CS

CM-P00057299

THE PARITY TRANSFORMATION FOR A SYMMETRY MODEL WHICH
USES UNITARY REPRESENTATIONS OF THE GROUP $SL(n, \mathbb{C})$

W. Rühl
CERN - Geneva

ABSTRACT

We study the parity operation for a model which involves infinite multiplets and is based on groups $SL(n, \mathbb{C})$. The postulate that the parity reflection should not double the representation space leads to restrictions on the invariants of the group. The investigation makes use of the theory of Gelfand and Neumark. The connection with the tensor calculus is discussed.

1. INTRODUCTION

Defining a parity transformation in connection with a non-compact group $SL(n, C)$ requires a physical interpretation of this group which justifies the notation "parity" for the reflection transformation considered. We have always a symmetry model in mind which is of the type proposed recently by Fronsdal ¹⁾ and is based on a group (n is an even number)

$$\mathcal{L} \approx (T_4 \otimes SL(n, C)).$$

$\mathcal{L}$ is meant here to be the homogeneous Lorentz group, T_4 is the Abelian group of translations in Minkowski space and $SL(n, C)$ is the dynamical symmetry group, the operations of which commute with the translations. The direct product of the translation group and the dynamical symmetry group is the invariant subgroup of the whole group.

There exists an important subgroup of the symmetry group $SL(n, C)$, namely $SL(2, C) \otimes SU(n/2)$ ²⁾. The factor $SL(2, C)$ is called the dynamical spin group. Its generators $S_{\mu\nu}$ commute with the generators $M_{\mu\nu}$ of the Lorentz group in the familiar way

$$[M_{\mu\nu}, S_{\xi\sigma}] = -i (g_{\mu\xi} S_{\nu\sigma} + g_{\nu\sigma} S_{\mu\xi} - g_{\mu\sigma} S_{\nu\xi} - g_{\nu\xi} S_{\mu\sigma}).$$

We introduce now a parity transformation

$$P, P^2 = 1$$

and require that

$$[S_{ke}, P]_- = 0, [S_{0k}, P]_+ = 0$$

$e, k = 1, 2, 3.$

This definition is in accord with the interpretation of the spin group $SL(2, C)$. Similarly, we require that for any $S \in SU(n/2)$

$$[S, P]_- = 0.$$

2.

As in reference ³⁾ we consider the set of generators of a representation of $SL(n, C)$

$$L_{ij}, \tilde{L}_{ij}$$

defined by the infinitesimal group element (ε_{ij} and η_{ij} are real) :

$$U_g = 1 + \sum_{ij} (\varepsilon_{ij} L_{ij} + \eta_{ij} \tilde{L}_{ij}).$$

In addition, we use the linear combinations

$$M_{ij} = \frac{1}{2} (L_{ij} + i \tilde{L}_{ij}),$$

$$N_{ij} = \frac{1}{2} (L_{ij} - i \tilde{L}_{ij}).$$

Extending the definition of P to the whole group, we postulate that the parity transformation commutes with the generators of the maximal compact subgroup

$$L_{ij} - L_{ji} = M_{ij} + N_{ij} - M_{ji} - N_{ji},$$

$$\tilde{L}_{ij} + \tilde{L}_{ji} = -i (M_{ij} - N_{ij} + M_{ji} - N_{ji}).$$

It anticommutes with the following generators of non-compact one-dimensional subgroups, which we shall call later on the "non-compact generators"

$$L_{ij} + L_{ji} = M_{ij} + N_{ij} + M_{ji} + N_{ji},$$

$$\tilde{L}_{ij} - \tilde{L}_{ji} = -i (M_{ij} - N_{ij} - M_{ji} + N_{ji}).$$

The commutation relations can be expressed in the simple form

$$M_{ij} P = - P N_{ji},$$

$$N_{ij} P = - P M_{ji}.$$

Let U_g be a unitary operator representing the matrix element $g \in SL(n, \mathbb{C})$. We shall consider throughout only unitary representations. If g is infinitesimal, U_g has the form

$$U_g = 1 + \sum_{i,j} \{ (\varepsilon_{ij} - i\eta_{ij}) M_{ij} + (\varepsilon_{ij} + i\eta_{ij}) N_{ij} \}$$

Applying the parity operation twice, we obtain

$$P U_g P = 1 + \sum_{i,j} \{ (\varepsilon'_{ij} - i\eta'_{ij}) M_{ij} + (\varepsilon'_{ij} + i\eta'_{ij}) N_{ij} \}$$

with

$$\varepsilon'_{ij} = -\varepsilon_{ji},$$

$$\eta'_{ij} = +\eta_{ji}.$$

Thus for general elements $g \in SL(n, \mathbb{C})$, we find

$$P U_g P = U_{(g^+)^{-1}}.$$

2. THE EQUIVALENCE THEOREM

We investigate now the following problem of purely mathematical character (n may now also be odd). When are the representations $V_g = U_{(g^+)^{-1}}$ and U_g equivalent?

We answer this question relying on the equivalence theorem contained in the book of Gelfand and Neumark⁴⁾. It states (for the non-degenerate principal series, the other cases are treated analogously):

"If we have two representations V_g, U_g of $SL(n, \mathbb{C})$ belonging to the sets of parameters m_k, φ_k and M_k, R_k , respectively, where k runs from 1 to n and the superficial parameters m_1, φ_1, M_1, R_1 are fixed by the conditions

$$\begin{aligned}\sum_{k=1}^n \varphi_k &= \sum_{k=1}^n R_k = 0, \\ 0 &\leq \sum_{k=1}^n m_k < n, \\ 0 &\leq \sum_{k=1}^n M_k < n,\end{aligned}$$

then these representations are equivalent if and only if a permutation of the n indices exists

$$k \xrightarrow{\pi} \pi(k) \quad k = 1, 2, \dots, n$$

such that

$$\begin{aligned}m_{\pi(k)} &= M_k, \\ \varphi_{\pi(k)} &= R_k.\end{aligned}$$

We emphasize that this theorem can only be applied to a pair of representations, if both are defined by a transformation formula of the Gelfand-Neumark type

$$\begin{aligned}U_g f(z_{pq}) &= \prod_{k=2}^n q_k^{-\frac{1}{2}(m_k - m_{k-1}) + \frac{i}{2}(\varphi_k - \varphi_{k-1}) - 1} \\ &\quad \cdot \bar{q}_k^{\frac{1}{2}(m_k - m_{k-1}) + \frac{i}{2}(\varphi_k - \varphi_{k-1}) - 1} \\ &\quad \cdot f(z'_{pq})\end{aligned}$$

[For the notations, see Ref. 4), the function $f(z_{pq})$ belongs to the Hilbert space $\mathcal{L}^2(R_{n(n-1)})$.] For an arbitrary pair U_g, V_g this must not be true and is not true for our particular example $V_g = U_{(g^+)-1}$. The set of all unitary representations which are equivalent to a given one is infinite. At most $n!$ ones of them can be defined by a transformation formula of the Gelfand-Neumark type. Therefore

we have to find a unitary similarity transformation which is constructed in such a way that it brings the representation V_g in a form which allows the application of the equivalence theorem. This is not very difficult.

First, we introduce the permutation S :

$$S = (-1)^{\frac{n-1}{2}} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & & 1 & 0 \\ \vdots & & & \vdots & \\ 0 & 1 & \dots & & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$S^+ = S^{-1}.$$

The phase factor $(-1)^{n-1/2}$ guarantees that

$$\det S = 1, \quad S \in SL(n, \mathbb{C}).$$

Thus $U_S = V_S$ exists and can be used as a unitary similarity transformation

$$V_g \rightarrow W_g = V_S V_g V_S^{-1} = U_{S(g^+)^{-1}} S^{-1}$$

The operators W_g and U_g are equivalent if and only if V_g and U_g are equivalent representations.

For the next step it is convenient to write down explicitly the arguments of the quantities z' and g_k appearing in the transformation formula

$$z' = z'(z, g), \quad g_k = g_k(z, g).$$

Then we perform a point transformation

$$z = S^{-1}(\xi^+)^{-1} S$$

(ξ is a matrix of the same type as z) which is accompanied by a Jacobian

$$|\det \frac{\partial z}{\partial \xi}|^2 = 1.$$

6.

It corresponds therefore to a unitary similarity transformation in the space $\mathcal{L}^2(\mathbb{R}_{n(n-1)})$. We define a new function h by

$$f(z) = h(\xi).$$

With the abbreviations

$$g' = S g S^{-1}, \quad S (k^+)^{-1} S^{-1} = k',$$

where k' is a matrix of the same type as k , namely upper triangular, we find

$$\begin{aligned} z \cdot g &= k \cdot z' \\ &= S^{-1} (\xi^+)^{-1} g' S \\ &= k \cdot S^{-1} (\xi'^+)^{-1} S \end{aligned}$$

or

$$\xi (g'^+)^{-1} = k' \cdot \xi'$$

and

$$\begin{aligned} g_k &= g_k (S^{-1} (\xi^+)^{-1} g' S) \\ &= \bar{g}_{n-k+1} (\xi (g'^+)^{-1}) \end{aligned}$$

The last equation can be proved after some algebraic calculation. Inserting these results into our original transformation formula for U_g we find

$$\begin{aligned} W_g h(z) &= \prod_{k=2}^n g_k^{-\frac{1}{2}(M_k - M_{k-1}) + \frac{i}{2}(R_k - R_{k-1}) - 1} \\ &\quad \cdot \bar{g}_k^{+\frac{1}{2}(M_k - M_{k-1}) + \frac{i}{2}(R_k - R_{k-1}) - 1} \\ &\quad \cdot h(z') \end{aligned}$$

where

$$z \cdot g = k' \cdot z'$$

and M_k, R_k are related to ρ_k, m_k by

$$M_{n-k+2} - M_{n-k+1} = -(m_k - m_{k-1})$$

$$R_{n-k+2} - R_{n-k+1} = +(\varphi_k - \varphi_{k-1}).$$

These equations can be solved uniquely and give

$$m_k = + M_{n+1-k},$$

$$\varphi_k = - R_{n+1-k}.$$

Thus the parameters M_k are always a permutation of the parameters m_k . With respect to the parameters φ_k there are several different cases to be discussed.

- 1) If all m_k are different, we find uniquely

$$\pi(k) = n+1-k.$$

Equivalence of both representations requires

$$\begin{aligned} \varphi_k &= R_{\pi(k)} = R_{n+1-k} \\ &= -R_{n+1-k} = 0. \end{aligned}$$

In this case U_g and $U_{(g^+)^{-1}}$ are equivalent if and only if all parameters φ_k vanish.

- 2) If some m_k are equal, non-trivial solutions for the parameters φ_k may exist. As an example we consider

$$m_1 = m_n = M_1 = M_n.$$

Besides $\pi(1) = n$, $\pi(n) = 1$ there exists the solution :

$$\pi(1) = 1, \quad \pi(n) = n,$$

which leads to

$$\varphi_1 = -R_n = R_{\pi(1)} = R_1 = -\varphi_n,$$

and Q_1 may thus be different from zero. We call these solutions "singular".

We add only the remark that from $Q_k = 0$ for all k follows that the degrees of homogeneity λ_k, μ_k for the generalized tensors [see Ref. 3)] are real and obey

$$\lambda_k + \mu_k = -2.$$

We turn now to a brief discussion of the principal degenerate series of representations. The transformations

$$z = S^{-1} (\xi^+)^{-1} S$$

$$k' = S (k^+)^{-1} S^{-1}$$

do not lead to matrices ξ, k' which are of the same type as z and k . Indeed, z and k are triangular matrices belonging to a partition $n_1, n_2, \dots, n_r$ of n , ξ, k' are similar matrices but belong to the partition of inversed order, $n_r, n_{r-1}, \dots, n_1$. But the permutation of n_k is part of the equivalence theorem for the degenerate representations. This theorem correlates the parameters of two equivalent representations by a permutation $\pi(k)$, where

$$m_k = M_{\pi(k)},$$

$$\rho_k = R_{\pi(k)},$$

$$n_k = N_{\pi(k)}$$

and k runs from 1 to r . After an analogous deduction as above, we find for U_g and W_g

$$m_k = + M_{r+1-k},$$

$$\rho_k = - R_{r+1-k},$$

$$n_k = + N_{r+1-k}.$$

If all the m_k are different, we find again uniquely

$$\pi(k) = \tau + 1 - k.$$

The same result can be obtained if all the n_k are different. It implies that the parameters φ_k vanish. But there exist also singular solutions.

If, for example, $m_1 = m_r$ and $n_1 = n_r$ we may have $\varphi_1 = -\varphi_r \neq 0$. We remark that $\varphi_k = 0$ for all k implies now λ_k, μ_k to be real and

$$\lambda_{2k} + \mu_{2k} = -(n_{k+1} + n_k), \quad \ell_k = \sum_{s=k+1}^{\tau} n_s, \\ k = 1, 2, \dots, \tau-1.$$

We discuss now these results from a more physical point of view. The reason why we are interested in the equivalence of U_g and $U_{(g^+)^{-1}}$ is that in the case of equivalence the parity operator P is a unitary operator in the representation space $\mathcal{L}^2$ of the representation U_g . If the representations U_g and $U_{(g^+)^{-1}}$ are inequivalent, we have to double the Hilbert space. This is quite similar as for the spinor representations for the homogeneous Lorentz group, but with one important difference. In the case of the Lorentz group we have indeed an inhomogeneous group which supplies us with field equations like the Dirac equation. These field equations couple the representations U_g and $U_{(g^+)^{-1}}$ in such a way that the particle multiplets are not doubled. One of the essentials of Fronsdal's theory is that he treats the dynamical symmetry group $SL(n, \mathbb{C})$ as a homogeneous group, field equations do therefore not exist and we have to forbid parity doubling, a priori.

Within a representation of $SL(n, \mathbb{C})$ the generators of the compact subgroup $SU(n)$ have matrix elements only between states belonging to one representation of $SU(n)$. States belonging to different representations can only be connected by the non-compact generators. But as the parity operation commutes with the compact generators, we can ascribe a parity eigenvalue to each representation of $SU(n)$. Since the parity operation anticommutes with the non-compact generators, two different representations of $SU(n)$ must have opposite parity

eigenvalue if a non-vanishing matrix element of a non-compact generator exists between two states of the pair of representations considered.

Thus, we are led to the following geometrical model. Take a unitary representation of $SL(n, \mathbb{C})$ and reduce it into the representations of the compact subgroup. Each such representation, take it as a vertex of a network. Lines between vertices describe the possibility of connecting the two representations by non-vanishing matrix elements of non-compact generators. Necessary and sufficient for equivalence of U_g and $U_{(g^+)^{-1}}$ is the non-existence of closed loops with an odd number of vertices in this network.

3. BILINEAR INVARIANT FORMS

If we consider the representation U_g of $SL(n, \mathbb{C})$ the space of which is spanned by certain square integrable functions $f(z_{pq}), g(z_{pq})$, the existence of a bilinear invariant form is trivially connected with the unitarity of the representation. For the principal series, we get

$$(g, f) = \int \bar{g}(z_{pq}) \cdot f(z_{pq}) \, dz$$

Using the concept of generalized tensors, this problem is somewhat more involved and intimately connected with the parity transformation.

We consider again the case of classical spinors of finite dimension. If ψ_α is a spinor, $\bar{\psi}_\alpha$ is its complex conjugate. To construct a bilinear term, we need instead a contravariant spinor of the type $\tilde{\psi}^\alpha$. It is well known that the transition from $\bar{\psi}_\alpha$ to $\tilde{\psi}^\alpha$ is managed by the matrix β which is connected with the parity operation (in a doubled representation space, because of the non-equivalence of $\bar{\psi}_\alpha$ and $\tilde{\psi}^\alpha$). We study now a unitary generalized tensor of a degenerate principal series with

$$r = 2, \quad n_1 + n_2 = n > 2, \quad n_1 = 1$$

$$\psi_{\alpha_1 \dots \alpha_\mu, \beta_1 \dots \beta_\lambda}$$

The complex conjugate spinor is obviously

$$\bar{\psi}_{\beta_1 \dots \beta_\lambda, \alpha_1 \dots \alpha_\mu}$$

We know that there exists an invariant expression formed from $\bar{\psi}$ and ψ , namely that one of the Gelfand-Neumark theory

$$(\psi, \psi)$$

On the other hand, a spinor of the form

$$\tilde{\psi}_{\alpha_1 \dots \alpha_\mu, \beta_1 \dots \beta_\lambda}$$

will also enable us to construct an invariant form with ψ . If there exists a raising operator β .

$$\tilde{\psi} = \bar{\psi} \beta$$

so that lower dotted indices are transformed into upper undotted ones and lower undotted ones into upper dotted indices, both invariants are identical. A necessary condition for this operator β to exist is that the number of indices has to remain the same during the raising process. This means

$$\lambda = \bar{\lambda}, \\ \mu = \bar{\mu}$$

or

$$g_1 = g_2 = 0.$$

Thus we have found the non-singular solution of Section 2 by this heuristic procedure. What about the singular solutions? These are due to the existence of an additional raising operator, namely the ε -tensor.

R E F E R E N C E S

- 1) C. Fronsdal, Trieste preprint IC/65/68 (1965).
- 2) P. Budini and C. Fronsdal, Phys.Rev.Letters 14, 968 (1965).
- 3) W. Rühl, CERN preprint TH-618 (1965).
- 4) J.M. Gelfand and M.A. Neumark, "Unitäre Darstellungen der klassischen Gruppen", Berlin (1957).

The more elegant way of deriving our results makes use of the theory of characters developed in this book. We prefer the equivalence theorem since it seems to be true by intuition, whereas the notion of characters is non-trivial for infinitely dimensional representations.