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MULTILINEAR INVARIANT FORMS BUILT OF AN ARBITRARY NUMBER
OF UNITARY REPRESENTATIONS OF $SL(n, \mathbb{C})$

W. Rühl
CERN - Geneva

A B S T R A C T

We study a method to construct multilinear invariant forms from a given set of unitary irreducible representations of the group $SL(n, \mathbb{C})$. We use a global technique based on the theory of Gelfand and Neumark.

We discuss a general method to construct Clebsch-Gordan kernels in the Gelfand-Neumark theory. These kernels $K(x)$ enable us to form multilinear invariant forms from an arbitrary given number of unitary representations of the group $SL(n, \mathbb{C})$ by

$$\int K(x) \prod_{\alpha} f_{\alpha}(x^{(\alpha)}) dx^{(\alpha)} = \text{invariant}$$

where α runs over the set of representations $\alpha = 1, 2, \dots, M$, $x^{(\alpha)}$ are the Gelfand-Neumark variables (matrix elements of triangular matrices) and f_{α} are vectors of representation spaces. We use the original notations of Gelfand and Neumark ¹⁾ and the additional notations introduced in Refs. ^{2), 3)}.

We emphasize that the problem to find invariants for a set of representations D_{α} is independent of the problem to reduce the tensor product $D_1 \times D_2 \times \dots \times D_M$ into irreducible parts and taking the one-dimensional part out of it. Indeed, if D_1 is a unitary representation of the principal series of $SL(2, \mathbb{C})$ and D_2 its complex conjugate D_1^* then the invariant is simply the scalar product. On the other hand, the decomposition of $D_1 \times D_2$ does not contain the one-dimensional representation (see, e.g., Ref. ⁴⁾). This is due to the fact that in the process of closing the product space of D_1 and D_2 δ functionals are excluded :

$$\int f(x) \bar{g}(y) \delta(x-y) dx dy = \int f(x) \bar{g}(x) dx$$

whereas

$$\int f(x, y) \delta(x-y) dx dy$$

does in general not exist, since δ is not square integrable.

This fact can be expressed more generally using the notion "trace". For the unitary infinite-dimensional representations a trace can be defined as a function on the group with a δ -like singularity at the identity element. The trace of a tensor product is the product of traces. Splitting off a one-dimensional representation would mean

subtracting from the tensor product an infinite-dimensional matrix proportional to the identity matrix. The trace of this matrix would not exist.

In general we cannot expect to find all the invariants with the ansatz

$$\int K(x) \prod_{\alpha=1}^M f_{\alpha}(x^{(\alpha)}) dx^{(\alpha)} = \text{invariant}.$$

From the outset such invariants as the scalar products in representations of the principal series are excluded. To obtain also these invariants we have to proceed as follows. The manifold of matrices $\{x^{(\alpha)}\}$ can be decomposed into transitive domains. These are defined by the two properties :

- 1 - if $\{x^{(\alpha)}\}$ is contained in a domain the transformed quantity $\{x^{(\alpha)}\}_{\bar{g}}$ should be in the same domain for each element g of $SL(n, \mathbb{C})$;
- 2 - if $\{x^{(\alpha)}\}, \{y^{(\alpha)}\}$ lie in one domain, an element $g \in SL(n, \mathbb{C})$ exists such that $\{x^{(\alpha)}\}_{\bar{g}} = \{y^{(\alpha)}\}$.

These domains may have lower dimension than the whole manifold. Taking instead of $\prod_{\alpha} dx^{(\alpha)}$ a measure on these lower dimensional domains we obtain such invariants as the scalar product of the principal series.

The determination of transitive domains is a complicated algebraic problem. The solution depends on the degeneracies of the representations but not on the representations themselves.

We continue now the discussion of the ansatz we made for the invariants. The set of solutions we obtain for the kernel is often continuous. Let us write it $K_{\lambda}(x)$ with a parameter λ varying in a certain domain of a many-dimensional complex space. Each integral

$$K(x; g) = \int g(\lambda) K_{\lambda}(x) d\lambda$$

defines an invariant if $g(\lambda)$ is chosen suitably. As we shall see the dependence on λ is such that $K(x;g)$ can be interpreted as a Fourier-Laplace transform of $g(\lambda)$. Therefore the class of allowed "functions" $g(\lambda)$ includes in general tempered distributions.

SOLUTION OF THE ANSATZ FOR K

We start by recalling the definition of the quantities

$$\Delta_{i_1 \dots i_k}^{(k)} (\xi^{(\alpha)})$$

They form a basis for a representation of $SL(n, \mathbb{C})$ described by a Young tableau of one column with k blocks, (see Refs. 2), 3). For each (α) only those k occur which are allowed by the degeneracy of the representation (α) . Let the degeneracy be defined by the following decomposition of n (the label n contained in $SL(n, \mathbb{C})$)

$$\sum_{j=1}^{\tau_\alpha} n_j^{(\alpha)} = n$$

Then the number k must be contained in the set

$$n_{\tau_\alpha}^{(\alpha)}, n_{\tau_\alpha}^{(\alpha)} + n_{\tau_\alpha - 1}^{(\alpha)}, \dots, n - n_1^{(\alpha)}$$

From all the $\Delta^{(k)}(\alpha)^*$ we construct all possible independent invariants with respect to $SL(n, \mathbb{C})$. This can be done by means of ε tensors

$$\varepsilon_{i_1 \dots i_n} \varepsilon_{i_{n+1} \dots i_{2n}} \dots \varepsilon_{i_{(s-1)n+1} \dots i_{sn}} \cdot \Delta_{i_1 \dots i_{k_1}}^{(k_1)}(1) \Delta_{i_{k_1+1} \dots i_{k_1+k_2}}^{(k_2)}(2) \dots \Delta_{i_{sn-k_M+1} \dots i_{sn}}^{(k_M)}(M)$$

*) From now on we write simply $\Delta(\alpha)$ instead of $\Delta(\xi^{(\alpha)})$.

It is not necessary that all representations (α) take part simultaneously in this invariant. If a representation (α) is missing we define

$$k_{\alpha} = 0, \quad \Delta^{(0)}(\alpha) = 1.$$

As the next step we eliminate the superficial variables $\Delta(\alpha)$ and express the invariants by the Gelfand-Neumark variables $x_{pq}^{(\alpha)}$. The connection between $\Delta(\alpha)$ and $x_{pq}^{(\alpha)}$ was studied in (H) and the necessary formulae are given there. We apply those formulae to a simple example to illustrate the general method.

We consider : $\alpha = 1, 2, 3, \quad n = 6$

$$X^{(1)} = \begin{pmatrix} 1 & & & & & \\ x_{21}^{(1)} & 1 & & & & \\ x_{31}^{(1)} & 0 & 1 & & & \\ x_{41}^{(1)} & 0 & 0 & 1 & & \\ x_{51}^{(1)} & 0 & 0 & 0 & 1 & \\ x_{61}^{(1)} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X^{(2)} = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & \\ x_{61}^{(2)} & x_{62}^{(2)} & x_{63}^{(2)} & x_{64}^{(2)} & x_{65}^{(2)} & 1 \end{pmatrix},$$

$$X^{(3)} = \begin{pmatrix} 1 & & & & & \\ x_{21}^{(3)} & 1 & & & & \\ x_{31}^{(3)} & 0 & 1 & & & \\ x_{41}^{(3)} & 0 & 0 & 1 & & \\ x_{51}^{(3)} & 0 & 0 & 0 & 1 & \\ x_{61}^{(3)} & x_{62}^{(3)} & x_{63}^{(3)} & x_{64}^{(3)} & x_{65}^{(3)} & 1 \end{pmatrix}.$$

$x^{(1)}, x^{(2)}, x^{(3)}$ are triangular matrices with the degeneracies $(1,5), (5,1), (1,4,1)$. They were introduced in (R) in connection with the baryon and meson representations.

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The Δ variables are

$$\Delta_{i_1 \dots i_5}^{(5)}(1), \Delta_i^{(1)}(2), \Delta_i^{(1)}(3), \Delta_{i_1 \dots i_5}^{(5)}(3).$$

There exist three independent $SL(6, \mathbb{C})$ invariants.

$$1 - \frac{1}{5!} \varepsilon_{i_1 \dots i_6} \Delta_{i_1}^{(1)}(2) \Delta_{i_2 \dots i_6}^{(5)}(1) = \Delta_i^{(1)}(2) \Delta_i^{(5)}(1),$$

where $\Delta_i^{(5)} = \frac{1}{5!} \varepsilon_{i i_1 \dots i_5} \Delta_{i_1 \dots i_5}^{(5)}(3).$

From (H) and (R) we obtain

$$\Delta^p(1) = -\Delta^1(1) \cdot x_{p1}^{(1)}, \quad p=2, \dots, 6,$$

$$\Delta_p^{(1)}(2) = +\Delta_6^{(1)}(2) \cdot x_{6p}^{(2)}, \quad p=1, \dots, 5.$$

Thus we can write the invariant as

$$\Delta^1(1) \Delta_6^{(1)}(2) \cdot [x_{61}^{(2)} - x_{62}^{(2)} x_{21}^{(1)} - x_{63}^{(2)} x_{31}^{(1)} - x_{64}^{(2)} x_{41}^{(1)} - x_{65}^{(2)} x_{51}^{(1)} - x_{61}^{(1)}].$$

Let us call the bracket $-r$ as was done in (R).

2 - In complete analogy, we find

$$\frac{1}{5!} \varepsilon_{i_1 \dots i_6} \Delta_{i_1}^{(1)}(3) \Delta_{i_2 \dots i_6}^{(5)}(1)$$

$$= \Delta^1(1) \Delta_6^{(1)}(3) [x_{61}^{(3)} - x_{62}^{(3)} x_{21}^{(1)} - x_{63}^{(3)} x_{31}^{(1)} - x_{64}^{(3)} x_{41}^{(1)} - x_{65}^{(3)} x_{51}^{(1)} - x_{61}^{(1)}]$$

$$= \Delta^1(1) \Delta_6^{(1)}(3) \cdot (-t).$$

$$3 - \frac{1}{5!} \varepsilon_{i_1 \dots i_6} \Delta_{i_1}^{(1)}(2) \Delta_{i_2 \dots i_6}^{(5)}(3).$$

This case is somewhat more complicated because we know that

$$\Delta^p(3) = -\Delta^1(3) \times_{p1}^{(3)} \quad \text{for } p = 2, 3, 4, 5$$

only. The case $p = 6$ is fixed by the subsidiary condition

$$\sum_{p=1}^6 \Delta^p(3) \Delta_p^{(1)}(3) = 0,$$

which implies

$$\frac{\Delta^6(3)}{\Delta^1(3)} = -x_{61}^{(3)} + \sum_{p=2}^5 x_{6p}^{(3)} x_{p1}^{(3)}.$$

Thus, our invariant takes on the form

$$\begin{aligned} & \Delta_6^{(1)}(2) \Delta^1(3) \cdot \left[x_{61}^{(2)} - x_{62}^{(2)} x_{21}^{(3)} - x_{63}^{(2)} x_{31}^{(3)} - x_{64}^{(2)} x_{41}^{(3)} - \right. \\ & \quad \left. - x_{65}^{(2)} x_{51}^{(3)} - x_{61}^{(3)} + \sum_{p=2}^5 x_{6p}^{(3)} x_{p1}^{(3)} \right] \\ & = \Delta_6^{(1)}(2) \Delta^1(3) \cdot (+S). \end{aligned}$$

Additional invariants like

$$\varepsilon_{i_1 \dots i_6} \varepsilon_{i_7 \dots i_{12}} \Delta_{i_1}(2) \Delta_{i_2}(3) \Delta_{i_3 \dots i_7}(1) \Delta_{i_8 \dots i_{12}}(3)$$

can simply be reduced to products of r, s, t by using the well-known identity

$$\begin{aligned} a_{i_1} \varepsilon_{i_2 \dots i_7} &= a_{i_2} \varepsilon_{i_1 i_3 \dots i_7} + a_{i_3} \varepsilon_{i_2 i_1 i_4 \dots i_7} + \dots \\ &\quad + a_{i_7} \varepsilon_{i_2 \dots i_6 i_1} \end{aligned}$$

In general, we obtain in this way a set of functions

$$F_{[k_\alpha]}(x).$$

In our example, we found

$$r = F_{[5,1,0]},$$

$$s = F_{[0,1,5]},$$

$$t = F_{[5,0,1]}.$$

By simple calculation, we find that under a group element of $SL(n, \mathbb{C})$ these functions $F(x)$ transform as

$$F_{[k_\alpha]}(x') = F_{[k_\alpha]}(x) \left[\prod_{\alpha=1}^M g_{n-k_\alpha+1}(x^{(\alpha)}) \right]^{-1}$$

If k_α is zero, we define $g_{n+1}=1$. The functions $g_m(x)$ have been introduced by Gelfand and Neumark. They depend on the group element and on the variables $x_{pq}^{(\alpha)}$. Postulating invariance of the integral

$$\int K(x) \prod_{\alpha=1}^M f_\alpha(x^{(\alpha)}) dx^{(\alpha)}$$

for arbitrary functions f_α leads to the quasi-invariance condition on $K(x)$

$$K(x') = K(x) \cdot \prod_{\alpha=1}^M \prod_{j=2}^{r_\alpha} \frac{g_{n-k_\alpha(j)+1}(x^{(\alpha)})^{\mu_j^{(\alpha)}}}{g_{n-k_\alpha(j)+1}(x^{(\alpha)})^{\lambda_j^{(\alpha)}}}$$

with

$$\mu_j^{(\alpha)} = -\frac{1}{2}(m_j^{(\alpha)} - m_{j-1}^{(\alpha)}) + \frac{i}{2}(p_j^{(\alpha)} - p_{j-1}^{(\alpha)}) + \frac{1}{2}(n_j^{(\alpha)} + n_{j-1}^{(\alpha)}),$$

$$\lambda_j^{(\alpha)} = +\frac{1}{2}(m_j^{(\alpha)} - m_{j-1}^{(\alpha)}) + \frac{i}{2}(p_j^{(\alpha)} - p_{j-1}^{(\alpha)}) + \frac{1}{2}(n_j^{(\alpha)} + n_{j-1}^{(\alpha)}),$$

$$k_\alpha(2) = n - n_1^{(\alpha)}, \quad k_\alpha(3) = n - n_1^{(\alpha)} - n_2^{(\alpha)}, \dots, \quad k_\alpha(r_\alpha) = n - n_{r_\alpha}^{(\alpha)},$$

and $m_1^{(\alpha)} = p_1^{(\alpha)} = 0$ by definition.

This functional equation for $K(x)$ leads us to the following ansatz for $K(x)$

$$K(x) = \prod_{\text{all independent } F} F[k_\alpha]^{-A[k_\alpha]} \cdot \bar{F}[k_\alpha]^{-B[k_\alpha]}$$

The powers $A[k_\alpha]$ and $B[k_\alpha]$ can be determined by inserting this ansatz into the quasi-invariance condition and using the relation for $F(x')$ and $F(x)$. This procedure yields linear equations for A and B . The equations for B are obviously obtained from those for A by substituting all m by their negatives.

Let us assume that one of the functions f_α is localized to a small region in the $x^{(\alpha)}$ space. Moving around with the support on closed curves we would expect that the invariant takes on values which lie also on a closed curve in the complex plane. But this is in general not true. Indeed, for the "univaluedness" of the invariant to hold it is necessary that the cuts introduced by non-integer powers of $F[k_\alpha]$ are cancelled by the cuts of $\bar{F}[k_\alpha]$. This implies an additional condition on A and B :

$$A[k_\alpha] - B[k_\alpha] = \text{integer number.}$$

This univaluedness condition is important, when we obtain a continuous set of solutions $K_\lambda(x)$.

We continue now the discussion of the example. The linear equations for A read

$$\begin{aligned} A[5,1,0] + A[5,0,1] &= -\frac{1}{2} m_2^{(1)} + \frac{i}{2} p_2^{(1)} + 3, \\ A[5,1,0] + A[0,1,5] &= -\frac{1}{2} m_2^{(2)} + \frac{i}{2} p_2^{(2)} + 3, \\ A[5,0,1] &= -\frac{1}{2} (m_3^{(3)} - m_2^{(3)}) + \frac{i}{2} (p_3^{(3)} - p_2^{(3)}) + \frac{5}{2}, \\ A[0,1,5] &= -\frac{1}{2} m_2^{(3)} + \frac{i}{2} p_2^{(3)} + \frac{5}{2}. \end{aligned}$$

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From this set of equations we deduce

$$A_{[5,1,0]} = -\frac{1}{2}(m_2^{(2)} - m_2^{(3)}) + \frac{1}{2}(\rho_2^{(2)} - \rho_2^{(3)}) + \frac{1}{2}$$

and two constraints, the ρ 's and m 's must satisfy, namely

$$m_2^{(1)} - m_2^{(2)} - m_3^{(3)} + 2m_2^{(3)} = 0,$$

$$\rho_2^{(1)} - \rho_2^{(2)} - \rho_3^{(3)} + 2\rho_2^{(3)} = 0.$$

We emphasize that these restrictions on possible values of m and ρ are a consequence of the degeneracy. For non-degenerate representations such restrictions do not exist. The invariant is univalued automatically in this case.

This example tells us that in general the integral over $dx^{(\alpha)}$ does not exist because of singularities in the kernel. The absolute value of the kernel is namely

$$|K(x)| = |F_{[5,0,1]}|^{-5} \cdot |F_{[0,1,5]}|^{-5} \cdot |F_{[5,0,1]}|^{-1}.$$

Therefore the curves on which $F_{[5,0,1]}$ or $F_{[0,1,5]}$ vanish are singular curves and bring in a divergence of the integral. This divergence is also a consequence of the degeneracy. It can be shown that the kernel for non-degenerate representations is free of such singularities, at least for a certain region of λ .

Let us investigate now another example with a continuous set of kernels $K_\lambda(x)$. Let us take

$$X^{(\alpha)} = \begin{pmatrix} 1 & & & & & \\ x_{21}^{(\alpha)} & 1 & & & & \\ x_{31}^{(\alpha)} & 0 & 1 & & & \\ x_{41}^{(\alpha)} & 0 & 0 & 1 & & \\ x_{51}^{(\alpha)} & 0 & 0 & 0 & 1 & \\ x_{61}^{(\alpha)} & x_{62}^{(\alpha)} & x_{63}^{(\alpha)} & x_{64}^{(\alpha)} & x_{65}^{(\alpha)} & 1 \end{pmatrix}, \alpha = 1, 2, 3.$$

The ansatz is now

$$K = F_{[5,0,1]}^{-A_{[5,0,1]}} \cdot F_{[0,1,5]}^{-A_{[0,1,5]}} \cdot F_{[1,5,0]}^{-A_{[1,5,0]}} \\ F_{[1,0,5]}^{-A_{[1,0,5]}} \cdot F_{[0,5,1]}^{-A_{[0,5,1]}} \cdot F_{[5,1,0]}^{-A_{[5,1,0]}} \\ \times \text{factors } \bar{F} \text{ with powers } -B.$$

This ansatz yields the set of linear equations

$$A_{[5,0,1]} + A_{[5,1,0]} = -\frac{1}{2} m_2^{(1)} + \frac{i}{2} p_2^{(1)} + \frac{5}{2},$$

$$A_{[1,5,0]} + A_{[0,5,1]} = -\frac{1}{2} m_2^{(2)} + \frac{i}{2} p_2^{(2)} + \frac{5}{2},$$

$$A_{[0,1,5]} + A_{[1,0,5]} = -\frac{1}{2} m_2^{(3)} + \frac{i}{2} p_2^{(3)} + \frac{5}{2},$$

$$A_{[5,0,1]} + A_{[0,5,1]} = -\frac{1}{2} (m_3^{(3)} - m_2^{(3)}) + \frac{i}{2} (p_3^{(3)} - p_2^{(3)}) + \frac{5}{2},$$

$$A_{[5,1,0]} + A_{[0,1,5]} = -\frac{1}{2} (m_3^{(2)} - m_2^{(2)}) + \frac{i}{2} (p_3^{(2)} - p_2^{(2)}) + \frac{5}{2},$$

$$A_{[1,5,0]} + A_{[1,0,5]} = -\frac{1}{2} (m_3^{(1)} - m_2^{(1)}) + \frac{i}{2} (p_3^{(1)} - p_2^{(1)}) + \frac{5}{2}.$$

Adding the first three equations and the fourth to sixth equations we find again a restriction on the m 's and ρ 's :

$$\sum_{\alpha} (2m_2^{(\alpha)} - m_3^{(\alpha)}) = 0,$$

$$\sum_{\alpha} (2\rho_2^{(\alpha)} - \rho_3^{(\alpha)}) = 0.$$

The linear equations can be solved by

$$A_{[5,1,0]} = -\frac{1}{6} (m_2^{(1)} - m_2^{(2)} + m_3^{(2)} - m_3^{(3)}) + \frac{1}{6} (\rho_2^{(1)} - \rho_2^{(2)} + \rho_3^{(2)} - \rho_3^{(3)}) + \lambda,$$

$$A_{[1,0,5]} = -\frac{1}{6} (m_2^{(3)} - m_2^{(1)} + m_3^{(1)} - m_3^{(2)}) + \frac{1}{6} (\rho_2^{(3)} - \rho_2^{(1)} + \rho_3^{(1)} - \rho_3^{(2)}) + \lambda,$$

$$A_{[0,5,1]} = -\frac{1}{6} (m_2^{(2)} - m_2^{(3)} + m_3^{(3)} - m_3^{(1)}) + \frac{1}{6} (\rho_2^{(2)} - \rho_2^{(3)} + \rho_3^{(3)} - \rho_3^{(1)}) + \lambda,$$

$$A_{[1,5,0]} = -\frac{1}{6} (m_2^{(2)} - m_2^{(1)} + m_3^{(1)} - m_3^{(3)}) + \frac{1}{6} (\rho_2^{(2)} - \rho_2^{(1)} + \rho_3^{(1)} - \rho_3^{(3)}) + \lambda',$$

$$A_{[5,0,1]} = -\frac{1}{6} (m_2^{(1)} - m_2^{(3)} + m_3^{(3)} - m_3^{(2)}) + \frac{1}{6} (\rho_2^{(1)} - \rho_2^{(3)} + \rho_3^{(3)} - \rho_3^{(2)}) + \lambda',$$

$$A_{[0,1,5]} = -\frac{1}{6} (m_2^{(3)} - m_2^{(2)} + m_3^{(2)} - m_3^{(1)}) + \frac{1}{6} (\rho_2^{(3)} - \rho_2^{(2)} + \rho_3^{(2)} - \rho_3^{(1)}) + \lambda'.$$

Putting these expressions into the original equations yields

$$\lambda + \lambda' = -\frac{1}{6} \sum_{\alpha} m_2^{(\alpha)} + \frac{1}{6} \sum_{\alpha} \rho_2^{(\alpha)} + \frac{5}{2}.$$

Similarly we find for the exponents B (by substituting $m \rightarrow -m$, $\lambda \rightarrow \mu$, $\lambda' \rightarrow \mu'$)

$$\mu + \mu' = +\frac{1}{6} \sum_{\alpha} m_2^{(\alpha)} + \frac{1}{6} \sum_{\alpha} \rho_2^{(\alpha)} + \frac{5}{2}.$$

We introduce new variables M, M', R, R' by

$$\begin{aligned}\lambda &= -\frac{1}{2} M + \frac{i}{2} R, \lambda' = -\frac{1}{2} M' + \frac{i}{2} R', \\ \mu &= +\frac{1}{2} M + \frac{i}{2} R, \mu' = +\frac{1}{2} M' + \frac{i}{2} R' .\end{aligned}$$

The condition of unvaluedness implies

$$\begin{aligned}M + \frac{1}{3} (m_2^{(\alpha-1)} - m_2^{(\alpha)} + m_3^{(\alpha)} - m_3^{(\alpha+1)}) &= \text{integer}, \\ M' + \frac{1}{3} (m_2^{(\alpha+1)} - m_2^{(\alpha)} + m_3^{(\alpha)} - m_3^{(\alpha-1)}) &= \text{integer} .\end{aligned}$$

for each value of α . R and R' are arbitrary complex. From the equations for $\lambda + \lambda'$ and $\mu + \mu'$ we deduce

$$\begin{aligned}M + M' &= \frac{1}{3} \sum_{\alpha} m_2^{(\alpha)}, \\ R + R' &= \frac{1}{3} \sum_{\alpha} \rho_2^{(\alpha)} - 5i .\end{aligned}$$

Thus, a univalued invariant exists if

$$\begin{aligned}\sum_{\alpha} m_2^{(\alpha)} - (m_2^{(\beta-1)} - m_2^{(\beta)} + m_3^{(\beta)} - m_3^{(\beta+1)}) \\ - (m_2^{(\gamma+1)} - m_2^{(\gamma)} + m_3^{(\gamma)} - m_3^{(\gamma-1)}) \equiv 0 \pmod{3}\end{aligned}$$

for each β, γ . This condition is equivalent to

$$\begin{aligned}\sum_{\alpha} m_2^{(\alpha)} &\equiv 0 \pmod{3}, \\ \sum_{\alpha} m_3^{(\alpha)} &\equiv 0 \pmod{3} .\end{aligned}$$

If these conditions are satisfied: M and R vary over a real discrete and a complex continuous manifold. If we allow multivalued invariants, λ and μ may range independently over the complex plane.

The singularities of the kernel are fixed by the numbers $\text{Re}(A+B)$. We obtain

$$\text{Re}(\lambda + \mu) = -\text{Im} R,$$

$$\text{Re}(\lambda' + \mu') = -\text{Im} R'.$$

Since

$$-\text{Im}(R + R') = +5$$

at least one of $-\text{Im} R$ and $-\text{Im} R'$ is $\geq \frac{5}{2}$. In general we would say that this implies a divergent integral. Absolute convergence needs indeed

$$\text{Re}(\lambda + \mu) < 2,$$

$$\text{Re}(\lambda' + \mu') < 2.$$

But in some cases simple convergence may occur as a consequence of oscillating terms. Let us write

$$z = A [5, 1, 0]$$

and our integral formally as

$$\int f(z) \cdot z^{+\frac{1}{6}(m) - \frac{i}{6}(p) + \frac{1}{2}M - \frac{i}{2}R} \cdot \bar{z}^{-\frac{1}{6}(m) - \frac{i}{6}(p) - \frac{1}{2}M - \frac{i}{2}R} dz$$

$$= \int f(z) |z|^{-\frac{1}{3}(p) - iR} e^{i\varphi_2(M + \frac{1}{3}(m))} |z| d|z| d\varphi_z.$$

(p) and (m) are abbreviations for

$$(m_2^{(1)} - m_2^{(2)} + m_3^{(2)} - m_3^{(3)}), \quad (p_2^{(1)} - p_2^{(2)} + p_3^{(2)} - p_3^{(3)}),$$

respectively. Convergence or divergence of the integral depends on the regularity of $f(z)$ at $z = 0$. If $f(z)$ allows an expansion

$$f(z) = f(0) + z f_z(0) + \bar{z} f_{\bar{z}}(0) + O(|z|)$$

where

$$\lim_{|z| \rightarrow 0} \frac{O(|z|)}{|z|} = 0,$$

we can choose M such that

$$M + \frac{1}{3}(m) \neq 0, \text{ integer}$$

Instead of $f(z)$ we insert $f(z) - f(0)$ into the integral, which lowers the degree of the singularity by one giving rise to a convergent integral. If a similar consideration can be done at all six singularities, we get a convergent integral if

$$2 < -\operatorname{Im} R < 3,$$

$$2 < -\operatorname{Im} R' < 3.$$

M, M' should be chosen so that $M + \frac{1}{3}(m)$, $M' + \frac{1}{3}(m)$ is different from zero for all six choices of (m) (three for M , three for M'). But the convergence depends on the order of integration.

The conditions for univaluedness

$$\sum_{\alpha} m_2^{(\alpha)} \equiv 0 \pmod{3},$$

$$\sum_{\alpha} m_3^{(\alpha)} \equiv 0 \pmod{3}$$

are not trivial in this example. Take all ρ 's equal to zero and

$$m_2^{(1)} = -3, \quad m_2^{(2)} = 0, \quad m_2^{(3)} = -1,$$

$$m_3^{(1)} = -3, \quad m_3^{(2)} = -3, \quad m_3^{(3)} = -2.$$

These are representations which can be ascribed to antibaryons, baryons and mesons. Their lowest SU(6) multiplets are $\overline{56}$, 56 and 35. They violate the conditions. If instead we use

$$m_2^{(3)} = 0, \quad m_3^{(3)} = 0$$

in which case the mesons start with an SU(6) singlet, a univalued invariant exists.

As a conclusion, we mention an additional condition which the invariants m must satisfy. It did not turn out in our formalism because it is a consistency requirement on the functional equation connecting $K(x')$ and $K(x)$. We find it if we transform with a group element of $SL(n, \mathbb{C})$ which is proportional to the unit matrix. Then necessarily $x' = x$ and the ratio $\frac{K(x')}{K(x)}$ is one. In our examples this condition would have given the results

$$m_2^{(1)} - m_2^{(2)} - m_3^{(3)} + 2m_2^{(3)} \cong 0,$$

or

$$\sum_{\alpha} m_2^{(\alpha)} \cong \sum_{\alpha} (m_3^{(\alpha)} - m_2^{(\alpha)}) \pmod{6},$$

which are covered by stronger restrictions. The weaker conditions "mod 6" are important for the coupling of non-degenerate representations.

R E F E R E N C E S

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