SUM RULES AND DYNAMICS

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ABSTRACT

We discuss the relation between current algebra sum rules and dynamical assumptions. We show that the unsubtractness assumption, which leads to the Fubini sum rule, is untenable in certain kinematical configurations. The discussion of a reggeized dynamics allows to justify such sum rules and to understand the peculiar singularities present in it.

It has been recently recognized ¹⁾ that the gauge invariance or - better - gauge covariance, is an alternative language to the current algebra approach. The point raised here is that the same gauge invariance or covariance gives rise to terms which, in certain kinematical configurations, violate the assumptions which are usually made (unsubtracted dispersion relations ²⁾ or finite mass intermediate states ³⁾) in order to derive sum rules from current algebra. This violation does not prevent to obtain sum rules, their form will of course depend on the dynamics which is assumed. For instance, if superconvergent sum rules are assumed, then the Fubini sum rule ²⁾ [Eq. (15)] is obtained.

In order to deal with a well-known gauge invariance, let us limit our discussion to the case in which one of the commuting currents is the electromagnetic one.

Due to the fact that we shall use elsewhere gauge invariance arguments, we shall rederive briefly current algebra relations with a language that shall be useful to us afterwards.

Let us consider the amplitude for photoproduction of a charged lepton pair. Just to avoid an extra index - which only complicates the formulae without changing the content of the arguments developed - we shall consider, in the following, a lepton pair coupled to a (charged) scalar current j^+ . The extension to vector or axial current is only a trivial matter which we shall illustrate later.

Let us call q, k, ε the lepton pair and photon momentum and photon polarization, respectively; p and p' the initial and final hadron (considered spinless) momenta and

$$\begin{aligned} \Delta &= p' - p = k - q, & t &= \Delta^2 \\ P &= p' + p, & \nu &= k \cdot P \end{aligned} \quad (1)$$

We consider now a real photon, i.e., $k^2 = 0$. We shall discuss later the extension to $k^2 \neq 0$ (electroproduction).

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If we call $\epsilon_\lambda T_\lambda$ the lepton pair photoproduction amplitude, gauge invariance implies

$$k_\lambda T_\lambda = 0 \quad (2)$$

T_λ can be written as

$$(2\pi)^4 \delta(p+k-p'-q) T_\lambda = \int e^{ikx} \langle p', (\text{lepton pair}) | j_\lambda^{e.m.}(x) | p \rangle d^4x \quad (3)$$

Let us separate from the total electromagnetic current $j_\lambda^{e.m.}(x)$ the pure leptonic part

$$j_\lambda^e(x) = i \bar{\psi}_e \gamma_\lambda \psi_e + i \bar{\psi}_\mu \gamma_\lambda \psi_\mu \quad (4)$$

and let us call *)

$$j_\lambda^R(x) = j_\lambda^{e.m.}(x) - j_\lambda^e(x) \quad (5)$$

Let us define

$$(2\pi)^4 \delta(p+k-p'-q) M_\lambda = \int e^{ikx} \langle p', (\text{lepton pair}) | j_\lambda^R(x) | p \rangle d^4x \quad (6)$$

and for L_λ an analogous expression with j_λ^e instead of j_λ^R . The equation

$$T_\lambda = M_\lambda + L_\lambda \quad (7)$$

can be represented graphically by

) We remark that j_λ^R is not necessarily the pure hadronic electromagnetic current j_λ^H . In the quark model $j_\lambda^R = j_\lambda^H$, but in a charged meson field Lagrangian model j_λ^R would contain also terms of the form $e \psi^ \psi A_\lambda$ which are proportional to the electromagnetic field itself.

Fig. 1

L_A can be explicitly calculated (in the zeroth order of the weak coupling constant), and one finds

$$k_A L_A = F(t) \quad (8)$$

where

$$F(t) = \langle p' | j_0^+ | p \rangle \quad (9)$$

The gauge invariant condition (2) then gives

$$k_A M_A = - F(t) \quad (10)$$

which is the usual equation derived from current algebra.

We write

$$M_A = P_A A(\nu, t, q^2) + \Delta_A B(\nu, t, q^2) + k_A C(\nu, t, q^2) \quad (11)$$

so that (10) implies

$$\nu A + \frac{1}{2}(t - q^2) B + k^2 C = - F(t) \quad (12)$$

The method used to obtain sum rules from Eq. (12) is to do the following "asymptotic assumption": A , B and C as functions of ν for fixed values of t and q^2 go to zero for $\nu \rightarrow \infty$.

This implies unsubtracted dispersion relations of the form

$$A(\nu, t, q^2) = \frac{1}{\pi} \int \frac{\text{Im} A(\nu', t, q^2) d\nu'}{\nu' - \nu}, \quad B(\nu, t, q^2) = \frac{1}{\pi} \int \frac{\text{Im} B(\nu', t, q^2) d\nu'}{\nu' - \nu} \quad (13)$$

and similar for C.

Then, from Eq. (12), A must go as $1/\nu$ for $\nu \rightarrow \infty$, i.e.,

$$\lim_{\nu \rightarrow \infty} \nu A(\nu, t, q^2) = -F(t) \quad (14)$$

or, from (13),

$$\frac{1}{\pi} \int \text{Im} A(\nu, t, q^2) d\nu = F(t) \quad (15)$$

which is the usual Fubini sum rule ²⁾.

We shall be interested in the singularities of $F(t)$. Let us begin by the study of a simple pole term, i.e.,

$$F(t) = \frac{g}{t - \mu^2} + \varphi(t) \quad (16)$$

where $\varphi(t)$ is free from singularities for t smaller than a certain t_0 ($t_0 > \mu^2$) and μ is the mass of a charged particle, scalar in this case [let us call it σ ^{*}] which can mediate for form factor of Eq. (9).

Fig. 2

The pole term of (16) can be represented by the diagram of Fig. 2.

We shall now show that, on the basis of gauge invariance, the "asymptotic assumption" is wrong at least for $t = \mu^2$. Indeed, if we have a charged particle which is coupled to the charged current, gauge invariance ensures that this particle interacts with the electromagnetic field.

^{*}) It would be the φ for a charged vector current.

This implies that M_λ shall contain a pole term M_λ^P which is represented in Fig. 3.

Fig. 3

M_λ^P has a pole for $t = \mu^2$, its residue is given by

$$\lim_{t \rightarrow \mu^2} (t - \mu^2) M_\lambda^P = (k - 2g)_\lambda \frac{g}{q^2 - \mu^2} \quad (17)$$

or

$$\lim_{t \rightarrow \mu^2} (t - \mu^2) A^P = 0, \quad \lim_{t \rightarrow \mu^2} (t - \mu^2) B^P = \frac{2g}{q^2 - \mu^2} \quad (18)$$

M_λ^P is not necessarily the only term which has a pole for $t = \mu^2$, but at least is the only one which shows a $1/(q^2 - \mu^2)$ dependence. It is therefore clear that B has a singularity for $t = \mu^2$ whose residue does not vanish for $V \rightarrow \infty$, so at least for $t = \mu^2$ the "asymptotic assumption" is violated.

If we separate out from B the pole term B^P in Eq. (12), i.e.,

$$VA + \frac{t - q^2}{2} (B - B^P) = -F(t) - \frac{t - q^2}{2} B^P \quad (19)$$

we see that the residues of $F(t)$ and B^P for $t = \mu^2$ exactly cancel out.

It is easy to realize that this cancellation is very general, i.e., that due to gauge invariance every singularity of $F(t)$ is compensated by an analogous singularity of B whose residue does not vanish at $V \rightarrow \infty$.

Indeed, the residue of $F(t)$ at a singularity t_n , corresponding to a state n , is the divergence of the diagram of Fig. 4, while the corresponding singularity of $k_\lambda M_\lambda$ represents the divergence of the amplitude of Fig. 5. Clearly, gauge invariance implies that the sum of the two divergences (i.e., sum of the two residues) vanishes. On the other hand, it is clear that the residue of the pole of M_λ , represented in Fig. 5, contributes only to B and C and not to A due to the fact that there is no trace of P_λ in it.

Fig. 4

Fig. 5

What happens for t away from the singularity depends only on the dynamics which is assumed. Let us go back to the simple pole situation. If the pole represents a non-reggeized particle, then the diagram of Fig. 3 contributes with

$$A^P = 0 \quad B^P = \frac{2g}{(q^2 - \mu^2)(t - \mu^2)} \quad (20)$$

so that B^P does not satisfy the "asymptotic assumption" for all values of t . Inserting (20) and (16) into (19), we find

$$\nu A + \frac{t - q^2}{2} (B - B^P) = -\frac{g}{q^2 - \mu^2} - \psi(t) \quad (21)$$

If B^P is the only term of B which violates the "asymptotic assumption", we obtain

$$-\lim_{\nu \rightarrow \infty} \nu A(\nu t q^2) = \frac{1}{\pi} \int \text{Im } A(\nu t q^2) d\nu = \frac{g}{q^2 - \mu^2} + \psi(t) \quad (22)$$

We see, therefore, that the non-reggeized contribution eliminates the corresponding t singularity of $F(t)$, introducing the q^2 one in its place. We note that for $t=q^2$ the divergence of the polar term M_{λ}^P is zero. Therefore, for $t=q^2$, one obtains the sum rule (15) even if the σ is not reggeized. In fact for the $k^2=0$ case we are discussing, the $t=q^2$ configuration is the one that would be obtained from the commutator of the charge with the current density j^+ .

Let us now discuss the situation when the pole is due to a reggeized contribution. Due to the fact that we need the asymptotic properties of the different invariants, we prefer to obtain them from a suitable model which gives a reggeized contribution. We shall therefore use the multi-peripheral approach ⁴⁾ in which the asymptotic properties of the absorptive parts of A^P and B^P are given by the ladder of Fig. 6 in which all particles of the chain are scalars.

Fig. 6

One obtains in that way the following asymptotic behaviours

$$\text{Im } A^P \sim f(t, q^2) \nu^{\alpha(t)-1} + O\left(\frac{1}{\nu^2}\right) \quad (23)$$

$$\text{Im } B^P \sim h(t, q^2) \nu^{\alpha(t)} + O\left(\frac{1}{\nu}\right) \quad (24)$$

where f and g are related to the eigenfunction of the scalar ladder.

We are interested here in $-1 < \alpha(t) < 0$ and we expect $\alpha(\mu^2)=0$. Indeed, it is clear that the dispersive integral [Eq. (13)] for B^P , using (24) for its imaginary part, diverges for $\alpha(t) \rightarrow 0$ (from below) reproducing in that way the pole of the particle (scalar in this case, called σ previously) exchanged in the t channel.

On the other hand, $\text{Im} A^P \sim 1/\nu$ for $\alpha = 0$, so that the dispersive integral for A^P will not develop the pole as was expected.

By direct integration we obtain the asymptotic behaviour ^{*)}

$$B^P \sim -h(t, q^2) \frac{\nu^\alpha}{2i\pi\alpha} \xi(t) + O\left(\frac{1}{\nu}\right) \quad (25)$$

$$A^P \sim -f(t, q^2) \frac{\nu^{\alpha-1}}{2i\pi\alpha} \xi(t) - \frac{1}{\nu} \frac{1}{\pi} \int \text{Im} A^P(\nu', t, q^2) d\nu' + O\left(\frac{1}{\nu^2}\right) \quad (26)$$

where ξ is the signature which depends on the crossing properties of the amplitude. For our case ²⁾, $b(\nu) = -b(-\nu)$, $a(\nu) = +a(-\nu)$, so that $\xi = e^{-i\pi\alpha} + 1$. From (26) we see explicitly that A^P has no pole for $t = \mu^2$. Indeed

$$\lim_{t \rightarrow \mu^2} (t - \mu^2)^{\frac{1}{\pi}} \int \text{Im} A^P(\nu', t, q^2) d\nu' = -f(\mu^2, q^2) \frac{\xi(\mu^2)}{\pi \alpha'(\mu^2)} \quad (27)$$

expression which cancels the residue of the pole developed by the first term in the right-hand side of (26) for $\alpha \rightarrow 0$.

If the ν^α behaviour is correctly given by the ladder, gauge invariance on the absorptive amplitudes [imaginary part of Eq. (12)] implies

$$f(t, q^2) = -\frac{(t - q^2)^{1/2}}{2} h(t, q^2) \quad (28)$$

^{*)} When this work was completed, we received a preprint by J.B. Bronzan, I.S. Gerstain, B.W. Lee and F.E. Low ("Current algebra and non-Regge behaviour of weak amplitudes", MIT - November 1966) in which an expression similar to that of Eq. (26) was obtained in order to conciliate the Regge behaviour with the Fubini sum rule.

The reggeized σ shall contribute to $F(t)$ with a pole term ⁵⁾
 $\gamma/(\alpha(t))$, i.e.,

$$F(t) = \frac{\gamma}{\alpha(t)} + \psi(t) \quad (29)$$

so that the residue at the pole for $t = \mu^2$ is given by

$$g = \frac{\gamma}{\alpha'(\mu^2)} \quad (30)$$

Gauge invariance condition (18) implies therefore

$$(q^2 - \mu^2) \frac{h(\mu^2, q^2)}{\pi \alpha'(\mu^2)} = g \quad (31)$$

From (25) we see that for $t < \mu^2$, B^P satisfies the "asymptotic assumption", so there is no obstacle to suppose that all B satisfy such an assumption and we obtain, therefore, the Fubini sum rule (15).

For t approaching μ^2 we must be more careful. The "asymptotic assumption" on all other contributions to A and B allows us to write the asymptotic expressions

$$A \sim -f(t, q^2) \frac{\nu^{\alpha-1}}{\sin \pi \alpha} \xi(t) - \frac{1}{\nu \pi} \int \text{Im } A(\nu, t, q^2) d\nu + \frac{R_a}{\nu} \quad (32)$$

$$B \sim -h(t, q^2) \frac{\nu^\alpha}{\sin \pi \alpha} \xi(t) + R_b \quad (33)$$

where R_a and R_b go to zero for $\nu \rightarrow \infty$ and have no singularity in the neighbourhood of $t = \mu^2$. We see that for $t = \mu^2$, B has a pole term whose residue does not vanish for $\nu \rightarrow \infty$ and besides $\lim_{\nu \rightarrow \infty} \nu A$ differs from $\int \text{Im } A d\nu$ by a singular pole term.

Indeed, the gauge invariance equation (12) gives

$$\nu A \sim -F(t) + \frac{t - q^2}{2} h(t, q^2) \frac{\nu^\alpha}{\sin \pi \alpha} \xi(t) \quad (34)$$

and, using (28) and (31), the Fubini sum rule

$$\frac{1}{\pi} \int \text{Im} A(\nu, t, q^2) d\nu = F(t) \quad (35)$$

for $t < t_0$, i.e., including $t = \mu^2$. We see that Eq. (31) ensures the cancellation of the pole in νA for $t \rightarrow \mu^2$ - which is a general statement as discussed previously - while $\int \text{Im} A d\nu$ shows such a pole. This fact is not surprising: the $\text{Im} A^P$ contribution to the integral over $\text{Im} A$ gives rise to a pole whose residue, due to (28), is equal to minus the corresponding residue of $([t - q^2]/2) B^P$ and this, due to (31), is just the residue of $F(t)$.

We expect our amplitude M_λ to have singularities in q^2 which are related to processes of photoproduction of states which can be coupled to the lepton pair. Let us fix our attention to the pole for $q^2 = \mu^2$ corresponding to the σ particle discussed before and call $M_\lambda^\sigma = P_\lambda A^\sigma + A_\lambda B^\sigma + k_\lambda C^\sigma$ the residue of M_λ . M_λ^σ is clearly the amplitude for photoproduction of a σ and, therefore, gauge invariance requires

$$k_\lambda M_\lambda^\sigma = \nu A^\sigma + \frac{t - \mu^2}{2} B^\sigma + k^2 C^\sigma = 0 \quad (36)$$

For the case of a non-reggeized σ , M_λ^σ shall have a photoelectric term which contributes to B^σ with a term that does not vanish for $\nu \rightarrow \infty$. Therefore, under the same dynamical assumptions made in order to obtain (22) from (12), we shall obtain from (36)

$$\frac{1}{\pi} \int \text{Im} A^\sigma(\nu, t) d\nu = g \quad (37)$$

For the case of the reggeized σ we discussed before, we shall instead obtain

$$\frac{1}{\pi} \int \text{Im} A^\sigma(\nu, t) d\nu = 0 \quad (38)$$

also for $t = \mu^2$.

On the other hand, if one considers the singularities of Eqs. (22) and (35) at $q^2 = \mu^2$, one obtains ⁶⁾ Eqs. (37) and (38), respectively. We see therefore that these so-called "strong interaction sum rules" are directly implied by the dynamical assumptions ⁷⁾ made in the usual treatment of current algebra, so they are an input, more than a result.

The extension of the preceding discussion to $k^2 \neq 0$ (virtual photon) and to a vector or axial charged current coupled to the leptons involves no fundamental differences. The crucial point of the cancellation of the singularities of the form factor (this time given by vector or axial states) from singularities of the invariants B , etc., is always true, as can be realized from the fact that the divergenceless condition holds even if $k^2 \neq 0$ or the leptons are coupled to vector or axial currents.

As is well known, the invariant A appearing in the sum rules will be the coefficient of $P_\lambda P_\nu$ and its imaginary part, in the case of a Regge pole, shall behave as $\nu^{\alpha-2}$ so that no pole shall be developed by the dispersive integral for $\alpha = 1$ (which is the value for α at the mass of the vector or axial particle). The form of A shall be therefore similar to that of Eq. (32) with α substituted by $\alpha - 1$ and with $\xi = e^{-i\pi\alpha} - 1$.

We have used until now the language of the dispersive approach to the current algebra. However, it is possible to understand the occurrence of the subtraction terms required by gauge invariance also in the $P \rightarrow \omega$ language ³⁾. Equation (15) is obtained inserting a complete set of intermediate states $\sum_n |n\rangle \langle n| = 1$ between the currents of the matrix element

$$\langle p' | [J_\lambda^\mu, J_\nu^\dagger] | p \rangle_{\chi_0 = k_0}$$

and going to the $P \rightarrow \omega$ system in the hypothesis that the states $|n\rangle$ with infinite mass do not contribute. It is easily seen that the diagram of Fig. 3, whose presence is due to the gauge invariance, comes from an intermediate state $|n\rangle$ with infinite mass and therefore violates that hypothesis. The point is that not all the states $|n\rangle$ give rise to singularities in the s channel. There exists a family of "disconnected"

intermediate states in which $|n\rangle$ decomposes into the initial (or final) state plus another state $|m\rangle$, so that for this family

$$\langle n | j^+ | p \rangle = \langle p | p \rangle \langle m | j^+ | 0 \rangle = \langle m | j^+ | 0 \rangle$$

In the $P \rightarrow \omega$ system the mass of the state $|n\rangle$ is infinite, even if the mass of the state $|m\rangle$ is finite. In the case of the diagram of Fig. 3, for instance, the state $|m\rangle$ in question is the σ particle.

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