0 + \mu z \\ \operatorname{Im} z &\neq 0 & \lambda, \mu &\text{real} \\ s_0 < 4 & \quad t_0 < 4 & \quad 4 - s_0 - t_0 < 4 \end{aligned}$$

The analyticity region one finds is almost orthogonal to the one we have obtained. For instance, this domain shrinks to zero when s tends to a real value $s > 4\mu^2$, which is not the case for us, but on the other hand, for any complex s the line $-\infty < \cos\theta_s < +\infty$ lies entirely inside the domain, and hence the partial wave analyticity domain is a cut plane. It seems to me that - though it may look disgusting to purists - it is worth investigating what happens when one makes the

union of the two domains. There are some rather promising features in sight. In particular, there is hope to prove dispersion relations for $4 < s < 16$, and one can show, by combination with the results of Section V, that for any s , $\text{Im } s > 0$. The region $\varepsilon < \text{Arg } t < \text{Arg}(s+28)$ for $|t|$ sufficiently large is inside the analyticity domain (ε arbitrarily small). In case it turns out that the final domain is Mandelstam representation, it is worth trying it.

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λ	c_λ
0	256
-2	324
-3	361
-4	362
-6	338
-8	319
-10	303
-12	292
-14	285
-16	281
-18	284
-20	281.7
-22	224.5
-24	158

T A B L E

The analyticity domains

$$|t-\lambda||u-\lambda| < c_\lambda$$

obtained by the Mandelstam method

R E F E R E N C E S

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FIGURE CAPTIONS

- Figure 1 : The real section of the analyticity domain of $A_s(s,t)$. This figure gives for each s the major axis of the ellipse of analyticity of $A_s(s,t)$ (shaded region). The continuous line indicates the part of the border of the double spectral function which cannot be reached. The dotted line indicates what part of the double spectral function region is on the border of the analyticity domain of A_s .
- Figure 2 : The region $\mathcal{D}'$, in the complex t plane, of validity of fixed t dispersion relations. The domain $\mathcal{D}$ obtained in I is indicated by the dotted line.
- Figure 3 : The points of the real s,t,u plane where the amplitude can be continued through a complex path (of course except in the triangle $s < 4\mu^2$, $t < 4\mu^2$, $u < 4\mu^2$, these points are on the border of the domain).
- Figure 4 : The principle of the analyticity completion carried out in IV.2. The analyticity domain is the union of pairs of shaded regions.
- Figure 5 : Analyticity domain for $F(s, \cos\theta_s)$, $-1 < \cos\theta_s < +1$ and for the partial wave amplitudes (the part from $\text{Re } s = 78$ to $+\infty$ is not indicated).
- Figure 6 : The domain of validity of quasi-dispersion relations.

FIG.1

FIG. 2

FIG. 3

FIG. 4

FIG.5

FIG. 6