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HOMOGENEOUS FUNCTIONS AS A BASIS OF UNITARY REPRESENTATIONS
OF GROUPS $SL(n, \mathbb{C})$

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A B S T R A C T

We construct a set of homogeneous functions which form a basis for unitary irreducible representations of groups $SL(n, \mathbb{C})$. The connection with the theory of Gelfand and Neumark is studied.

1. THE LIE ALGEBRA OF $SL(n, \mathbb{C})$

We are going to discuss a method of constructing unitary representations of groups $SL(n, \mathbb{C})$ which was originally proposed by Fronsda1¹⁾. We investigate the connection of this method with the one known from the book of Gelfand and Neumark²⁾.

The advantage of Fronsda1's treatment lies in its simplicity and heuristic procedure. It allows to introduce the concept of tensors in a generalized version. The use of tensors is hoped to simplify the construction of invariant forms. In addition the physical language makes use of the tensor notation in many cases. In this context the subsidiary conditions play a particular role, they project out a certain irreducible representation. We shall introduce them for the generalized tensors.

The real Lie algebra of $SL(n, \mathbb{C})$ is defined by the basis

$$\begin{aligned} (l_{ij})_{mn} &= \delta_{im} \delta_{jn} - \frac{1}{n} \delta_{ij} \delta_{mn}, \\ (\tilde{l}_{ij})_{mn} &= i \delta_{im} \delta_{jn} - \frac{i}{n} \delta_{ij} \delta_{mn}. \end{aligned}$$

Let S be an infinitesimal element of $SL(n, \mathbb{C})$

$$S = 1 + \sum_{i,j} (\epsilon_{ij} l_{ij} + \eta_{ij} \tilde{l}_{ij})$$

with real parameters ϵ_{ij} and η_{ij} . The generators l_{ij} and $\tilde{l}_{ij}$ obey therefore the commutation relations

$$\begin{aligned} [l_{ij}, l_{ke}] &= \delta_{jk} l_{ie} - \delta_{ie} l_{kj}, \\ [l_{ij}, \tilde{l}_{ke}] &= \delta_{jk} \tilde{l}_{ie} - \delta_{ie} \tilde{l}_{kj}, \\ [\tilde{l}_{ij}, \tilde{l}_{ke}] &= -\delta_{jk} l_{ie} + \delta_{ie} l_{kj}. \end{aligned}$$

2.

A representation is defined by a mapping

$$\begin{aligned} e_{ij} &\rightarrow L_{ij}, \\ \tilde{e}_{ij} &\rightarrow \tilde{L}_{ij} \end{aligned}$$

which preserves the commutation relations. It is a unitary representation, if

$$\begin{aligned} L_{ij}^\dagger &= -L_{ij}, \\ \tilde{L}_{ij}^\dagger &= -\tilde{L}_{ij}. \end{aligned}$$

The compact subgroup $SU(n)$ of $SL(n, \mathbb{C})$ is generated by the combinations

$$e_{ij} - e_{ji}, \quad \tilde{e}_{ij} + \tilde{e}_{ji}.$$

Given a representation we can define the linear combinations

$$\begin{aligned} M_{ij} &= \frac{1}{2} (L_{ij} + i\tilde{L}_{ij}), \\ N_{ij} &= \frac{1}{2} (L_{ij} - i\tilde{L}_{ij}). \end{aligned}$$

They commute

$$[M_{ij}, N_{kl}] = 0$$

and generate complex Lie algebras $SL(n, \mathbb{C})$ as can be seen from their commutation relations

$$\begin{aligned} [M_{ij}, M_{km}] &= \delta_{jk} M_{im} - \delta_{im} M_{kj}, \\ [N_{ij}, N_{km}] &= \delta_{jk} N_{im} - \delta_{im} N_{kj}. \end{aligned}$$

For an infinitesimal element $S \in SL(n, \mathbb{C})$ and its complex conjugate $\bar{S}$ the representing operators $V_S, V_{\bar{S}}$ are

$$\begin{aligned}
V_S &= 1 + \sum_{ij} (\varepsilon_{ij} L_{ij} + \eta_{ij} \tilde{L}_{ij}) \\
&= 1 + \sum_{ij} \{ (\varepsilon_{ij} - i\eta_{ij}) M_{ij} + (\varepsilon_{ij} + i\eta_{ij}) N_{ij} \}, \\
V_{\bar{S}} &= 1 + \sum_{ij} (\varepsilon_{ij} L_{ij} - \eta_{ij} \tilde{L}_{ij}) \\
&= 1 + \sum_{ij} \{ (\varepsilon_{ij} + i\eta_{ij}) M_{ij} + (\varepsilon_{ij} - i\eta_{ij}) N_{ij} \}.
\end{aligned}$$

This shows that under conjugation

$$S \rightarrow \bar{S}$$

we have an exchange

$$M_{ij} \longleftrightarrow N_{ij}$$

Let η be a fundamental representation of the algebra N_{ij} ,

$$\eta = (\eta_1, \eta_2, \dots, \eta_n)$$

As the expression for V_S shows (put $M = 0$) it transforms like

$$\eta' = \eta \cdot S, \quad N_{ij} = \eta_i \frac{\partial}{\partial \eta_j}.$$

Similarly, let $\bar{\xi}$ be a fundamental representation of the algebra spanned by M_{ij} , then

$$\bar{\xi}' = \bar{\xi} \cdot \bar{S}, \quad M_{ij} = \bar{\xi}_i \frac{\partial}{\partial \bar{\xi}_j}.$$

Thus the fundamental representation of the N algebra transforms as a spinor basis η_i , the fundamental representation of the M algebra as a spinor basis $\bar{\xi}_i = \bar{\eta}_i$.

4.

2. REPRESENTATIONS OF SU(n)

In this Section we repeat some well-known facts of the representation theory for groups SU(n). We do so since we shall use representations for these groups in a special form which is convenient for our purposes. In addition we take the opportunity to introduce some notations.

The irreducible representations of SU(n) are finite dimensional and equivalent to unitary ones. They are uniquely determined by their highest weights. There exists a set of n-1 fundamental representations so that each highest weight can be decomposed into a sum of highest weights of the fundamental representations. We construct the fundamental representations as follows. Let $\xi = (\xi_{ij})$ be a matrix of SU(n) and $S \in \text{SU}(n)$. Then

$$\xi'_{ij} = \xi_{ik} S_{kj}$$

defines a reducible representation of SU(n), a regular representation. The first row of ξ with elements

$$\Delta_i^{(1)}(\xi) = \xi_{1i}$$

serves as a basis for the first fundamental representation. Its highest weight is

$$M^{(1)} = (1 - \frac{1}{n}, -\frac{1}{n}, \dots, -\frac{1}{n})$$

Then consider the determinants

$$\Delta_{ij}^{(2)}(\xi) = \begin{vmatrix} \xi_{1i} & \xi_{1j} \\ \xi_{2i} & \xi_{2j} \end{vmatrix}$$

They span the space for the second fundamental representation with the highest weight

$$M^{(2)} = \left(1 - \frac{2}{n}, 1 - \frac{2}{n}, -\frac{2}{n}, \dots, -\frac{2}{n} \right)$$

Proceeding in this way we can define $n-1$ fundamental representations, spanned by

$$\Delta_{i_1 \dots i_k}^{(k)}(\xi) = \begin{vmatrix} \xi_{1 i_1} & \xi_{1 i_k} \\ \vdots & \vdots \\ \xi_{k i_1} & \xi_{k i_k} \end{vmatrix}, \quad k=1, 2, \dots, n-1$$

and possessing as the highest weight

$$M^{(k)} = \underbrace{\left(1 - \frac{k}{n}, 1 - \frac{k}{n}, \dots \right)}_{k \text{ times}} \underbrace{\left(-\frac{k}{n}, -\frac{k}{n} \right)}_{n-k \text{ times}}$$

To each of these fundamental representations we adjoin a Young tableau of a column of k boxes. Every irreducible representation can be constructed by taking the tensorial product of fundamental representations. Let the highest weight be

$$M = \sum_{k=1}^{n-1} \lambda_k M^{(k)} \quad (\lambda_k \text{ non-negative integer})$$

Then take the most symmetrical part of a tensor product which contains the k^{th} fundamental representation λ_k times. The symmetry type of the tensor is of the form described by the Young tableau

The symmetrization is automatically carried out if we use the basis $\Delta^{(k)}(\xi)$ with the same ξ for all k . Only the most symmetric part survives.

Thus we arrive at a basis for the representation characterized by the $(n-1)$ -tuple

$$(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$$

the elements of which have the form of homogeneous polynomials in the parameters ξ_{ij} ,

$$P_{[m]}^{[\lambda]}(\xi) = \prod_{k=1}^{n-1} \prod (\Delta_{i_1 i_2 \dots i_k}^{(k)} m_{i_1 \dots i_k})$$

The second product extends over all sets of indices $i_1 \dots i_k$, which contain only different indices. Two sets of indices which differ only by their order may be considered identical. The exponents $m_{i_1 \dots i_k}$ are non-negative integers which are subject to the condition

$$\sum_{k \text{ fixed}} m_{i_1 \dots i_k} = \lambda_k$$

Certainly not all of these polynomials $P_{[m]}^{[\lambda]}$ are independent. We define a set of "displacement" operators

$$Q_{ij} = \sum_{k=1}^n \xi_{ik} \frac{\partial}{\partial \xi_{jk}}$$

They commute with all operators of the type

$$N_{ij} = \sum_{k=1}^{n-1} \xi_{ki} \frac{\partial}{\partial \xi_{kj}}$$

Indeed,

$$\begin{aligned}
 [N_{ij}, Q_{km}] &= \sum_{p=1}^n \sum_{q=1}^{u-1} \left(\delta_{kp} \delta_{qj} \xi_{qi} \frac{\partial}{\partial \xi_{mp}} - \right. \\
 &\quad \left. - \delta_{qm} \delta_{ip} \xi_{kp} \frac{\partial}{\partial \xi_{qj}} \right) \\
 &= \xi_{ki} \frac{\partial}{\partial \xi_{mj}} - \xi_{ki} \frac{\partial}{\partial \xi_{mj}} = 0.
 \end{aligned}$$

By applying these operators to the polynomial basis we get

$$Q_{ij} P_{[m]}^{[\lambda]}(\xi) = 0 \quad \text{if } i < j,$$

which are invariant restrictions on the elements of the basis. These equations can be solved ³⁾.

The solution tells us that the basis elements can be constructed by means of the following parameters only

$$\begin{array}{ccccccc}
 \Delta_1^{(1)} & \Delta_2^{(1)} & \Delta_3^{(1)} & \dots & \Delta_n^{(1)} \\
 & \Delta_{12}^{(2)} & \Delta_{13}^{(2)} & \dots & \Delta_{1n}^{(2)} \\
 & & \Delta_{123}^{(3)} & \dots & \Delta_{12n}^{(3)} \\
 & & & \vdots & \vdots \\
 & & & \Delta_{12\dots n-2,n-1}^{(n-1)} & \Delta_{12\dots n-2,n-1}^{(n-1)}
 \end{array}$$

Their number is $\frac{1}{2}(n-1)(n+2)$ in accord with $n(n-1)$ parameters subject to $\frac{1}{2}(n-1)(n-2)$ conditions Q_{ij} .

We can then rewrite the polynomials as

$$\begin{aligned}
 P_{[m]}^{[\lambda]}(\xi) &= \Delta_1^{\lambda_1} \Delta_{12}^{\lambda_2} \Delta_{123}^{\lambda_3} \dots \Delta_{123\dots n-1}^{\lambda_{n-1}} \\
 &\quad \cdot \left(\frac{\Delta_2}{\Delta_1} \dots \frac{\Delta_n}{\Delta_1}, \frac{\Delta_{13}}{\Delta_{12}} \dots \frac{\Delta_{1n}}{\Delta_{12}}, \dots, \frac{\Delta_{123\dots n-2,n}}{\Delta_{123\dots n-2,n-1}} \right).
 \end{aligned}$$

8.

Z is a polynomial in its indicated arguments. $P_{[m]}^{\{\lambda\}}$ is now no longer an entire function of the reduced set of variables Δ . Indeed by the elimination procedure we may introduce negative powers of $\Delta_1, \Delta_{12}, \dots, \Delta_{1\dots n-1}$.

Example : $SU(3), \lambda_1 = \lambda_2 = 1$.

The condition on the

$$Q_{12} P = 0$$

can be re-expressed as a condition in the Δ 's

$$\Delta_1 \Delta_{23} + \Delta_2 \Delta_{31} + \Delta_3 \Delta_{12} = 0.$$

As a basis we take the nine different products

$$\Delta_i \Delta_{jk}$$

If we would eliminate Δ_{23} only in the basis element

$$\Delta_1 \Delta_{23}$$

we were left with entire functions. But by a complete elimination we get expressions like

$$\Delta_2 \Delta_{23} = \frac{\Delta_2^2}{\Delta_1} \Delta_{13} - \frac{\Delta_2 \Delta_3}{\Delta_1} \Delta_{12}$$

which are non-entire.

We shall not go into the details of solving the differential equations of Q_{ij} applied to P . Instead we show how we can derive all the conditions necessary for an elimination of superficial Δ 's. For this purpose we consider the determinant of ξ

$$|\xi_{ij}| = 1.$$

Applying the operator $Q_{1,n}$ we find

$$\Delta_1^{(1)} \Delta_{2,3 \dots n-1,n}^{(n-1)} - \Delta_2^{(1)} \Delta_{1,3 \dots n-1,n}^{(n-1)} \pm \dots = 0.$$

Then we apply $Q_{1,n-1}$ and treat the elements of the last row ξ_{ni} as unknowns. Thus we obtain a linear expression in the unknowns, the coefficients of which must vanish.

$$\sum_{k=1}^n a_k \xi_{nk} = 0$$

$$a_1 = 0 : \Delta_2^{(1)} \Delta_{3,4 \dots n}^{(n-2)} - \Delta_3^{(1)} \Delta_{2,4 \dots n}^{(n-2)} \pm \dots = 0,$$

$$a_2 = 0 : \Delta_1^{(1)} \Delta_{3,4 \dots n}^{(n-2)} - \Delta_3^{(1)} \Delta_{1,4 \dots n}^{(n-2)} \pm \dots = 0,$$

$$a_r = 0 : \Delta_1^{(1)} \Delta_{2,3 \dots n-1}^{(n-2)} - \Delta_2^{(1)} \Delta_{1,3 \dots n-1}^{(n-2)} \pm \dots = 0.$$

Continuing this procedure we apply $Q_{1,k}$.

We use the expansion theorem of Laplace with respect to the first k and the last $n-k$ rows. We treat the determinants formed from the last $n-k$ rows as unknowns and get a linear expression of the type

$$\sum a_{i_1 i_2 \dots i_{n-k}} \begin{vmatrix} \xi_{k+1, i_1} & \xi_{k+1, i_{n-k}} \\ \xi_n, i_1 & \xi_n, i_{n-k} \end{vmatrix} = 0.$$

Each coefficient $a_{i_1 \dots i_{n-k}}$ must vanish. Consider, e.g., the coefficient $a_{k+1, k+2, \dots, n}$. Expanding this determinant of k rows we get

$$\pm a_{k+1, k+2, \dots, n} = \Delta_1^{(1)} \Delta_{2,3 \dots k}^{(k-1)} - \Delta_2^{(1)} \Delta_{1,3 \dots k}^{(k-1)} \pm \dots = 0.$$

10.

The index k runs from 3 to n . For each k we get $\binom{n}{n-k}$ conditions. The total number of conditions we get is therefore

$$2^n - 1 - \binom{n}{1} - \binom{n}{2}$$

The number of Δ 's is

$$\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} = 2^n - 2$$

The residual set of Δ 's is of the number

$$\binom{n}{2} + \binom{n}{1} - 1 = \frac{1}{2} (n-1)(n+2)$$

as we would have expected. It is a simple task to show that there are no further independent conditions on the Δ 's than those derived above.

3. GENERALIZED TENSORS FOR A GROUP $SL(n, \mathbb{C})$

Our technique of constructing representations for $SU(n)$ made use of homogeneous functions of the components of the matrix ξ . For a given set of λ 's, λ_k was the degree of homogeneity for variables $\Delta^{(k)}(\xi)$. The operators M_{ij} span the complex Lie algebra of $SL(n, \mathbb{C})$. We construct its representations by means of a matrix ξ . Then the differential operator form of M_{ij} is

$$M_{ij} = \sum_{k=1}^{n-1} \xi_{ki} \frac{\partial}{\partial \xi_{kj}}$$

Similarly we use a matrix η to construct representations for the algebra spanned by N_{ij} and get

$$N_{ij} = \sum_{k=1}^{n-1} \eta_{ki} \frac{\partial}{\partial \eta_{kj}}$$

Let the degree of homogeneity in the variables $\Delta^{(k)}(\gamma)$ be μ_k . These parameters λ_k, μ_k are invariants of the representations. If they are non-negative integers, the basis of the representation must be finite. Finite dimensional representations of $SL(n, \mathbb{C})$ are all non-unitary (except the trivial one). Thus we must let the λ_k and μ_k take non-integer, in general complex values. Doing this we get representations which can be made unitary under certain conditions and which we shall call generalized tensors. In the most general case they may have the following properties :

- 1) they can be reducible but not completely reducible,
- 2) they can be non-unitary,
- 3) sometimes it is impossible to "integrate" these representations in order to get a representation for the group.

As a simple example we study the group $SL(2, \mathbb{C})$.

4. UNITARY REPRESENTATIONS OF $SL(2, \mathbb{C})$

In the case of the group $SL(2, \mathbb{C})$ it is straightforward to investigate unitary representations starting from generalized tensors. We do this following Fronsdaal's work ¹⁾.

We introduce the spinors (dropping the first index)

$$\begin{aligned}\Delta_1^{(1)}(\xi) &= \xi_1, & \Delta_1^{(1)}(\gamma) &= \gamma_1, \\ \Delta_2^{(1)}(\xi) &= \xi_2, & \Delta_2^{(1)}(\gamma) &= \gamma_2.\end{aligned}$$

The generators of $SL(2, \mathbb{C})$ are usually written as

$$\left. \begin{aligned} A_1 &= \frac{1}{2} \sigma_1 \\ A_2 &= \frac{1}{2} \sigma_2 \\ A_3 &= \frac{1}{2} \sigma_3 \end{aligned} \right\}$$

as generators of
the maximal
compact subgroup

[note that $A_1 = -\frac{i}{2} (\tilde{e}_{12} + \tilde{e}_{21})$].

$$\left. \begin{aligned} B_1 &= \frac{i}{2} \sigma_1 \\ B_2 &= \frac{i}{2} \sigma_2 \\ B_3 &= \frac{i}{2} \sigma_3 \end{aligned} \right\}$$

generating one-
dimensional
non-compact subgroup

We define

$$A^\pm = A_1 \pm iA_2, \quad B^\pm = B_1 \pm iB_2$$

We express these operators as differential operators in ξ, η

$$\begin{aligned} A_1 &= \frac{1}{2} \left(-\xi_1 \frac{\partial}{\partial \xi_2} - \xi_2 \frac{\partial}{\partial \xi_1} + \eta_1 \frac{\partial}{\partial \eta_2} + \eta_2 \frac{\partial}{\partial \eta_1} \right), \\ A_2 &= \frac{i}{2} \left(-\xi_1 \frac{\partial}{\partial \xi_2} + \xi_2 \frac{\partial}{\partial \xi_1} - \eta_1 \frac{\partial}{\partial \eta_2} + \eta_2 \frac{\partial}{\partial \eta_1} \right), \\ A_3 &= \frac{1}{2} \left(-\xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} + \eta_1 \frac{\partial}{\partial \eta_1} - \eta_2 \frac{\partial}{\partial \eta_2} \right), \\ B_1 &= \frac{i}{2} \left(+\xi_1 \frac{\partial}{\partial \xi_2} + \xi_2 \frac{\partial}{\partial \xi_1} + \eta_1 \frac{\partial}{\partial \eta_2} + \eta_2 \frac{\partial}{\partial \eta_1} \right), \\ B_2 &= \frac{1}{2} \left(-\xi_1 \frac{\partial}{\partial \xi_2} + \xi_2 \frac{\partial}{\partial \xi_1} + \eta_1 \frac{\partial}{\partial \eta_2} - \eta_2 \frac{\partial}{\partial \eta_1} \right), \\ B_3 &= \frac{i}{2} \left(+\xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2} + \eta_1 \frac{\partial}{\partial \eta_1} - \eta_2 \frac{\partial}{\partial \eta_2} \right), \end{aligned}$$

$$\begin{aligned} A^+ &= -\xi_2 \frac{\partial}{\partial \xi_1} + \eta_1 \frac{\partial}{\partial \eta_2}, \\ A^- &= -\xi_1 \frac{\partial}{\partial \xi_2} + \eta_2 \frac{\partial}{\partial \eta_1}, \\ B^+ &= +i\xi_2 \frac{\partial}{\partial \xi_1} + i\eta_1 \frac{\partial}{\partial \eta_2}, \\ B^- &= +i\xi_1 \frac{\partial}{\partial \xi_2} + i\eta_2 \frac{\partial}{\partial \eta_1}. \end{aligned}$$

Let us consider the maximal compact subgroup of $SL(2, \mathbb{C}) : SU(2)$. Unitary representations of $SL(2, \mathbb{C})$ should decompose into unitary representations of $SU(2)$, which are finite dimensional and possess a vector of highest weight. Let Ψ be a generalized tensor. Then the vectors of highest weight for the compact subgroup are determined by

$$A^+ \Psi = \left(-\xi_2 \frac{\partial}{\partial \xi_1} + \eta_1 \frac{\partial}{\partial \eta_2} \right) \Psi(\xi_1, \xi_2, \eta_1, \eta_2) = 0$$

Solving this equation we see that Ψ must be a linear combination of basis vectors

$$\Psi(\alpha, \text{highest}) = n_\alpha \cdot \eta_1^{\mu-\alpha} \xi_2^{\lambda-\alpha} (\eta_1 \xi_1 + \eta_2 \xi_2)^\alpha$$

We have introduced already a normalizing factor n_α . We apply the lowering operator A^- several times. Since the expression

$$\eta_1 \xi_1 + \eta_2 \xi_2$$

is invariant under $SU(2)$, $\mu - \alpha$ and $\lambda - \alpha$ must be non-negative integers if the representation of the compact subgroup is to be finite dimensional. We write

$$\begin{aligned} \mu - \alpha &= n_1, \\ \lambda - \alpha &= n_2, \\ n_1 - n_2 &= \mu - \lambda = m. \end{aligned}$$

In addition we introduce the parameters

$$j = \frac{1}{2}(n_1 + n_2) = \frac{1}{2}(\mu + \lambda) - \alpha$$

which characterizes the representation generated by the highest weight (j is the spin) and a second invariant parameter besides m by

$$\frac{i}{2} \rho = 1 + j + \alpha = \frac{1}{2}(\mu + \lambda) + 1.$$

The parameter m is obviously an integer. We write now

$$\begin{aligned}\Psi(\alpha, \text{highest}) &= \Psi(j, j) \\ &= n_j \eta_1^{j+\frac{1}{2}m} \xi_2^{j-\frac{1}{2}m} (\eta_1 \xi_1 + \eta_2 \xi_2)^{\frac{1}{2}\varphi-1-j} \\ \Psi(j, j_z) &= N_{jj_z} (A^-)^{j-j_z} \Psi(j, j)\end{aligned}$$

with the well-known normalizing factor

$$N_{jj_z} = \left[\frac{(j+j_z)!}{(2j)!(j-j_z)!} \right]^{\frac{1}{2}}$$

If we apply B_3 to $\Psi(j, j_z)$ we get a linear combination of $\Psi(j+1, j_z)$, $\Psi(j, j_z)$ and $\Psi(j-1, j_z)$ because of the Wigner-Eckart theorem

$$\begin{aligned}-iB_3\Psi(j, j_z) &= \frac{n_j}{n_{j+1}} \frac{N_{jj_z}}{N_{j+1, j_z}} \frac{(\frac{1}{2}\varphi-j-1)(j+j_z+1)}{(j+1)(2j+1)} \Psi(j+1, j_z) \\ &+ \frac{m}{2} \frac{i\varphi}{2} \frac{j_z}{j(j+1)} \Psi(j, j_z) \\ &+ \frac{n_j}{n_{j-1}} \frac{N_{jj_z}}{N_{j-1, j_z}} \frac{(j^2 - \frac{m^2}{4})(\frac{1}{2}\varphi+j)(j-j_z)}{j(2j+1)} \Psi(j-1, j_z)\end{aligned}$$

Hermiticity of B_3 requires

$$m \cdot \varphi \quad \text{to be purely real,} \quad (1)$$

$$\begin{aligned}&\left\{ i \frac{n_j}{n_{j+1}} \frac{N_{jj_z}}{N_{j+1, j_z}} \frac{(\frac{1}{2}\varphi-j-1)(j+j_z+1)}{(j+1)(2j+1)} \right\} \\ &= i \frac{n_{j+1}}{n_j} \frac{N_{j+1, j_z}}{N_{jj_z}} \frac{[(j+1)^2 - \frac{m^2}{4}](\frac{1}{2}\varphi+j+1)(j-j_z+1)}{(j+1)(2j+3)}\end{aligned} \quad (2)$$

Condition (2) leads to

$$\frac{|n_{j+1}|^2}{|n_j|^2} = \frac{\frac{i}{2}\varphi + j+1}{\frac{i}{2}\varphi + j+1} \frac{(2j+2)(2j+3)}{(j+1)^2 - \frac{m^2}{4}}$$

The right-hand side must be positive, this is equivalent with

$$\frac{\varphi^2}{4} + (j+1)^2 > 0 \quad (2')$$

The conditions (1) and (2') are fulfilled in the two cases

- a) φ is real, m is arbitrary integer,
- b) $\varphi = i\sigma$ is purely imaginary, $0 < \sigma < 2$, m is zero.

The set a) of representations is known as the principal series, the other set as the supplementary series.

These representations are irreducible if the parameter j starts off with $|\frac{m}{2}|$,

$$j = |\frac{m}{2}| + n, \quad n = 0, 1, 2, 3, \dots$$

We find indeed that the operator B_j couples different j states for equal j_z and the operators $A^\pm$ couple different j_z states for equal j . Thus all states can be reached starting say with $j = |\frac{m}{2}|$, $j_z = j$.

5. COMPARISON WITH THE THEORY OF GELFAND AND NEUMARK ⁴⁾

There exist different ways to compare the results of the foregoing Section with the theory of Gelfand and Neumark. In this Section we relate the infinitesimal operators in both treatments and show that the correspondence between our parameters m , φ and the parameters of Gelfand and Neumark is an identity. In the next Section we use a more fundamental procedure to derive generalized tensors of a group $SL(n, \mathbb{C})$ with general n from the theory of Gelfand and Neumark. An intimate relationship will then show up.

Gelfand and Neumark span their representation space by functions $f(z)$ where z characterizes a class of matrices of $SL(2, \mathbb{C})$. Exactly, take $a \in SL(2, \mathbb{C})$, k as an upper triangular matrix and write (if possible, but then uniquely)

$$a = k \cdot z, \quad z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

The triangular matrices k form a subgroup. Its right cosets can be described by the complex numbers z with exception of a set of z measure zero in the complex z plane. Let the coset corresponding to a matrix za be described by $z\bar{a} = z'$. With

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

we find

$$z' = \frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}$$

The results of Gelfand and Neumark are :

- 1) If φ is real and m any integer we get a unitary representation $a \rightarrow V_a$ by defining

$$V_a f(z) = (a_{12}z + a_{22})^{\frac{m}{2} + \frac{\varphi}{2} - 1} (\bar{a}_{12}\bar{z} + \bar{a}_{22})^{-\frac{m}{2} + \frac{\varphi}{2} - 1} \cdot f\left(\frac{a_{11}z + a_{21}}{a_{12}z + a_{22}}\right)$$

The functions $f(z)$ span the Hilbert space of square integrable functions over the complex z plane, $\mathcal{L}^2(\mathbb{R}_2)$.

- 2) Let $\varphi = i\sigma$, $0 < \sigma < 2$, be purely imaginary and $m = 0$. Take the same formula as above. We get a unitary representation if $f(z)$ belongs to a Hilbert space H_σ defined by a scalar product ($z = x + iy$)

$$(f_1, f_2) = \int dx_1 dx_2 dy_1 dy_2 f_1(z_1) f_2(z_2) \cdot |z_1 - z_2|^{-2 + \sigma}$$

- 3) Let φ be arbitrary, m integer. Our transformation formula defines a representation in a Hilbert space (not only in a Banach space!).

The representations of cases 1) and 2) are irreducible. In case 3) we can split off a finite dimensional representation if φ obeys the condition

$$\varphi^2 = -(1+m+2n)^2, \quad n \text{ integer}$$

In this case the degrees of homogeneity λ, μ are connected with m, φ by the relations

$$\begin{aligned} \mu &= +\frac{m}{2} + i\frac{\varphi}{2} - 1 & (\eta) \\ \lambda &= -\frac{m}{2} + i\frac{\varphi}{2} - 1 & (\xi = \bar{\eta}) \end{aligned}$$

which are identical with the relations of Section 4. If $\varphi^2 \neq -(1+m+2n)^2$ the representations of case 3) are completely irreducible.

We consider now the set of functions $f(z)$ either belonging to $\mathcal{L}^2(R_2)$ or H_σ which are in addition differentiable. Then we can express the infinitesimal operators by differential operators

$$\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \bar{z}}$$

For infinitesimal $\varepsilon_i, \eta_i, i = 1, 2, 3$ we find

$$V_a = 1 + i\varepsilon_i A_i + i\eta_i B_i$$

$$a_{11} = 1 + i(\varepsilon_3 + i\eta_3),$$

$$a_{22} = 1 - i(\varepsilon_3 + i\eta_3),$$

$$a_{21} = i(\varepsilon_1 + i\varepsilon_2 + i\eta_1 - \eta_2),$$

$$a_{12} = i(\varepsilon_1 - i\varepsilon_2 + i\eta_1 + \eta_2),$$

$$\begin{aligned} -i(z' - z) &= -z^2(\varepsilon_1 - i\varepsilon_2 + i\eta_1 + \eta_2) + 2z(\varepsilon_3 + i\eta_3) \\ &\quad + (\varepsilon_1 + i\varepsilon_2 + i\eta_1 - \eta_2). \end{aligned}$$

This leads to

$$\begin{aligned}
 A_1 &= +\frac{1}{2} (1-z^2) \frac{\partial}{\partial z} - \frac{1}{2} (1-\bar{z}^2) \frac{\partial}{\partial \bar{z}} \\
 &\quad + \frac{1}{2} \left(\frac{m}{2} + i\frac{\varphi}{2} - 1 \right) z - \frac{1}{2} \left(-\frac{m}{2} + i\frac{\varphi}{2} - 1 \right) \bar{z}, \\
 A_2 &= +\frac{i}{2} (1+z^2) \frac{\partial}{\partial z} + \frac{i}{2} (1+\bar{z}^2) \frac{\partial}{\partial \bar{z}} \\
 &\quad - \frac{i}{2} \left(\frac{m}{2} + i\frac{\varphi}{2} - 1 \right) z - \frac{i}{2} \left(-\frac{m}{2} + i\frac{\varphi}{2} - 1 \right) \bar{z}, \\
 A_3 &= z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} - \frac{m}{2},
 \end{aligned}$$

$$\begin{aligned}
 B_1 &= \frac{i}{2} (1-z^2) \frac{\partial}{\partial z} + \frac{i}{2} (1-\bar{z}^2) \frac{\partial}{\partial \bar{z}} \\
 &\quad + \frac{i}{2} \left(\frac{m}{2} + i\frac{\varphi}{2} - 1 \right) z + \frac{i}{2} \left(-\frac{m}{2} + i\frac{\varphi}{2} - 1 \right) \bar{z}, \\
 B_2 &= -\frac{1}{2} (1+z^2) \frac{\partial}{\partial z} + \frac{1}{2} (1+\bar{z}^2) \frac{\partial}{\partial \bar{z}} \\
 &\quad + \frac{1}{2} \left(\frac{m}{2} + i\frac{\varphi}{2} - 1 \right) z - \frac{1}{2} \left(-\frac{m}{2} + i\frac{\varphi}{2} - 1 \right) \bar{z}, \\
 B_3 &= +iz \frac{\partial}{\partial z} + i\bar{z} \frac{\partial}{\partial \bar{z}} - i \left(i\frac{\varphi}{2} - 1 \right).
 \end{aligned}$$

Now we consider representations based on homogeneous functions. In this case Euler's identity for homogeneous functions tells us that

$$\begin{aligned}
 \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} &= \lambda, \\
 \eta_1 \frac{\partial}{\partial \eta_1} + \eta_2 \frac{\partial}{\partial \eta_2} &= \mu
 \end{aligned}$$

with λ and μ being related to m, φ by

$$\begin{aligned}
 \lambda &= -\frac{m}{2} + i\frac{\varphi}{2} - 1, \\
 \mu &= +\frac{m}{2} + i\frac{\varphi}{2} - 1.
 \end{aligned}$$

We eliminate the operations $\partial/\partial \xi_2, \partial/\partial \eta_2$ by these identities and rewrite the expressions for the generators A_i, B_i with

$$\bar{z} = \frac{\eta_1}{\eta_2}$$

$$\bar{z} = \frac{\xi_1}{\xi_2}$$

Our result is then identical with the one of Gelfand and Neumark.

6. UNITARY REPRESENTATIONS FOR $SL(n, \mathbb{C})$

The regular representation

$$\forall_a f(\eta) = f(\eta a), \quad \eta \in SL(n, \mathbb{C})$$

is in general a reducible representation. We can obtain irreducible representations if we restrict the class of functions $f(\eta)$ to homogeneous functions with certain degrees of homogeneity. Another way of restricting the variety $\{f(\eta)\}$ consists in postulating constancy of the functions on each right coset of a fixed subgroup of matrices η . This is the method of Gelfand and Neumark. Their subgroup consists of upper triangular matrices

$$k = \begin{pmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ 0 & k_{22} & k_{23} & k_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & k_{nn} \end{pmatrix}$$

Let $\eta \in SL(n, \mathbb{C})$. We decompose it into

$$\eta = k \cdot z$$

(if possible) with z of the form

$$z = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ z_{21} & 1 & 0 & \dots & 0 & 0 \\ z_{31} & z_{32} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{n1} & & & & z_{nn-1} & 1 \end{pmatrix}$$

The right coset can thus be described by the complex parameters z_{pq} , $p > q$, $p, q = 1, 2, \dots, n$. If we then have a transformation $a \in SL(n, \mathbb{C})$

$$za = k_1 z'$$

the transformed class is defined to belong to z' . The explicit dependence of z' on z and " a " can be found in the book of Gelfand and Neumark ²⁾.

The transformation formula for the non-degenerate principal series looks like

$$\begin{aligned} V_a f(z_{pq}) = & g_2^{-\frac{m_1}{2} + i\frac{\rho_1}{2} - 1} g_3^{-\frac{m_2 - m_1}{2} + i\frac{\rho_2 - \rho_1}{2} - 1} \dots \\ & \cdot g_n^{-\frac{m_{n-1} - m_{n-2}}{2} + i\frac{\rho_{n-1} - \rho_{n-2}}{2} - 1} \\ & \bar{g}_2^{+\frac{m_1}{2} + i\frac{\rho_1}{2} - 1} \bar{g}_3^{+\frac{m_2 - m_1}{2} + i\frac{\rho_2 - \rho_1}{2} - 1} \dots \\ & \cdot \bar{g}_n^{+\frac{m_{n-1} - m_{n-2}}{2} + i\frac{\rho_{n-1} - \rho_{n-2}}{2} - 1} \\ & \cdot f(z'_{pq}) \end{aligned}$$

$f(z_{pq})$ is a square integrable function in the complex space of all parameters z_{pq} . ρ_k, m_k , $k = 1, 2, \dots, n-1$ are real and integer numbers respectively. The quantities g_k are subdeterminants taken from the matrix g

$$g = za = k^{-1} \eta',$$

$$g_k = \begin{vmatrix} g_{kk} & g_{k, k+1} & \dots & g_{kn} \\ g_{k+1, k} & & & \\ \vdots & & & \\ g_{nk} & & & g_{nn} \end{vmatrix}$$

Our aim is to express z_{pq} and g_k by the parameters $\Delta_{i_1 \dots i_k}^{(k)}(\eta)$. This task is quite simple for z_{pq} . A formula of Gelfand and Neumark ²⁾ [Chapter 4, Eq. (3.15)] tells us that

$$z_{pq} = \frac{\Delta_{q,p+1,\dots,n}^{(n-p+1)}(\eta)}{\Delta_{p,p+1,\dots,n}^{(n-p+1)}(\eta)}.$$

The parameters Δ used here are defined slightly differently than in Section 2. We must take the rows of η in the inverse order when we construct the fundamental representations

$$\begin{aligned}\Delta_i^{(1)}(\eta) &= \eta_{ni}, \\ \Delta_{ij}^{(2)}(\eta) &= \begin{vmatrix} \eta_{n-1,i} & \eta_{n-1,j} \\ \eta_{ni,i} & \eta_{ni,j} \end{vmatrix}, \text{ etc.}\end{aligned}$$

The problem to write g_p in terms of Δ 's is somewhat more laborious. We use a trick for its solution.

The values z'_{pq} must be independent of the matrix elements of the matrix k appearing in the decomposition

$$\eta = k \cdot z.$$

Therefore the following equation holds

$$z'_{pq} = \frac{\Delta_{q,p+1,\dots,n}^{(n-p+1)}(\eta')}{\Delta_{p,p+1,\dots,n}^{(n-p+1)}(\eta')} = \frac{\Delta_{q,p+1,\dots,n}^{(n-p+1)}(g)}{\Delta_{p,p+1,\dots,n}^{(n-p+1)}(g)},$$

where we put $k = 1$ in the transition from η' to g . By definition we have

$$g_p = \Delta_{p,p+1,\dots,n}^{(n-p+1)}(g).$$

This leads us to the ansatz

$$g_p = \text{const.} \Delta_{p, p+1 \dots n}^{(n-p+1)}(\gamma')$$

where const. does not depend on "a". For the identity matrix "a" we have

$$g_p = 1 = \text{const.} \Delta_{p, p+1 \dots n}^{(n-p+1)}(\gamma),$$

so we get the result

$$g_p = \frac{\Delta_{p, p+1 \dots n}^{(n-p+1)}(\gamma')}{\Delta_{p, p+1 \dots n}^{(n-p+1)}(\gamma)}.$$

The other consequence of our ansatz, namely

$$\Delta_{q, p+1 \dots n}^{(n-p+1)}(g) = \text{const.} \Delta_{q, p+1 \dots n}^{(n-p+1)}(\gamma')$$

implies in the limit $a \rightarrow 1$ the expression for z_{pq} in terms of Δ 's which we quoted above.

Inserting the results for z_{pq} and g_p into the transformation formula we find

$$V_a F(\gamma) = F(\gamma a)$$

with (take $m_0 = p_0 = 0$)

$$F(\gamma) = \prod_{p=1}^{n-1} \left[\Delta_{p+1, p+2 \dots n}^{(n-p)}(\gamma) \right]^{-\frac{m_p - m_{p-1}}{2} + i \frac{p_p - p_{p-1}}{2} - 1} \left[\Delta_{p+1, p+2 \dots n}^{(n-p)}(\bar{\gamma}) \right]^{+\frac{m_p - m_{p-1}}{2} + i \frac{p_p - p_{p-1}}{2} - 1} \\ \cdot f \left(\frac{\Delta_{q, r+2 \dots n}^{(n-r)}(\gamma)}{\Delta_{r+1, r+2 \dots n}^{(n-r)}(\gamma)}, \frac{\Delta_{s, t+2 \dots n}^{(n-t)}(\bar{\gamma})}{\Delta_{t+1, t+2 \dots n}^{(n-t)}(\bar{\gamma})} \right)$$

where q and s vary between

$$1 \leq q \leq r, \quad 1 \leq s \leq t.$$

The function $F(\eta)$ is the immediate generalization of the polynomials $P_{\{\lambda\}}^{\{\lambda\}}$ found in Section 2 to constitute a basis for representations of the group $SU(n)$. There are only some minor differences in the notation. The change in the definition of Δ 's (starting now with the last row of η) was already pointed out above. There is a similar change which concerns the order of the columns of η . The order of columns entered the expression $P_{\{\lambda\}}^{\{\lambda\}}$ during the process of eliminating superficial Δ 's. It can be altered arbitrarily.

Remembering our discussion of generalized tensors in Section 3 we expect a basis for the generalized tensors of the form (now in the new notation of this Section)

$$P^{\{\mu, \lambda\}}(\eta, \bar{\eta}) = \Delta_n^{(1)}(\eta)^{\mu_1} \Delta_{n-1, n}^{(2)}(\eta)^{\mu_2} \dots \Delta_{23 \dots n}^{(n-1)}(\eta)^{\mu_{n-1}} \\ \cdot \Delta_n^{(1)}(\bar{\eta})^{\lambda_1} \Delta_{n-1, n}^{(2)}(\bar{\eta})^{\lambda_2} \dots \Delta_{23 \dots n}^{(n-1)}(\bar{\eta})^{\lambda_{n-1}} \\ \cdot Z \left(\frac{\Delta_{n-1}^{(1)}(\eta)}{\Delta_n^{(1)}(\eta)}, \dots, \frac{\Delta_{134 \dots n}^{(n-1)}(\eta)}{\Delta_{234 \dots n}^{(n-1)}(\eta)}, \right. \\ \left. \frac{\Delta_{n-1}^{(1)}(\bar{\eta})}{\Delta_n^{(1)}(\bar{\eta})}, \dots, \frac{\Delta_{134 \dots n}^{(n-1)}(\bar{\eta})}{\Delta_{234 \dots n}^{(n-1)}(\bar{\eta})} \right),$$

where Z is a certain polynomial in both sets of variables and λ_k, μ_k are arbitrary complex numbers. Instead, from the theory of Gelfand and Neumark we found an expression where the degrees of homogeneity λ_k, μ_k are restricted by

$$\mu_k = -\frac{1}{2} (m_{n-k} - m_{n-k-1}) + \frac{i}{2} (\vartheta_{n-k} - \vartheta_{n-k-1}) - 1, \\ \lambda_k = +\frac{1}{2} (m_{n-k} - m_{n-k-1}) + \frac{i}{2} (\vartheta_{n-k} - \vartheta_{n-k-1}) - 1.$$

Together with the replacement of the polynomial Z by the square integrable function f these relations guarantee unitarity of our representations.

Now we consider rather briefly the principal degenerate series of representations. Let $n_1, n_2, \dots, n_r$ be a partition of n

$$\sum_{k=1}^r n_k = n.$$

Assume z to be a matrix composed of submatrices $z^{(p,q)}$ of n_p rows and n_q columns with

$$\begin{aligned} z^{(p,q)} &= 0 \quad \text{if } p < q, \\ z^{(p,q)} &= \text{identity matrix if } p = q. \end{aligned}$$

The elements of the big matrix will be denoted z_{st} as we did above. Similarly the matrix k consists of submatrices $k^{(p,q)}$ which are subject to the condition

$$k^{(p,q)} = 0 \quad \text{if } p > q.$$

Let η be decomposed in a product

$$\eta = k \cdot z$$

which is again nearly always possible. If $f(z_{st})$ is a function belonging to the Hilbert space of square integrable functions over the space of all the complex variables z_{st} , we define a representation by

$$\begin{aligned} \forall_a f(z_{st}) &= \prod_{k=1}^{r-1} \left[g_{n-e_k+1} \right]^{-\frac{m_k - m_{k-1}}{2} + i \frac{p_k - p_{k-1}}{2} - \frac{n_{k+1} + n_k}{2}} \\ &\quad \left[\bar{g}_{n-e_k+1} \right]^{+\frac{m_k - m_{k-1}}{2} + i \frac{p_k - p_{k-1}}{2} - \frac{n_{k+1} + n_k}{2}} \\ &\quad \cdot f(z'_{st}) \end{aligned}$$

The quantities g_k are the same as we used them above

$$g_k = \frac{\Delta_{k, k+1, \dots, n}^{(n-k+1)}(\eta')}{\Delta_{k, k+1, \dots, n}^{(n-k+1)}(\eta)}$$

The index l_k means

$$l_k = \sum_{s=k+1}^r n_s, \quad k = 1, 2, \dots, r-1.$$

The transformed matrix z' is defined by

$$\eta a = k z a = k_1 z'$$

A rather lengthy calculation allows us to express the parameters z_{st} by the Δ 's. We find

$$z_{n-n_r+p, q} = (-1)^{p+1} \frac{\Delta_{q, n-n_r+1, \dots, n}^{(n_r)}(\eta)}{\Delta_{n-n_r+1, \dots, n}^{(n_r)}(\eta)}$$

with the index $n - n_r + p$ left out in the numerator and p, q varying between

$$1 \leq p \leq n_r, \quad 1 \leq q \leq n - n_r;$$

$$z_{n-n_r-n_{r-1}+p, q} = (-1)^{p+1} \frac{\Delta_{q, n-n_r-n_{r-1}+1, \dots, n}^{(n_r+n_{r-1})}(\eta)}{\Delta_{n-n_r-n_{r-1}+1, \dots, n}^{(n_r+n_{r-1})}(\eta)},$$

with the index $n - n_r - n_{r-1} + p$ left out in the numerator and p, q varying between

$$1 \leq p \leq n_{r-1}, \quad 1 \leq q \leq n - n_r - n_{r-1}$$

etc.

The general rule can certainly be inspected from these examples. We insert the results in the transformation formula and get

$$\forall a \quad F(\eta) = F(\eta a)$$

with

$$F(\eta) = \prod_{p=1}^{r-1} \left[\Delta_{n-l_p+1, \dots, n}^{(\ell_p)}(\eta) \right]^{-\frac{m_p - m_{p-1}}{2} + i \frac{\rho_p - \rho_{p-1}}{2} - \frac{n_{p+1} + n_p}{2}} \\ \cdot \left[\Delta_{n-l_p+1, \dots, n}^{(\ell_p)}(\bar{\eta}) \right]^{+\frac{m_p - m_{p-1}}{2} + i \frac{\rho_p - \rho_{p-1}}{2} - \frac{n_{p+1} + n_p}{2}} \\ \cdot f(\dots)$$

We see that only $\Delta^{(\ell_p)}(\eta)$, $\Delta^{(\ell_p)}(\bar{\eta})$, $p = 1, 2, \dots, r-1$ are used for the construction of the basis. In the case $r = n$ we get back the formula for the non-degenerate case. We shall not discuss the supplementary series of representations. The final results are quite analogous.

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