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A RELATIVISTICALLY INVARIANT MODEL OF $SU(6)$ SYMMETRY
WHICH INVOLVES INFINITE REPRESENTATIONS OF THE
INTERNAL SYMMETRY GROUP

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A B S T R A C T

We study a relativistic $SU(6)$ symmetry model based on the group $SL(6, C)$. It uses infinite dimensional, unitary representations of this group as they are known from the theory of Gelfand and Neumark. We construct the states for mesons and baryons and compute the integral kernel which enables us to couple meson and baryon representations to an invariant vertex. There are no troubles with the unitarity of the S matrix in this model.

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I.- INTRODUCTION

This work is the third one in a series of papers ^{1),2)} devoted to a study of a relativistically invariant model of $SU(6)$ symmetry, which does not enlarge the dimension of the momentum space but instead introduces a non-compact symmetry group. The underlying group structure is of the type

$$\mathcal{L} \boxtimes (T_4 \otimes S)$$

which was proposed by Budini and Fronsdal ³⁾ and discussed to a certain extent with heuristic means by Fronsdal ⁴⁾. We understand this notation as follows. $\mathcal{L}$ is the homogeneous Lorentz group; T_4 the four dimensional group of translations which commutes with the internal symmetry group S .

The direct product of internal symmetry group S and translation group T_4 is the invariant subgroup of the complete group.

In our investigation we shall make use of a symmetry group

$$S = SL(6, C)$$

so that we can apply the rigorous mathematical means developed for $SL(n, C)$ groups by Gelfand and Neumark ⁵⁾.

In Ref. ¹⁾ we studied the notion of generalized tensors which was introduced by Fronsdal. We investigated its connection with the G.-N. theory thus putting generalized tensors on a rigorous mathematical basis. In Ref. ²⁾ we studied the restrictions on the eigenvalues of the invariants of the group $SL(n, C)$ which are implied by the parity postulate. This postulate states that under parity reflections the $SU(6)$ multiplets contained in one infinite representation

of $SL(6, C)$ should be eigenstates for the parity operator with an eigenvalue common to each multiplet. This assumption implies that the representation of $SL(6, C)$ need not be doubled for a parity reflection operation. Thus, parity belongs to a unitary operator in the representation space. We proved that in the case of the principal series this postulate leads to restrictions on the continuous parameters ρ . Apart from some singular cases they must vanish.

In this paper we start with an extensive discussion of the group structure of the model, its physical interpretation and the origin of the spin-orbit coupling. Most of the ideas of this part go back to the work of Fronsdal, Ref. 4). But we are here mainly interested in a technique to extract physical information from the abstract group model. First we turn to the problem of ascribing physical particles to states of representations. We construct explicitly the square integrable functions appearing in the G.-N. theory which belong to the states of baryons, antibaryons and mesons. By means of an integral kernel, a Clebsch-Gordan kernel, we couple the baryon functions to the meson function in an invariant manner. The Clebsch-Gordan kernel is computed in the Appendix. All our arguments can be generalized to other examples of representations and symmetry groups of the $SL(n, C)$ type, as for example $SL(2, C)$ and $SL(4, C)$. One of the most interesting problems to be studied in future is the relation of the present model with the old $SL(6, C)$ model ⁶⁾. So we suggest that also in this model hybrid subgroups for special kinematical conditions may exist.

II.- THE GROUP AND ITS REPRESENTATIONS

Let us write the generators of the internal symmetry group $SL(6, \mathbb{C})$ as

$$B_{\mu\nu, i} = S_{\mu\nu} \cdot \lambda_i, \quad i = 0, 1, \dots, 8. \quad (1)$$

If the generators of the homogeneous Lorentz group are written as $M_{\mu\nu}$ then the commutation relations are

$$[M_{\mu\nu}, B_{\lambda\kappa, i}] = -i (g_{\mu\lambda} B_{\nu\kappa, i} + g_{\nu\kappa} B_{\mu\lambda, i} - g_{\mu\kappa} B_{\nu\lambda, i} - g_{\nu\lambda} B_{\mu\kappa, i}). \quad (2)$$

From these relations, it follows that the differences

$$L_{\mu\nu} = M_{\mu\nu} - S_{\mu\nu} \quad (3)$$

commute with the generators of the internal symmetry group $SL(6, \mathbb{C})$. Let $M(p)$, be an operator which transforms a particle state from rest to the four momentum p . Then this operator is an element of the homogeneous Lorentz group. It can be decomposed into a product of two commuting factors

$$M(p) = \Lambda(p) \cdot \Sigma(p) \quad (4)$$

where $\Sigma(p)$ is an element of the group $SL(6, \mathbb{C})$ and $\Lambda(p)$ belong to the group $\mathcal{L}'$ which is generated by the $L_{\mu\nu}$'s and is isomorphic

4.

to the homogeneous Lorentz group $\mathcal{L}$. Such a factor decomposition can be performed in general and the group can then be written as

$$(\mathcal{L}' \cong T_4) \otimes SL(6, \mathbb{C}). \quad (5)$$

This way of writing the invariance group tells us how to find all its unitary representations. We form the Cartesian product of unitary representations of a group P' of the structure of the Poincaré group

$$P' = \mathcal{L}' \cong T_4 \quad (6)$$

and unitary representations of the group $SL(6, \mathbb{C})$ which are all known from Gelfand and Neumark.

At this point the question arises how the two different spin quantities which are introduced by the groups $\mathcal{L}'$ and $SL(2, \mathbb{C}) \subset SL(6, \mathbb{C})$ can be combined to one known kind of physical spin. We shall follow Fronsdaal at this point ^{4), 7)} and consider only scalar representations of the group P' . Thus we have representations

$$f(p, z)$$

where p is the four momentum and z an element of a subgroup of triangular matrices of $SL(6, \mathbb{C})$ on which the G.-N. theory is based. Under an operation of the invariance group consisting of

$$L \in \mathcal{L}'$$

$$a \in T_4$$

$$g \in SL(6, \mathbb{C})$$

we have a transformation

$$U(L, a, g) f(p, z) = e^{-i p a} \alpha(g, z) \cdot f(L^{-1} p, z'). \quad (7)$$

Here, $\alpha(g, z)$ is the so-called multiplier of the G.-N. theory. Besides the arguments indicated it depends on the invariant parameters m_k, φ_k of $SL(6, \mathbb{C})$. Let $n_k, k = 1, 2, \dots, r$ be a decomposition of 6.

$$\sum_{k=1}^r n_k = 6. \quad (8)$$

This set of numbers characterizes the degeneracy of the representations. This means that the matrices z are composed of block matrices $z^{(p,q)}$, $p, q = 1, 2, \dots, r$ consisting of n_p rows and n_q columns, with

$$z^{(p,q)} = 0, \quad \text{if } p < q,$$

$$z^{(p,q)} = \text{unit matrix}, \quad \text{if } p = q.$$

The transformed element z' is obtained by multiplying z from the right with g and splitting off an upper triangular matrix, k ,

$$z \cdot g = k \cdot z'$$

which consists of corresponding block matrices submitted to the condition

$$k^{(p,q)} = 0 \quad \text{if } p > q.$$

We introduce the determinants

$$g_k = \begin{vmatrix} (zg)_{kk} \cdots & (zg)_{kn} \\ \vdots & \vdots \\ (zg)_{nk} \cdots & (zg)_{nn} \end{vmatrix} \quad (9)$$

Then the multipliers $\alpha(z, g)$ takes the form

$$\alpha(z, g) = \prod_{k=1}^{r-1} g_{n-l_k+1}^{-\frac{1}{2}(m_{k+1}-m_k) + \frac{i}{2}(\varphi_{k+1}-\varphi_k) - \frac{1}{2}(n_{k+1}+n_k)} \cdot g_{n-l_k+1}^{+\frac{1}{2}(m_{k+1}-m_k) + \frac{i}{2}(\varphi_{k+1}-\varphi_k) - \frac{1}{2}(n_{k+1}+n_k)} \quad (10)$$

Here m_k are integers and φ_k real numbers, they are conveniently normalized by

$$m_1 = \varphi_1 = 0.$$

The index l_k is defined by

$$l_k = \sum_{s=k+1}^r n_s \quad (11)$$

The function $f(p, z)$ is square integrable in the sense that

$$\int |f(p, z)|^2 \delta(p^2 - m^2) d^4 p dz < \infty \quad (12)$$

The homogeneous Lorentz group does not commute with the symmetry groups $SL(6, \mathbb{C})$. Thus we consider a state in the system where $\vec{p} = 0$, the rest system. Then we define the physical $SU(6)$ multiplets to belong to the representations of the maximally compact subgroup $SU(6)$ in the rest system. This definition is relativistically invariant.

In the rest system the subgroup $SU(2)$ of $SU(6)$ transforms the spin variables. Take one spin $-\frac{1}{2}$ multiplet out of the infinite set of spin states. Then we can ascribe to it a Pauli spinor. If we apply a Lorentz transformation to this Pauli spinor which transforms the state in rest into a moving state, it is in general possible to describe the state thus reached by a Dirac spinor. But in our model such Lorentz transformations belong to operators $\sum_i (p_i)$ introduced in (4). They connect different $SU(6)$ multiplets and thus states with different spins

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \text{ etc.}$$

but do not touch the $SU(3)$ properties. From this deduction we see that the Dirac spinor will in our model appear decomposed into a series of angular momentum states which are all contained in the infinite multiplet. Let us assume first that only one $SU(6)$ multiplet in the rest system is filled with physical states. Then the other multiplets of the moving system are repetitions of this multiplet in higher orbital angular momentum states. This angular momentum is external. The ladder thus generated must not be confused with the spectrum generated by non-compact groups according to the idea of Barut⁸⁾. In models of the type Barut proposed the series of orbital angular momenta contained in one infinite representation must be considered as internal angular momentum of bound states, which does not vanish if the centre-of-mass of the bound state is at rest.

Thus, if we insist on one $SU(6)$ multiplet in the rest system, we get a relativistic form of $SU(6)$ symmetry which is rather analogous to the old theory. Since some states of an infinite multiplet remain empty, it violates unitarity. To avoid this defect we adapt Barut's point of view and consider all the $SU(6)$ multiplets of the rest system filled with physical particles. Of course, they have all the same mass. In a moving state we have then a superposition of an infinite number of ladders, e.g., superimposed a spin $\frac{1}{2}^+$ nucleon in a P state and a $\frac{3}{2}^-$ baryon resonance in an S state.

Let us again start our interpretation at a different point. If we have a static $SU(6)$ theory of the Gürsey-Radicati type ⁹⁾ based on the group $SO_3 \boxtimes SU(6)$, we must embed the rotation group $SO(3)$ into the homogeneous Lorentz group if we want a relativistically invariant model. Since $SU(6)$ is the invariant subgroup, closing the algebra forces us to enlarge the group $SU(6)$ to $SL(6, C)$. Therefore, the matrix elements of non-compact operators of $SL(6, C)$ between different $SU(6)$ multiplets are of the same order as the matrix elements of non-compact operators of the homogeneous Lorentz group between different spin multiplets, i.e., of order $\frac{v}{c}$. In the static limit the different $SU(6)$ multiplets decouple and we get a true $SU(6)$ theory with finite multiplets.

After this discussion, which shows that the model considered is intuitively suggestive to a high degree, we have to argue that some very important processes of strong interactions like

$$\begin{aligned}\rho &\rightarrow \pi + \pi, \\ N^* &\rightarrow N + \pi\end{aligned}$$

are really describable within our model. The spin-orbit coupling necessary for these decays comes in through the operators $\sum_i (p_i) \in SL(6, C)$. The separate spin conservation of the group

SU(6) if all particles are at rest, is destroyed by accelerating the pions applying the operator $\sum (p)$ to them. This operator transforms the pions into a linear combination of angular momentum states 0, 1, 2... which makes the coupling possible.

III.- THE CHOICE OF REPRESENTATIONS AND THE CONSTRUCTION OF STATES FOR MESONS AND BARYONS

In choosing the representations for mesons and baryons, we are led by the following principles :

- a) We take the principal series of representations. The reason is that only this series of representations allows all $SU(6)$ multiplets to be starting multiplets ¹⁰⁾ of unitary representations of $SL(6, C)$. The supplementary series involves also some mathematical troubles.
- b) We take the representations which contain the physical $SU(6)$ multiplet as a starting multiplet. After the discussion of Section II, this is plausible. This postulate fixes the parameters m_k .
- c) We take the maximally degenerate representation which is consistent with requirement (b). The reason is purely technical. We shall see that even in this case the dimension of the subgroups we have to integrate over is very high (38 real dimensions for the $\bar{B} B M$ - vertex, some of the integrations are certainly trivial ¹¹⁾).
- d) To fix the parameters φ_k we make use of the parity-non-doubling postulate.

The following results of Ref. ¹⁾ and Ref. ²⁾ are used explicitly. Given a representation m_k , φ_k with a certain degeneracy n_k , $k = 1, 2, \dots, r$ we can define a certain homogeneous function $F(\eta)$ (generalized tensor) corresponding to the square integrable function $f(z)$ of the G.-N. theory in the following way :

$$F(\eta) = \prod_{p=1}^{r-1} \left[\Delta_{n-\ell_p+1, \dots, n}^{(\ell_p)}(\eta) \right]^{-\frac{1}{2}(m_{p+1}-m_p) + \frac{i}{2}(\vartheta_{p+1}-\vartheta_p) - \frac{1}{2}(u_{p+1}+u_p)} \\ \cdot \left[\Delta_{n-\ell_p+1, \dots, n}^{(\ell_p)}(\bar{\eta}) \right]^{+\frac{1}{2}(m_{p+1}-m_p) + \frac{i}{2}(\vartheta_{p+1}-\vartheta_p) - \frac{1}{2}(u_{p+1}+u_p)} \\ \cdot f(z). \quad (13)$$

This relation is the main tool used for the construction of particle states. Let us discuss it. The indices ℓ_p and the exponents are the same as appearing in the multiplier function $\alpha(z, g)$ of Section II, Eqs. (10), (11). η is a matrix of $SL(6, 0)$ which is related to z by

$$\eta = k \cdot z$$

with arbitrary k . The Δ 's are constructed from η resp. $\bar{\eta}$ as

$$\Delta_{i_1 i_2 \dots i_k}^{(k)}(\eta) = \begin{vmatrix} \eta_{n-k+1, i_1} & \dots & \eta_{n-k+1, i_k} \\ \vdots & & \vdots \\ \eta_{n, i_1} & \dots & \eta_{n, i_k} \end{vmatrix} \quad (14)$$

The quantities Δ and z are related as follows :

$$z_{n-n_r+p, q} = (-1)^{p+1} \frac{\Delta_{q, n-n_r+1, \dots, n}^{(n_r)}(\eta)}{\Delta_{n-n_r+1, \dots, n}^{(n_r)}(\eta)} \quad (15)$$

where the index $n - n_r + p$ is left out in the numerator and the indices p, q run over

$$1 \leq p \leq n_r, \quad 1 \leq q \leq n - n_r;$$

$$Z_{n-n_r-n_{r-1}+p, q} = (-1)^{p+1} \frac{\Delta_{q, n-n_r-n_{r-1}+1, \dots, n}^{(n_r+n_{r-1})}(\eta)}{\Delta_{n-n_r-n_{r-1}+1, \dots, n}^{(n_r+n_{r-1})}(\eta)}, \quad (16)$$

where the index $n - n_r - n_{r-1} + p$ is left out in the numerator and the indices p, q run over

$$1 \leq p \leq n_{r-1}, \quad 1 \leq q \leq n - n_r - n_{r-1}.$$

It is obvious how these relations must be continued.

The "parity-non-doubling" postulate ends up in the following necessary and sufficient condition. There must exist a permutation of the indices $i = 1, 2, \dots, r$ such that

$$\begin{aligned} i &\rightarrow \pi(i) \\ m_{\pi(i)} &= +m_{r+1-i}, \\ m_{\pi(i)} &= +m_{r+1-i}, \\ g_{\pi(i)} &= -g_{r+1-i} \end{aligned} \quad (17)$$

where the parameters m, g are submitted to the condition

$$0 \leq \sum_{k=1}^r m_k < n, \quad \sum_{k=1}^r g_k = 0$$

This normalization differs from the usual one

$$m_1 = g_1 = 0.$$

Now we construct the particle states in their rest system. We start with the baryon functions. Quarks are of no heuristic value in our model. We shall not discuss them.

Baryons

For the baryons we expect a generalized tensor of the type

$$F_{(B)}(\eta) = \chi^{ABC} F_{ABC}(\eta), \quad (18)$$

where χ^{ABC} is the $SU(6)$ tensor of the rest system. We construct the tensor basis $F_{ABC}(\eta)$ and the corresponding square integrable functions $f_{ABC}(z)$. We determine the parameters m_k, ρ_k, n_k .

Under the compact subgroup $SU(6)$ the quantities $\Delta_i^{(1)}(\eta)$ transform as a basis for the first fundamental representation. Thus we guess that $F_{A_1 A_2 A_3}(\eta)$ is proportional to

$$\Delta_{A_1}^{(1)}(\eta) \Delta_{A_2}^{(1)}(\eta) \Delta_{A_3}^{(1)}(\eta).$$

The so far undetermined factor is necessarily invariant under $SU(6)$. This structure of $F_{A_1 A_2 A_3}(\eta)$ allows a maximal degeneracy [compare Eq. (13)].

$$r = 2, \ell_1 = 1, n_1 = 5, n_2 = 1.$$

The variables z_{st} are contained in a matrix

$$Z = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & 1 \end{pmatrix}.$$

Because $n_1 \neq n_2$ we find from the parity postulate

$$n_{\pi(1)} = n_2, \quad \pi(1) = 2,$$

which leads to

$$\vartheta_2 = \vartheta_{\pi(1)} = -\vartheta_2 = \vartheta_1 = 0.$$

The only possibility to construct an $SU(6)$ invariant expression for this degeneracy is

$$\sum_{i=1}^6 \Delta_i^{(1)}(\eta) \Delta_i^{(1)}(\bar{\eta}).$$

The difference in the degrees of homogeneity for $\Delta^{(1)}(\eta)$ and $\Delta^{(1)}(\bar{\eta})$ is three, this tells us that

$$m_1 = 0, \quad m_2 = -3.$$

If we introduce the parameters $\tilde{m}$ by

$$\begin{aligned} m_{p+1} &= \tilde{m}_{7-e_p}, \quad p=1,2,\dots,r-1, \\ \tilde{m}_{6-e_p+q} &= \tilde{m}_{7-e_p}, \quad q=1,2,\dots,n_{p+1}, \end{aligned} \quad (19)$$

we see that the Young tableau of the starting multiplet is in general

$-\tilde{m}_6$					
$-\tilde{m}_5$					
$-\tilde{m}_4$					
$-\tilde{m}_3$					
$-\tilde{m}_2$					

where $m_1 = m_1 = 0$ by normalization. We call the representations determined by this Young tableau "f type representations" or "f type states", (see below).

We arrive at the result

$$F_{A_1 A_2 A_3}(\eta) = \Delta_{A_1}^{(1)}(\eta) \Delta_{A_2}^{(1)}(\eta) \Delta_{A_3}^{(1)}(\eta) \cdot \left\{ \sum_{i=1}^6 \Delta_i^{(1)}(\eta) \Delta_i^{(1)}(\bar{\eta}) \right\}^{-\frac{9}{2}} \quad (20)$$

From this expression we go over to the square integrable function $f_{A_1 A_2 A_3}(z)$. From Eq. (15) we obtain

$$z_{6q} = \frac{\Delta_q^{(1)}(\eta)}{\Delta_6^{(1)}(\eta)} \quad (21)$$

Introducing this into Eq. (20), and comparing with Eq. (13), we find

$$F_{A_1 A_2 A_3}(\eta) = \Delta_6^{(1)}(\eta)^{-\frac{3}{2}} \Delta_6^{(1)}(\bar{\eta})^{-\frac{9}{2}} \cdot f_{A_1 A_2 A_3}(z) \quad (22)$$

where

$$f_{A_1 A_2 A_3}(z) = \frac{\prod_{k=1}^3 [\sigma_{6A_k} + (1 - \sigma_{6A_k}) z_{6A_k}]}{[1 + \sum_{q=1}^5 |z_{6q}|^2]^{\frac{9}{2}}} \quad (23)$$

It is a trivial task to prove that this function is indeed square integrable. The asymptotic behaviour at infinity is at most

$$\int_0^{\infty} \frac{r^6 \cdot r^9 dr}{r^{18}} = \int_0^{\infty} \frac{dr}{r^3} \quad (24)$$

But the expression (23) is not the only one which can describe baryon states. If we define

$$\Delta^i(\eta) = \frac{1}{5!} \varepsilon^{ij_1 j_2 \dots j_5} \Delta_{j_1 j_2 \dots j_5}^{(5)}(\eta) \quad (25)$$

the quantities $\Delta^i(\bar{\eta})$ will transform under $SU(6)$ equally well as a basis for the first fundamental representation. By means of them we construct a function $G_{(B)}(\eta)$.

$$G_{(B)}(\eta) = \chi^{ABC} G_{ABC}(\eta) \quad (26)$$

where $G_{ABC}(\eta)$ is now proportional to $\Delta^A(\bar{\eta}) \Delta^B(\bar{\eta}) \Delta^C(\bar{\eta})$

The maximal degeneracy is therefore given by

$$r=2, \quad \ell_1=5, \quad n_1=1, \quad n_2=5.$$

As above we find

$$Q_1 = Q_2 = 0.$$

For these functions (g type functions) the Young tableau of the starting multiplet is given by

$\tilde{m}_6 - \tilde{m}_1 = \tilde{m}_6$
$\tilde{m}_6 - \tilde{m}_2$
$\tilde{m}_6 - \tilde{m}_3$
$\tilde{m}_6 - \tilde{m}_4$
$\tilde{m}_6 - \tilde{m}_5$

Thus, we get

$$m_1 = 0, m_2 = +3,$$

and finally

$$G_{A_1 A_2 A_3}(\eta) = \Delta^{A_1}(\bar{\eta}) \Delta^{A_2}(\bar{\eta}) \Delta^{A_3}(\bar{\eta}) \cdot \left\{ \sum_{i=1}^6 \Delta^i(\eta) \Delta^i(\bar{\eta}) \right\}^{-\frac{9}{2}} \quad (27)$$

Introducing the parameters z contained in the matrix

$$Z = \begin{pmatrix} 1 & & & & & \\ z_{21} & 1 & & & & \\ z_{31} & & 1 & & & \\ z_{41} & & & 1 & & \\ z_{51} & & & & 1 & \\ z_{61} & & & & & 1 \end{pmatrix},$$

which according to Eq. (15) are related to the Δ 's by

$$z_{p,1} = (-1)^p \frac{\Delta_{1,2,\dots,p-1,p+1,\dots,6}^{(5)}(\eta)}{\Delta_{2,\dots,6}^{(5)}(\eta)} \quad (28)$$

$$= - \frac{\Delta^p(\eta)}{\Delta^1(\eta)}, \quad (29)$$

we obtain

$$g_{A_1 A_2 A_3}(z) = \frac{\prod_{k=1}^3 [\sigma_{A_k 1} - (1 - \sigma_{A_k 1}) \bar{z}_{A_k 1}]}{[1 + \sum_{q=2}^6 |z_{q1}|^2]^{\frac{9}{2}}}. \quad (30)$$

The connection between the f and g type functions is obvious. We can transform them into another by substituting

$$\begin{aligned} \Delta_a(\eta) &\rightarrow \Delta^a(\bar{\eta}), \\ \Delta_a(\bar{\eta}) &\rightarrow \Delta^a(\eta). \end{aligned}$$

For finite spinors this means

$$\begin{aligned} \psi^a &\rightarrow \psi_{\dot{a}}, \\ \psi_{\dot{a}} &\rightarrow \psi_a. \end{aligned}$$

This is a parity transformation

$$T_g \rightarrow T_{(g^+)^{-1}} = P \cdot T_g \cdot P^{-1}$$

We remark here that if we drop the $SU(6)$ invariant factors in $F(\eta)$ or $G(\eta)$, we get a basis for finite non-unitary spinor representations

$$F \rightarrow \psi^{ABC},$$

$$G \rightarrow \psi^{AB\dot{C}},$$

which were considered in the old $SL(6, C)$ model, Ref. 6). We shall denote the $SU(6)$ invariant factors therefore sometimes as the "unitarizing factors".

Antibaryons

From the theory of finite spinors (e.g., Dirac spinors), we know that an invariant form can be constructed from a spinor ψ and its parity transformed conjugate spinor $\tilde{\psi} = \bar{\psi} \gamma_0$. We take this as a heuristic guide.

The antibaryon function of $\tilde{\psi}$ type (which is now the f type

$$\begin{aligned} \psi, \tilde{\psi} &: f\text{-type} \\ \gamma_0 \psi, \bar{\psi} &: g\text{-type} \end{aligned}$$

belongs to a degeneracy

$$r = 2, \ell_1 = 5, n_1 = 1, n_2 = 5.$$

and to the invariants

$$m_1 = 0, m_2 = -3,$$

$$q_1 = q_2 = 0,$$

$$F_{(\beta)}(\eta) = \bar{\chi}_{ABC} F^{ABC}(\eta) \quad (31)$$

$$F^{A_1 A_2 A_3}(\eta) = \Delta^{A_1}(\eta) \Delta^{A_2}(\eta) \Delta^{A_3}(\eta) \cdot \left\{ \sum_{i=1}^6 \Delta^i(\eta) \Delta^i(\bar{\eta}) \right\}^{-\frac{q}{2}}, \quad (32)$$

$$f^{A_1 A_2 A_3}(z) = \frac{\frac{3}{1!1!} [\sigma_{A_k 1} - (1 - \sigma_{A_k 1}) z_{A_k 1}]}{[1 + \sum_{q=2}^6 |z_{q1}|^2]^{\frac{q}{2}}} \quad (33)$$

Analogously we obtain the g type functions

$$r = 2, \ell_1 = 1, n_1 = 5, n_2 = 1,$$

$$m_1 = 0, m_2 = +3,$$

$$q_1 = q_2 = 0,$$

$$G_{(\beta)}(\eta) = \bar{\chi}_{ABC} G^{ABC}(\eta), \quad (34)$$

$$G^{A_1 A_2 A_3}(\eta) = \Delta_{A_1}(\bar{\eta}) \cdot \Delta_{A_2}(\bar{\eta}) \cdot \Delta_{A_3}(\bar{\eta}) \cdot \left\{ \sum_{i=1}^6 \Delta_i(\eta) \Delta_i(\bar{\eta}) \right\}^{-\frac{9}{2}}$$
(35)

$$g^{A_1 A_2 A_3}(z) = \frac{\prod_{k=1}^3 [\delta_{6A_k} + (1 - \delta_{6A_k}) \bar{z}_{6A_k}]}{\left[1 + \sum_{q=1}^5 |z_{6q}|^2 \right]^{\frac{9}{2}}}$$
(36)

Mesons

We construct the meson functions. For the f type we find

$$r = 3, \quad n_1 = 1, \quad n_2 = 4, \quad n_3 = 1,$$

$$l_1 = 5, \quad l_2 = 1,$$

$$m_1 = 0, \quad m_2 = -1, \quad m_3 = -2,$$

$$q_1 = q_2 = q_3 = 0.$$

The values for q_k given are again necessary.

$$F_{(M)}(\eta) = \phi^A_B F^B_A(\eta)$$
(37)

$$F^B_A(\eta) = \Delta^B(\eta) \Delta_A(\eta) \cdot \left\{ \sum_{l=1}^6 \Delta^l(\eta) \Delta^l(\bar{\eta}) \right\}^{-3} \left\{ \sum_{k=1}^6 \Delta_k(\eta) \Delta_k(\bar{\eta}) \right\}^{-3}$$
(38)

The elimination of Δ 's in favor of z 's is now somewhat more complicated.

From Eq. (15) we find

$$z_{6,q} = \frac{\Delta_q^{(1)}(\eta)}{\Delta_6^{(1)}(\eta)}, \quad q = 1, 2, \dots, 5, \quad (39)$$

and from Eq. (16)

$$z_{p,1} = - \frac{\Delta^p(\eta)}{\Delta^1(\eta)}, \quad p = 2, 3, 4, 5. \quad (40)$$

But, in addition, the relation

$$\sum_{i=1}^6 \Delta^i(\eta) \Delta_i(\eta) =$$

$$\sum_{i=1}^6 \Delta^i(\bar{\eta}) \Delta_i(\bar{\eta}) = 0 \quad (41)$$

has to be taken into account. The result is

$$f^B_A(z) = \left[\sigma_{B1} + \sigma_{B6} (-z_{61} + \sum_{k=2}^5 z_{6k} z_{k1}) \right. \\ \left. - (1 - \sigma_{B1})(1 - \sigma_{B6}) z_{B1} \right] \cdot \\ \cdot \left[\sigma_{6A} + (1 - \sigma_{6A}) z_{6A} \right] \cdot \quad (42) \\ \cdot \left\{ 1 + \sum_{k=1}^5 |z_{6k}|^2 \right\}^{-3} \\ \cdot \left\{ 1 + \sum_{k=2}^5 |z_{k1}|^2 + |z_{61}|^2 - \sum_{k=2}^5 z_{6k} z_{k1} \right\}^{-3}$$

As a proof we may look at the trace $f^A_A(z)$. We find indeed

$$f^A_A(z) = 0. \quad (43)$$

We give now the parity transformed states of g type. We find

$$n_1 = 1, n_2 = 4, n_3 = 1,$$

$$m_1 = 0, m_2 = 1, m_3 = 2,$$

$$p_1 = p_2 = p_3 = 0,$$

and

$$G_{(M)}(\eta) = \Phi^A_B G^B_A(\eta), \quad (44)$$

$$G^B_A(\eta) = \Delta_B(\bar{\eta}) \Delta^A(\eta).$$

$$\left\{ \sum_{\ell=1}^6 \Delta^\ell(\eta) \Delta^\ell(\bar{\eta}) \right\}^{-3} \cdot \left\{ \sum_{k=1}^6 \Delta_k(\eta) \Delta_k(\bar{\eta}) \right\}^{-3}, \quad (45)$$

$$g^B_A(z) = \left[\sigma_{A1} + \sigma_{A6} (-\bar{z}_{61} + \sum_{k=2}^5 \bar{z}_{6k} \bar{z}_{k1}) - (1 - \sigma_{A1})(1 - \sigma_{A6}) \bar{z}_{A1} \right]$$

$$\left[\sigma_{6B} + (1 - \sigma_{6B}) \bar{z}_{6B} \right]$$

$$\cdot \left\{ 1 + \sum_{k=1}^5 |z_{6k}|^2 \right\}^{-3} \quad (46)$$

$$\cdot \left\{ 1 + \sum_{k=2}^5 |z_{k1}|^2 + |z_{61} - \sum_{k=2}^5 z_{6k} z_{k1}|^2 \right\}^{-3}$$

At this point we turn our attention to the relation between our present model and the old $SL(6, C)$ theory (Ref. ⁶). According to our remark made above there is a correspondence between both models if the "unitarizing factors" are dropped. This correspondence relates

$$G^B_A \text{ with } \phi^B_A,$$

and

$$F^B_A \text{ with } \phi^A_B,$$

where the spinors ϕ are finite spinors of the old $SL(6, C)$ model. Why are the quantities $F(\eta)$ corresponding to spinors

$$\phi_{BA}, \phi^{BA}$$

missing? Such functions $F(\eta)$ would be proportional to

$$\Delta^B(\eta) \Delta^A(\bar{\eta}), \Delta_B(\bar{\eta}) \Delta_A(\eta).$$

This implies that the parameters m_k of these maximally degenerate representations vanish, and consequently the starting multiplet is an $SU(6)$ singlet and not a 35-plet. Indeed, we know that ϕ_{BA} , ϕ^{BA} contain also a singlet, but this is eliminated by transversality conditions. Such conditions do not exist in the model we are discussing here and the representations starting with a singlet must be excluded. A similar situation is met in the case of baryons and antibaryons. This limits the invariant couplings from six in the old $SL(6, C)$ theory to only one in our present model. This single invariant will be discussed in Section V.

IV.- MOVING STATES

We face now the problem of moving states. We get them by applying a rotation-free Lorentz transformation $\Sigma(p)$ to the states in rest. Let the momentum point into the direction of the third axis,

$$p_1 = p_2 = 0, \quad p_3 = p, \quad p_0^2 - p^2 = m^2.$$

Then the operator

$$\Sigma(p) \in SL(6, \mathbb{C})$$

introduced in Section II, Eq. (4), has the matrix form

$$\Sigma(p) = \begin{pmatrix} \alpha & & & & & \\ & \alpha^{-1} & & & & \\ & & \alpha & & & \\ & & & \alpha^{-1} & & \\ 0 & & & & \alpha & \\ & & & & & \alpha^{-1} \end{pmatrix}, \quad (47)$$

where

$$\alpha = \sqrt{\frac{p_0 + p}{m}}$$

The usual block form (see e.g., Ref. ⁶⁾) is assumed here. We apply the corresponding operator $T_{\Sigma(p)}$ of the unitary representation to the state function $f(z)$,

$$T_{\Sigma(p)} f(z) = \alpha(z, \Sigma(p)) f(z') = f(p, z). \quad (48)$$

As an example we determine the functions $f(p, z)$ for the baryons and antibaryons. The mesons may be assumed to be at rest. We must compute the multipliers and the transformed arguments z' . The general formulae are given in the Appendix. We find for the baryons, f type:

$$\begin{aligned} z_{G1}' &= \alpha^2 z_{G1}, \quad z_{G3}' = \alpha^2 z_{G3}, \quad z_{G5}' = \alpha^2 z_{G5} \\ z_{G2}' &= z_{G2}, \quad z_{G4}' = z_{G4}, \\ g_G &= \alpha^{-1}, \quad \alpha(z, \Sigma(p)) = \alpha^{+6}, \end{aligned}$$

and finally

$$f_{A_1 A_2 A_3}(p, z) = \alpha^6 \frac{\prod_{k=1}^3 [\delta_{GA_k} + (1 - \delta_{GA_k}) \binom{1}{\alpha^2} \cdot z_{GA_k}]}{[1 + \alpha^4 \sum_{k=1,3,5} |z_{Gk}|^2 + \sum_{k=2,4} |z_{Gk}|^2]^{\frac{9}{2}}} \quad (49)$$

where the symbol

$$\binom{1}{\alpha^2} \cdot z_{GA_k}$$

means : z_{GA_k} if A_k is even, $z_{GA_k} \cdot \alpha^2$ if A_k is odd.

Similarly we find for

antibaryons, f type:

$$f_{A_1 A_2 A_3}(p, z) = \alpha^6 \frac{\prod_{k=1}^3 [\delta_{A_k 1} - (1 - \delta_{A_k 1}) \binom{\alpha^2}{1} z_{A_k 1}]}{[1 + \alpha^4 \sum_{k=2,4,6} |z_{k1}|^2 + \sum_{k=3,5} |z_{k1}|^2]^{\frac{9}{2}}} \quad (50)$$

where

$$\begin{pmatrix} \alpha^2 \\ 1 \end{pmatrix} \mathbb{Z}_{A_k 1}$$

means analogously : $\alpha^2 z_{A_k 1}$ if A_k is even, $z_{A_k 1}$ if A_k is odd.

V.- THE MESON-BARYON VERTEX

The invariant matrix element for the meson-baryon vertex is expected to have the following form

$$\begin{aligned} \mathcal{J} = & \int K(x, y, z) f_{(\bar{B})}(p_1, x) f_{(B)}(p_2, y) f_{(M)}(p_3, z) \\ & \cdot dx dy dz \\ & + \eta_{\bar{B}} \eta_B \eta_M \int K'(x, y, z) g_{(\bar{B})}(p_1, x) g_{(B)}(p_2, y) \\ & \cdot g_{(M)}(p_3, z) dx dy dz. \end{aligned} \quad (51)$$

The factors $\eta_{\bar{B}}$, η_B , η_M are the over-all parity eigenvalues of the infinite multiplets which are fixed by the values for the physical SU(6) multiplet

$$\begin{aligned} \eta_B &= \eta_{\bar{B}} = +1, \\ \eta_M &= -1. \end{aligned}$$

The integration extends over all the complex parameters x, y, z and over the complete complex plane. This means that we have a 38-fold integral over real parameters. One of the main tasks for future work will be finding a way to handle these integrals. Here we analyze only the structure of the integral kernels and explain how physical information can be extracted from the integrals. The physical information consists in relations between form factors belonging to the invariance subgroup $P \otimes SU(3)$.

The construction of the Clebsch-Gordan kernels $K(x, y, z)$ respectively $K'(x, y, z)$ is performed in the Appendix by a method which can easily be generalized to other representations. The kernels are uniquely fixed by functional equations, which express the invariance of the integral.

We quote here only the result for the vertex $\bar{B} B M$ and $SL(6, C)$ and the representations introduced in the preceding Sections III, IV. As variables we use the determinants

$$\tau = \begin{vmatrix} x_{21} & 1 & 0 & 0 & 0 \\ x_{31} & 0 & 1 & 0 & 0 \\ x_{41} & 0 & 0 & 1 & 0 \\ x_{51} & 0 & 0 & 0 & 1 \\ x_{61} - \gamma_{61}, -\gamma_{62}, -\gamma_{63}, -\gamma_{64}, -\gamma_{65} \end{vmatrix}$$

$$-S = \begin{vmatrix} z_{21} & 1 & 0 & 0 & 0 \\ z_{31} & 0 & 1 & 0 & 0 \\ z_{41} & 0 & 0 & 1 & 0 \\ z_{51} & 0 & 0 & 0 & 1 \\ z_{61} - \gamma_{61}, z_{62} - \gamma_{62}, z_{63} - \gamma_{63}, z_{64} - \gamma_{64}, z_{65} - \gamma_{65} \end{vmatrix}$$

$$t = \begin{vmatrix} x_{21} & 1 & 0 & 0 & 0 \\ x_{31} & 0 & 1 & 0 & 0 \\ x_{41} & 0 & 0 & 1 & 0 \\ x_{51} & 0 & 0 & 0 & 1 \\ x_{61} - z_{61}, -z_{62}, -z_{63}, -z_{64}, -z_{65} \end{vmatrix}$$

(52), (53), (54)

Then $K(x, y, z)$ and $K'(x, y, z)$ have the form

$$K(x, y, z) = C \cdot \tau^{-\frac{3}{2}} S^{-3} t^{-3} \tau^{+\frac{1}{2}} \bar{S}^{-2} \bar{t}^{-2} \quad (55)$$

$$K'(x, y, z) = C \cdot \tau^{+\frac{1}{2}} S^{-2} t^{-2} \tau^{-\frac{3}{2}} \bar{S}^{-3} \bar{t}^{-3} \quad (56)$$

The undetermined factor C which is common to both kernels depends only on the invariants of the group, i.e., it is an over-all form factor.

Now we discuss how physical information is contained in the invariant. In general we can write it as

$$J = \bar{\chi}_{ABC} \chi^{DEF} \bigoplus_{H}^G T^{ABC, DEF, G}{}^H \quad (57)$$

But it is not necessary to compute each component of the matrix T . Since the Poincaré group and the unitary symmetry group are subgroups of our invariance group, the matrix T can be decomposed into projection operators belonging to the group $P \otimes SU(3)$. Since the spinors $\bar{\chi}$, χ and ϕ correspond to Pauli spinors, it is natural to write the projection operators belonging to Poincaré invariants in Pauli spinor form. Thus we get formally

$$T^{ABC, DEF, G}{}^H = \sum_{x, y} a_{xy} P_{(x)}^{\alpha\beta\gamma} P_{(y)}^{\alpha\beta\gamma} \delta_{\alpha\beta\gamma}^{\eta} \quad (58)$$

where $a, b, \dots$ are $SU(2)$ indices and $\alpha, \beta, \dots$ $SU(3)$ indices. We solve Eq. (58) for the form factors a_{xy} by taking suitable traces on both sides. Then the form factors a_{xy} are the integrals which must be computed. If we are not interested in the full information, we need also not know all the projection operators. Let us assume for example that we want to obtain the D/F ratio of pseudoscalar mesons and nucleons. We split the $SU(6)$ representations in the familiar manner

$$\chi^{ABC} = D^{\alpha\beta\gamma, abc} + \frac{1}{113} \sum_{\substack{\text{cycl.} \\ \text{perm.}}} \varepsilon^{\alpha\beta\gamma} \varepsilon^{abc} N_{\delta}^{\gamma c}, \quad (59)$$

$$\phi^A_B = P^{\alpha}_{\beta} \delta^a_b + \vec{V}^{\alpha}_{\beta} \vec{\sigma}^a_b. \quad (60)$$

The part $\bar{N} N P$ is then of the form

$$\bar{N}^{\alpha}_{\beta, a} N_{\delta}^{\gamma, b} P^{\varepsilon}_{\xi} \tilde{T}^{\beta\delta\xi, a}_{\alpha\gamma\varepsilon, b} \quad (61)$$

where $\tilde{T}$ is (in the annihilation channel)

$$\begin{aligned} \tilde{T}^{\beta\delta\xi, a}_{\alpha\gamma\varepsilon, b} = & \delta^a_b \left\{ (\delta^{\beta}_{\varepsilon} \delta^{\xi}_{\gamma} \delta^{\delta}_{\alpha} - \delta^{\delta}_{\varepsilon} \delta^{\xi}_{\alpha} \delta^{\beta}_{\gamma}) a_F \right. \\ & \left. + (\delta^{\beta}_{\varepsilon} \delta^{\xi}_{\gamma} \delta^{\delta}_{\alpha} + \delta^{\delta}_{\varepsilon} \delta^{\xi}_{\alpha} \delta^{\beta}_{\gamma}) a_D \right\} \end{aligned} \quad (62)$$

In the explicit calculation some difficulties arise because of the unsymmetrical treatment of indices in the variables z . It is worth while to investigate the more symmetrical method which instead of triangular matrices z uses unitary matrices u of the maximal compact subgroup $SU(6)$.

Another critical problem which must be solved besides the computation of the integrals, consists in continuing analytically the integrals, particularly in the meson mass. We started from real masses for all three particles. But in order to obtain information about form factors in general, especially about such quantities as the anomalous magnetic moment of the proton or the limit

$$\lim_{q^2 \rightarrow \infty} \frac{G_c(q^2)}{G_m(q^2)}$$

for the proton form factors which are both fixed in our model, an analytic continuation is necessary.

A P P E N D I X

We derive now the integral kernels $K(x, y, z)$. For this we consider certain degenerate representations of the principal series of $SL(n, \mathbb{C})$, defined by the decompositions :

- i) $n_1 = 1, \quad n_2 = n-1,$
- ii) $n_1 = n-1, \quad n_2 = 1,$
- iii) $n_1 = 1, \quad n_2 = n-2, \quad n_3 = 1.$

This covers both cases K and K' . We do the actual calculation for the group $SL(4, \mathbb{C})$. The case $SL(6, \mathbb{C})$ is completely analogous, the result will be valid for all $SL(n, \mathbb{C})$ with $n=3, 4, 5, 6$. It is quite remarkable that for $n \geq 8$ difficulties seem to arise (see below) .

We consider the representations defined by

$$i) \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 \\ x_{31} & 0 & 1 & 0 \\ x_{41} & 0 & 0 & 1 \end{pmatrix}$$

$$T_g f(x) = g_2(x)^{-\frac{m}{2} + i\frac{s}{2} - 2} \overline{g_2(x)}^{+\frac{m}{2} + i\frac{s}{2} - 2} \cdot f(x')$$

$$ii) \quad Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y_{41} & y_{42} & y_{43} & 1 \end{pmatrix}$$

$$T_g f(y) = g_4(y)^{-\frac{M}{2} + i\frac{R}{2} - 2} \overline{g_4(y)}^{+\frac{M}{2} + i\frac{R}{2} - 2} \cdot f(y')$$

iii)

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ z_{41} & z_{42} & z_{43} & 1 \end{pmatrix}$$

$$T_g f(z) = \frac{g_2(z) \left(-\frac{m_2}{2} + i\frac{\rho_2}{2} - \frac{3}{2} \right) + \frac{m_2}{2} + i\frac{\rho_2}{2} - \frac{3}{2}}{g_4(z) \left(-\frac{m_3 - m_2}{2} + i\frac{\rho_3 - \rho_2}{2} - \frac{3}{2} \right) + \frac{m_3 - m_2}{2} + i\frac{\rho_3 - \rho_2}{2} - \frac{3}{2}} \cdot f(z')$$

The invariant parameters

$$m, M, m_2, m_3, \rho, R, \rho_2, \rho_3$$

are arbitrary. Later it will turn out that an invariant coupling is possible only if the following relations are satisfied :

$$\begin{aligned} m - M + 2m_2 - m_3 &= 0, \\ \rho - R + 2\rho_2 - \rho_3 &= 0. \end{aligned}$$

They are satisfied by our meson and baryon representations :

$$\text{f type : } m = -3, M = -3, m_2 = -1, m_3 = -2,$$

$$\text{g type : } m = +3, M = +3, m_2 = +1, m_3 = +2,$$

$$\text{for both types : } \rho = R = \rho_2 = \rho_3 = 0.$$

We write down the explicit expressions for the transformed arguments x', y', z' and the factors $g_2(x), g_4(y), g_2(z), g_4(z)$ contained in the multipliers

i)

$$g_2(x) = \begin{vmatrix} 1 & g_{12} & g_{13} & g_{14} \\ -x_{21} & g_{22} & g_{23} & g_{24} \\ -x_{31} & g_{32} & g_{33} & g_{34} \\ -x_{41} & g_{42} & g_{43} & g_{44} \end{vmatrix},$$

$$x'_{21} = [g_2(x)]^{-1} \begin{vmatrix} g_{11} & -1 & g_{13} & g_{14} \\ g_{21} & x_{21} & g_{23} & g_{24} \\ g_{31} & x_{31} & g_{33} & g_{34} \\ g_{41} & x_{41} & g_{43} & g_{44} \end{vmatrix},$$

$$x'_{31} = [g_2(x)]^{-1} \begin{vmatrix} g_{11} & g_{12} & -1 & g_{14} \\ g_{21} & g_{22} & x_{21} & g_{24} \\ g_{31} & g_{32} & x_{31} & g_{34} \\ g_{41} & g_{42} & x_{41} & g_{44} \end{vmatrix},$$

$$x'_{41} = [g_2(x)]^{-1} \begin{vmatrix} g_{11} & g_{12} & g_{13} & -1 \\ g_{21} & g_{22} & g_{23} & x_{21} \\ g_{31} & g_{32} & g_{33} & x_{31} \\ g_{41} & g_{42} & g_{43} & x_{41} \end{vmatrix},$$

$$\text{ii)} \quad g_4(y) = g_{44} + y_{43} g_{34} + y_{42} g_{24} + y_{41} g_{14},$$

$$y'_{41} = [g_4(y)]^{-1} [g_{41} + y_{43} g_{31} + y_{42} g_{21} + y_{41} g_{11}],$$

$$y'_{42} = [g_4(y)]^{-1} [g_{42} + y_{43} g_{32} + y_{42} g_{22} + y_{41} g_{12}],$$

$$y'_{43} = [g_4(y)]^{-1} [g_{43} + y_{43} g_{33} + y_{42} g_{23} + y_{41} g_{13}],$$

$$\text{iii)} \quad g_4(z) = g_{44} + z_{43} g_{34} + z_{42} g_{24} + z_{41} g_{14},$$

$$z'_{41} = [g_4(z)]^{-1} [g_{41} + z_{43} g_{31} + z_{42} g_{21} + z_{41} g_{11}],$$

$$\begin{aligned} z'_{42} &= [g_4(z)]^{-1} [g_{42} + z_{43}g_{32} + z_{42}g_{22} + z_{41}g_{12}], \\ z'_{43} &= [g_4(z)]^{-1} [g_{43} + z_{43}g_{33} + z_{42}g_{23} + z_{41}g_{13}], \end{aligned}$$

$$g_2(z) = \begin{vmatrix} +1 & , & g_{12}, g_{13}, g_{14} \\ -z_{21} & , & g_{22}, g_{23}, g_{24} \\ -z_{31} & , & g_{32}, g_{33}, g_{34} \\ -z_{41} + z_{42}z_{21} + z_{43}z_{31} & , & g_{42}, g_{43}, g_{44} \end{vmatrix},$$

$$z'_{21} = [g_2(z)]^{-1} \begin{vmatrix} g_{11} & -1 & , & g_{13} & g_{14} \\ g_{21} & +z_{21} & , & g_{23} & g_{24} \\ g_{31} & +z_{31} & , & g_{33} & g_{34} \\ g_{41}, z_{41} - z_{42}z_{21} - z_{43}z_{31} & , & g_{43} & g_{44} \end{vmatrix},$$

$$z'_{31} = [g_2(z)]^{-1} \begin{vmatrix} g_{11} & g_{12} & -1 & , & g_{14} \\ g_{21} & g_{22} & +z_{21} & , & g_{24} \\ g_{31} & g_{32} & +z_{31} & , & g_{34} \\ g_{41}, g_{42}, +z_{41} - z_{42}z_{21} - z_{43}z_{31} & , & g_{44} \end{vmatrix},$$

The functional equation for K derived from the invariance of the integral

$$\int K(x, y, z) f(x) g(y) h(z) dx dy dz$$

for general f, g, h , has the form

$$\begin{aligned} K(x', y', z') &= K(x, y, z) \cdot \\ &\cdot g_2(x)^{-\frac{m}{2} + i(\frac{p}{2} + 2)} \frac{\overline{g_2(x)} + \frac{m}{2} + i(\frac{p}{2} + 2)}{g_2(x)} \\ &\cdot g_4(y)^{-\frac{M}{2} + i(\frac{R}{2} + 2)} \frac{\overline{g_4(y)} + \frac{M}{2} + i(\frac{R}{2} + 2)}{g_4(y)} \\ &\cdot g_2(z)^{-\frac{m_2}{2} + i(\frac{p_2}{2} + \frac{3}{2})} \frac{\overline{g_2(z)} + \frac{m_2}{2} + i(\frac{p_2}{2} + \frac{3}{2})}{g_2(z)} \\ &\cdot g_4(z)^{-\frac{1}{2}(m_3 - m_2) + \frac{i}{2}(p_3 - p_2) + \frac{3}{2}} \\ &\cdot \frac{\overline{g_4(z)} + \frac{1}{2}(m_3 - m_2) + \frac{i}{2}(p_3 - p_2) + \frac{3}{2}}{g_4(z)} \end{aligned}$$

First we take $g \in \text{SL}(6, \mathbb{C})$ in triangular form

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_{21} & 1 & 0 & 0 \\ v_{31} & v_{32} & 1 & 0 \\ v_{41} & v_{42} & v_{43} & 1 \end{pmatrix}$$

and find

$$K(x, y, z) = K(x', y', z')$$

This means that K depends only on a reduced number of variables, which are invariant against triangular transformations. For general n we have $4n-5$ complex variables x, y, z for the degeneracies considered. The restriction on K due to triangular elements g reduces the possible number of variables of K by $\frac{1}{2}n(n-1)$. We get the following numbers of variables invariant against triangular transformations.

$n =$	2	3	4	5	6	7	8
number of variables	2	4	5	5	4	2	-1
							...

In the case of $\text{SL}(4, \mathbb{C})$ we have the five "triangular" invariant variables

$$u = x_{21} - z_{21},$$

$$v = y_{43} - z_{43},$$

$$s = y_{41} - z_{41} - (y_{42} - z_{42})z_{21} - (y_{43} - z_{43})z_{31},$$

$$t = x_{41} - z_{41} + z_{42}x_{21} + z_{43}x_{31},$$

$$w = (y_{42} - z_{42})(x_{21} - z_{21}) + (x_{31} - z_{31})(y_{43} - z_{43}).$$

In addition, we introduce :

$$r = w - s + t.$$

If g is diagonal, we find the transformed quantities

$$u' = \frac{g_{11}}{g_{22}} u$$

$$v' = \frac{g_{33}}{g_{44}} v$$

$$s' = \frac{g_{11}}{g_{44}} s$$

$$t' = \frac{g_{11}}{g_{44}} t$$

$$w' = \frac{g_{11}}{g_{44}} w$$

Since $g_{11} \cdot g_{22} \cdot g_{33} \cdot g_{44} = 1$ these relations allow us to eliminate three further variables. We assume

$$u = \frac{g_{22}}{g_{11}}, \quad v = \frac{g_{44}}{g_{33}}, \quad w = \frac{g_{44}}{g_{11}}$$

or, inversed,

$$g_{11} = u^{-\frac{1}{4}} v^{+\frac{1}{4}} w^{-\frac{1}{2}}$$

$$g_{22} = u^{+\frac{3}{4}} v^{+\frac{1}{4}} w^{-\frac{1}{2}}$$

$$g_{33} = u^{-\frac{1}{4}} v^{-\frac{3}{4}} w^{+\frac{1}{2}}$$

$$g_{44} = u^{-\frac{1}{4}} v^{+\frac{1}{4}} w^{+\frac{1}{2}}$$

Then we put

$$K(u', v', s', t', w') = K(1, 1, \frac{s}{w}, \frac{t}{w}, 1) = \tilde{K}(\frac{s}{w}, \frac{t}{w})$$

$\tilde{K}(\frac{s}{w}, \frac{t}{w})$ is related to $K(u, v, s, t, w)$ by

$$\tilde{K}(\frac{s}{w}, \frac{t}{w}) = w^{-\frac{1}{4}} (m + M + m_3) + \frac{i}{4} (p + R + s_3) + \frac{7}{2}$$

$$K(u, v, s, t, w)$$

Here we dropped already a factor which because of the restriction on m and ρ found later (quoted above), is equal to one. The functional equation for $\tilde{K}$ reads

$$\frac{\tilde{K}\left(\frac{s'}{w}, \frac{t'}{w}\right)}{\tilde{K}\left(\frac{s}{w}, \frac{t}{w}\right)} = \left(\frac{w'}{w}\right)^{-\frac{1}{4}(m+M+m_3) + \frac{1}{2}(\rho+R+\rho_3) + \frac{7}{2}} \cdot \frac{K(u', v', s', t', w')}{K(u, v, s, t, w)}$$

The latter ratio is a function of $g_2(x)$, $g_4(y)$, $g_2(z)$, $g_4(z)$. So far in our deduction no loss of generality has occurred. If we find a function $\tilde{K}$ obeying its functional equation, the corresponding kernel K will also do.

We now specialize to $z=0$ and

$$w = s = t.$$

Then we find with

$$\sigma = \frac{g_2(z)}{g_2(x)}, \quad \tau = \frac{g_4(y)}{g_4(z)},$$

$$\frac{w'}{w} = [g_4(y) g_2(z)]^{-1} (1 + \sigma - \tau),$$

$$\frac{s'}{s} = [g_4(y) g_2(z)]^{-1}$$

$$\frac{t'}{t} = [g_4(y) g_2(z)]^{-1} \tau$$

Inserting into the function $\tilde{K}$ we obtain

$$\tilde{K}\left(\frac{1}{1+\sigma-\tau}, \frac{\tau}{1+\sigma-\tau}\right) = \tilde{K}(1,1) \cdot \sigma^{\frac{1}{2}m - \frac{1}{2}\rho - 2} \left(\frac{\tau}{\sigma}\right)^{-\frac{1}{2}(m_2 - m_3) + \frac{1}{2}(\rho_2 - \rho_3) - \frac{3}{2}} (1+\sigma-\tau)^{-\frac{1}{4}(m+M+m_3) + \frac{1}{2}(\rho+R+\rho_3) + \frac{7}{2}}$$

x. (comp. conj. with $m \rightarrow -m$).

All the factors on the right-hand side which did not depend on $\sigma, \tau, \bar{\sigma}, \bar{\tau}$ must be constant. This condition leads to the restriction on the m 's and ρ 's.

From the equation thus obtained we can get $\tilde{K}(\xi, \eta)$ by solving for ξ, η . Inserting the function thus found into the relation between $\tilde{K}$ and K we find

$$K = \text{const. } \tau^a s^b t^c \bar{\tau}^{a'} \bar{s}^{b'} \bar{t}^{c'}$$

with

$$a = +\frac{1}{2} (m_1 + m_2 - m_3) - \frac{1}{2} (\rho_1 + \rho_2 - \rho_3) - \frac{1}{2},$$

$$b = +\frac{1}{2} m_2 - \frac{1}{2} \rho_2 - \frac{3}{2},$$

$$c = +\frac{1}{2} (m_3 - m_2) - \frac{1}{2} (\rho_3 - \rho_2) - \frac{3}{2},$$

$$a' = -\frac{1}{2} (m_1 + m_2 - m_3) - \frac{1}{2} (\rho_1 + \rho_2 - \rho_3) - \frac{1}{2},$$

$$b' = -\frac{1}{2} m_2 - \frac{1}{2} \rho_2 - \frac{3}{2},$$

$$c' = -\frac{1}{2} (m_3 - m_2) - \frac{1}{2} (\rho_3 - \rho_2) - \frac{3}{2}.$$

We can generalize these expressions to the case $SL(n, \mathbb{C})$, $n=3,5,6$ by substituting in a, b, c, a', b', c' for the constant ratios

$$\text{in } a, a' : \quad -\frac{1}{2} \text{ for } -\frac{1}{2},$$

$$\text{in } b, b' : \quad -\frac{n-1}{2} \text{ for } -\frac{3}{2},$$

$$\text{in } c, c' : \quad -\frac{n-1}{2} \text{ for } -\frac{3}{2}.$$

The sufficiency of the form found for $\tilde{K}(\xi, \eta)$ can also be proved

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- 3) P. Budini, C. Fronsdal - Phys.Rev.Letters 14, 968 (1965).
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- 5) J.M. Gelfand, M.A. Neumark - Unitäre Darstellungen der klassischen Gruppen, Berlin (1957). Their theory will be referred to as G.-N. theory.
- 6) See, e.g., W. Rühl, The $SL(6, \mathbb{C})$ model as a relativistically invariant formulation of $SU(6)$ symmetry, Seminar on High-Energy Physics and Elementary Particles, Trieste (1965), to be published, and the literature quoted in this paper.
- 7) We feel obliged to quote here an unpublished work of J. Wess and B. Zumino, who studied a similar model under analogous assumptions. They were the first to find the non-local structure of this type of models.
- 8) A.O. Barut - Phys.Rev. 135, B839 (1964). Later on this idea was popularized particularly by C. Fronsdal and M. Gell-Mann.
- 9) F. Gürsey, L.A. Radicati - Phys.Rev.Letters 13, 173 (1964).
- 10) We shall use this notation throughout. The starting multiplet is the smallest representation of the maximal compact subgroup contained in the unitary representation of $SL(6, \mathbb{C})$. It is uniquely determined by the parameters m_k .
- 11) The notations "maximally degenerate representation" and "tower representations" are probably equivalent.