2HDMs predictions for $\bar{B} \rightarrow X_s \gamma$ in NLO QCD

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ABSTRACT

The decay $\bar{B} \rightarrow X_s \gamma$ is studied at the Next to Leading Order in QCD in a class of models containing at least two Higgs doublets and with only one charged Higgs boson non–decoupled at low–energy. The two–loop matching condition is calculated and it is found to agree with existing results. The complete dependence of the Wilson coefficients on the matching scale is given. The size of the Next to Leading Order corrections is extensively discussed. Results for branching ratios, possible CP asymmetries and lower bounds on the charged Higgs mass are presented when the convergence of the perturbative series appears fast enough to yield reliable predictions. Regions in the parameter space of these models where the Next to Leading Order calculation is still not a good approximation of the final result for these observables are singled out.

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1. Introduction

It is well known that charged Higgs contributions to the branching ratio for the decay $\bar{B} \rightarrow X_s \gamma$, $\text{BR}(\bar{B} \rightarrow X_s \gamma)$, decouple very slowly from the Standard Model (SM) one. Hence, in the absence of any experimental evidence for a charged Higgs boson, this decay may provide a powerful tool to limit the range of unknown parameters in models where such a particle is present and other – non standard – contributions to $\text{BR}(\bar{B} \rightarrow X_s \gamma)$ are subleading [1,2].

Supersymmetric models constitute, perhaps, the best motivated extension of the SM where a second Higgs doublet, and therefore a charged Higgs boson $H^\pm$, is necessary for internal consistency [3]. Simpler extensions where only Higgs doublets are added to the SM are, however, important on their own right. They are also excellent pedagogical tools to understand the subtleties that the Next to Leading Order (NLO) calculation of $\text{BR}(\bar{B} \rightarrow X_s \gamma)$ entails, and which may be hidden in the SM results.

The simplest class of such extensions, with two Higgs doublets, is usually denoted as 2HDMs. This class contains the well–known Type I and Type II models in which the same or the two different Higgs fields couple to up– and down–type quarks. For what concerns us here, Multi–Higgs Doublet models can be included in this class, provided only one charged Higgs boson remains light enough to be relevant for the process $\bar{B} \rightarrow X_s \gamma$. This generalization allows a simultaneous study of $\bar{B} \rightarrow X_s \gamma$ in different models, including Type I and Type II, by a continuous variation of the (generally complex) charged Higgs couplings to fermions. (No tree–level flavour violating neutral couplings are assumed in the present paper.) It allows also a more complete investigation of the question whether the measurement of $\text{BR}(\bar{B} \rightarrow X_s \gamma)$ closes the possibility of a relatively light $H^\pm$ not embedded in a supersymmetric model [4].

At present, a measurement of this decay rate by the CLEO Collaboration is available [5]:

$$\text{BR}(\bar{B} \rightarrow X_s \gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4}. \quad (1)$$

There exists also a preliminary result by the ALEPH Collaboration with a larger central value [6]:

$$\text{BR}(\bar{B} \rightarrow X_s \gamma) = (3.29 \pm 0.71 \pm 0.68) \times 10^{-4}. \quad (2)$$

Adding statistical and systematical errors in quadrature, one obtains 90% C.L. lower and upper limits on $\text{BR}(\bar{B} \rightarrow X_s \gamma)$ of $1.22 \times 10^{-4}$ and $3.42 \times 10^{-4}$ from the CLEO measurement and $1.68 \times 10^{-4}$, $4.90 \times 10^{-4}$ from the ALEPH result. The band of allowed values, corresponding to a more conservative estimate of the systematic error, is reported by CLEO to be $(1.0 - 4.2) \times 10^{-4}$ [5].

The theoretical situation within the SM, is at the moment far better settled than the experimental one. After the original observation that QCD corrections to the decay $\bar{B} \rightarrow X_s \gamma$ are a substantial fraction of its rate [7], a collective theoretical effort of almost a decade has led to the determination of $\text{BR}(\bar{B} \rightarrow X_s \gamma)$ at the NLO in QCD, and to a
considerable reduction of the theoretical error. For the Leading Order (LO) calculation, several groups contributed to the evaluation of the elements of the the anomalous dimension matrix [8,9] and provided phenomenological analyses with partial inclusion of some NLO contributions (e.g. bremsstrahlung corrections) [10–13]. The two–loop matching condition, needed for a complete NLO calculation, was first obtained in [14] and later confirmed in [15–17], using different techniques. The two–loop corrections to the matrix elements were calculated [18] and the determination of the $O(\alpha_s^2)$ elements of the anomalous dimension matrix, started already since some time [19,20], has been completed only recently [21,22]. In addition, non–perturbative contributions to $\text{BR}(\overline{B} \to X_s\gamma)$, scaling like $1/m_H^2$ [23–25] and $1/m_c^2$ [26], were also computed. The issue of the dependence of the branching ratio on scales, first raised at the LO level in [11,13], was addressed for the NLO calculation, and discussed in detail, in [27]. In the SM, uncertainties due to sensitivity of $\text{BR}(\overline{B} \to X_s\gamma)$ on such scales are, undisputedly, rather small and the theoretical estimate for this observable suffers mainly from the large experimental errors of the input parameters.

It would be desirable to have similarly precise calculations also in extensions of the SM. Thus, more accurate experimental measurements, when available, could provide stringent constraints on the free parameters of these models. We present here a detailed study of $\overline{B} \to X_s\gamma$ at the NLO in QCD, in the class of models specified above, aiming, in particular, to an assessment of the reliability of the theoretical calculation. Our results are, in general, less optimistic than one could have foreseen. Indeed, we find unexpectedly large NLO corrections and scale dependences in the Higgs contributions to $\text{BR}(\overline{B} \to X_s\gamma)$, irrespectively of the value of the charged Higgs couplings to fermions. This feature remains undetected in Type II models, where the SM contribution to $\text{BR}(\overline{B} \to X_s\gamma)$ is always larger than, and in phase with, the Higgs contributions. It can, however, produce unacceptably large scale uncertainties for certain ranges of these couplings and, at times, completely ill–defined results. We single out combinations of couplings and of values of the charged Higgs mass, $m_H$, for which the branching ratio can be reliably predicted. In this case, a comparison between theoretical and experimental results for $\text{BR}(\overline{B} \to X_s\gamma)$ allows to conclude that values of $m_H = O(M_W)$ can be excluded, in general, only in Type II models, but are otherwise allowed.

Previous LO analyses had dealt with Type I and Type II models [8,28–30] and the generalized class of models considered here [31,32]. LO calculations are known to have large scale uncertainties. Consequently, they are not particularly good arenas to distinguish the quantitative differences between QCD corrections to charged Higgs and SM contributions. These differences stand out clearly at the NLO level. Two NLO calculations have been performed recently [17,33]. They deal with Type I and Type II models and the issue of scale uncertainty is addressed in [17] for Type II models. The fact that the NLO corrections to the Higgs contributions are large is, therefore, understandably missed, since the branching ratio is dominated by the SM contribution. From the technical point of view, our calculation of the two–loop matching condition agrees with that reported in [17,33]. We disagree, however, with the analytic dependence of $\text{BR}(\overline{B} \to X_s\gamma)$ on the
the matching scale given in the literature for the SM [27] and for 2HDMs [17].

The reminder of this paper is organized as follows. In Sect. 2 we define the class of 2HDMs studied. In Sect. 3 we outline the main steps of the calculation; to keep this section readable, we relegate some parts of our results to appendices. In Sect. 4 we present our phenomenological analysis of the decay $\bar{B} \to X_s \gamma$ in the class of models considered. After giving our branching ratio prediction for the SM, we discuss, in 4.1, the size of the NLO corrections to the charged Higgs contributions at the amplitude level. In Sec. 4.2, we give results for $\text{BR}(\bar{B} \to X_s \gamma)$, and discuss their reliability, for various ranges of real couplings of the charged Higgs to fermions. NLO branching ratios for Type I and Type II models are in particular studied in Sec. 4.3, where lower bounds for $m_H$ are given within Type II models. In Sec. 4.4, we consider the decay $\bar{B} \to X_s \gamma$ when complex couplings are involved and we investigate the impact of their phases on CP rate asymmetries of $\bar{B} \to X_s \gamma$. We give also specific examples of couplings for which the theoretical prediction of the branching ratio is reliable and compatible with the CLEO measurement, even for $m_H = \mathcal{O}(M_W)$. Finally, our concluding remarks are in Sect. 5.

2. Two Higgs Doublet Models

Models with more than one Higgs doublet have generically a Yukawa Lagrangian of the form:

$$-\mathcal{L} = h_{ij}^d \bar{q}_L^i \phi_1 d^i_{Rj} + h_{ij}^u \bar{q}_L^i \tilde{\phi}_2 u^i_{Rj} + h_{ij}^l \bar{l}_L^i \phi_3 e^i_{Rj} + \text{h.c.},$$

where $d_L$, $u_L$, $e_L$, $(i = 1, 2, 3)$ are SU(2) doublets ($\tilde{\phi}_2 = i\sigma^2 \phi_*^i$), $u^i_R$, $d^i_R$, and $e^i_R$ are SU(2) singlets and $h_d^i$, $h_u^i$, and $h_l^i$ denote $3 \times 3$ Yukawa matrices. Gauge invariance imposes the value of hypercharge $Y(\phi_i) = 1/2$. Suitable discrete symmetries are usually invoked to forbid additional terms like $h_{ij}^d \bar{q}_L^i \phi_2 d^i_{Rj}$ which would induce flavour changing neutral couplings at the tree–level. Indeed, to avoid them altogether, it is sufficient to impose that no more than one Higgs doublet couples to the same right–handed field [34].

When only two Higgs doublets are present, $\phi_1$ and $\phi_2$, it is in general $\phi_3 = \phi_1$ (or $\phi_3 = \phi_2$). Nevertheless, for the sake of generality, we leave the symbol $\phi_3$ distinct from the other two. We indeed include in our discussion, as “effective” 2HDMs, also models where an $n$–number of sequential Higgs doublets is present and we assume that the additional charged Higg bosons other than the lightest one become heavy enough to decouple from our problem.

After rotating the fermion fields from the current eigenstate to the mass eigenstate basis, the charged Higgs component in eq. (3) becomes:

$$-\mathcal{L} = \frac{m_{d_i}}{\langle \phi_0 \rangle^2} \bar{\nu}_{Lj} V_{ji} d_{Ri} \phi_1^+ - \frac{m_{u_i}}{\langle \phi_0 \rangle^2} \bar{\nu}_{Ri} V_{ij} d_{Lj} \phi_2^+ + \frac{m_{l_i}}{\langle \phi_3 \rangle^2} \bar{\nu}_{Lj} e_{Ri} \phi_3^+ + \text{h.c.},$$

where $V_{ji}$ are elements of the Cabibbo–Kobayashi–Maskawa (CKM) matrix. A further
rotation of the charged Higgs fields to their physical basis through a unitary matrix \( U \),

\[
\begin{pmatrix}
\phi_1^+ \\
\phi_2^+ \\
\phi_3^+ \\
\vdots \\
\phi_n^+
\end{pmatrix}
\quad = \quad
U
\begin{pmatrix}
\phi^+ \\
H^+ \\
H_1^+ \\
\vdots \\
H_{n-2}^+
\end{pmatrix}
\] (5)

yields the following Yukawa interaction for \( H^+ \):

\[
\mathcal{L} = \frac{g}{\sqrt{2}} \left\{ \left( \frac{m_{d_1}}{M_W} \right) X u_{L_j} V_{ji} d_{R_i} + \left( \frac{m_{u_1}}{M_W} \right) Y \overline{u}_{R_i} V_{ij} d_{L_j} + \left( \frac{m_{l_i}}{M_W} \right) Z \overline{\nu}_{L_i} e_{R_i} \right\} H^+ + \text{h.c.} \quad \text{(6)}
\]

All fields \( H_i^+ \) are supposed to be heavy enough to become irrelevant for phenomenology at the electroweak scale. In (6), the symbols X and Y are defined in terms of elements of the matrix \( U \) \([35–37,31,32]\):

\[
X = -\frac{U_{12}}{U_{11}}; \quad Y = \frac{U_{22}}{U_{21}}.
\] (7)

The symbol Z has a similar definition, i.e. \( Z = -U_{32}/U_{31} \) if \( \phi_3 \neq \phi_1, \phi_2 \) or coincides with \( X (-Y) \) if \( \phi_3 = \phi_1 (\phi_3 = \phi_2) \). Notice that X, Y, and Z are in general complex numbers and therefore potential sources of CP violating effects \([38,39]\). Their values are only very loosely constrained by the requirement of perturbativity and low–energy processes such as the \( B \rightarrow \overline{B} \) mixing \([32]\).

When only two doublets are considered, the diagonalization matrix \( U \) is a \( 2 \times 2 \) orthogonal matrix

\[
\begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix}
\] (8)

Although both doublets are present in the theory, one still has the freedom of selecting \( \phi_1 = \phi_2 \) in eq. (3). This choice gives rise to the 2HDM of Type I, to be distinguished from the Type II in which both doublets contribute to the Yukawa interactions \([40]\). It is easy to see that in these two cases the couplings X and Y are real and given by:

\[
X = -\cot \beta, \quad Y = \cot \beta \quad \text{(Type I)}
\]
\[
X = \tan \beta, \quad Y = \cot \beta \quad \text{(Type II)}.
\] (9)

Note that the coupling of the Goldstone boson \( \phi^+ \) to matter fields is independent of the number \( n \) of Higgs doublets considered, and always equal to:

\[
\mathcal{L} = -\frac{g}{\sqrt{2}} \left\{ \left( \frac{m_{d_1}}{M_W} \right) \overline{u}_{L_j} V_{ji} d_{R_i} - \left( \frac{m_{u_1}}{M_W} \right) \overline{u}_{R_i} V_{ij} d_{L_j} + \left( \frac{m_{l_i}}{M_W} \right) \overline{\nu}_{L_i} e_{R_i} \right\} \phi^+ + \text{h.c.}, \quad \text{(10)}
\]

The calculation of the \( B \rightarrow X_s \gamma \) decay rate in these models, therefore, is modified simply by the addition of charged Higgs contributions to the usual SM one. The results can be described in terms of \( m_H \) and the two complex parameters X and Y. The presence of phases in the couplings X and Y allows for CP asymmetries in the decay rate for \( B \rightarrow X_s \gamma \).
3. The Calculation

We use the framework of an effective low–energy theory with five quarks, obtained by integrating out the heavy degrees of freedom, which in the present case are the $t$–quark, the $W$–boson and the charged Higgs boson. As in the SM calculations, we only take into account operators up to dimension six and we put $m_s = 0$. In this approximation the effective Hamiltonian relevant for radiative $B$–decays (with $|\Delta B| = |\Delta S| = 1$)

$$H_{eff} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu) O_i(\mu)$$

consists precisely of the same operators $O_i(\mu)$ used in the SM case, weighted by the Wilson coefficients $C_i(\mu)$. They read:

$$O_1 = (\bar{s}_L\gamma_\mu T^a c_L)(\bar{c}_L\gamma^\mu T_a b_L), \quad O_2 = (\bar{s}_L\gamma_\mu c_L)(\bar{c}_L\gamma^\mu b_L),$$

$$O_3 = (\bar{s}_L\gamma_\mu b_L) \sum_{q}(\bar{q}\gamma^\mu q), \quad O_4 = (\bar{s}_L\gamma_\mu T^a b_L) \sum_{q}(\bar{q}\gamma^\mu T_a q),$$

$$O_5 = (\bar{s}_L\gamma_\mu \gamma^\nu \gamma_\rho b_L) \sum_{q}(\bar{q}\gamma^\mu \gamma^\nu \gamma_\rho q), \quad O_6 = (\bar{s}_L\gamma_\mu \gamma^\nu \gamma_\rho T_a b_L) \sum_{q}(\bar{q}\gamma^\mu \gamma^\nu \gamma_\rho T_a q),$$

$$O_7 = \frac{g_s}{16\pi} \overline{m}_b(\mu) (\bar{s}_L\sigma^{\mu\nu} b_R) F_{\mu\nu}, \quad O_8 = \frac{g}{16\pi} \overline{m}_b(\mu) (\bar{s}_L\sigma^{\mu\nu} T^a b_R) G_{\mu\nu}^a,$$

where $T^a$ ($a = 1, 8$) are $SU(3)$ colour generators; $g_s$ and $g$, the strong and electromagnetic coupling constants. In eq. (12), $\overline{m}_b(\mu)$ is the running $b$–quark mass in the $\overline{MS}$ scheme at the renormalization scale $\mu$. Henceforth, $\overline{m}_q(\mu)$ and $m_q$ denote $\overline{MS}$ running and pole masses, respectively. To first order in $\alpha_s$, these masses are related through:

$$\overline{m}_q(\mu) = m_q \left(1 + \frac{\alpha_s(\mu)}{\pi} \ln \frac{m_q^2}{\mu^2} - \frac{4\alpha_s(\mu)}{3\pi}\right).$$

Note that the equations of motion have been used when writing down the list of operators in eq. (12). This is sufficient since we are interested in on–shell matrix elements [41] and since we choose to perform the matching by comparing on–shell amplitudes obtained in the effective theory with the corresponding amplitudes in the full theory. The reader who is interested in doing off–shell matching (and therefore working in a (larger) off–shell operator basis) is referred to ref. [17].

Working to NLO precision means that one is resumming all the terms of the form $\alpha_s^a(m_b) \ln^a(m_b/M)$, as well as $\alpha_s(m_b) (\alpha_s^a(m_b) \ln^a(m_b/M))$, where $M$ stands for one of the heavy masses $M_W$, $M_t$ or $M_H$. This is achieved by performing the following 3 steps:

1) One matches the full standard model theory with the effective theory at a scale $\mu_w$, of order $M$. At this scale, the matrix elements of the operators in the effective theory lead to the same logarithms as the calculation in the full theory. Consequently, the Wilson coefficients $C_i(\mu_w)$ only pick up “small” QCD corrections, which can be calculated in fixed order perturbation theory. In the NLO program, the matching has to be worked out at the $O(\alpha_s)$ level.
2) The evolution of these Wilson coefficients from $\mu = \mu_W$ down to $\mu = \mu_b$, where $\mu_b$ is of the order of $m_b$, is obtained by solving the appropriate Renormalization Group Equation (RGE). As the matrix elements of the operators evaluated at the low scale $\mu_b$ are free of large logarithms, the latter are contained in resummed form in the Wilson coefficients. For a NLO calculation, this step has to be performed using the anomalous dimension matrix up to order $\alpha_s^2$.

3) The corrections to the matrix elements of the operators $\langle s\gamma|O_i(\mu)|b\rangle$ at the scale $\mu = \mu_b$ have to be calculated to order $\alpha_s^3$ precision.

The charged Higgs boson enters the NLO calculation only via step 1). The Higgs boson contribution to the matching condition is obtained in the same way as the SM one [15]; therefore, we do not repeat any technical details. A general remark, however, is in order. In the procedure described above all heavy particles ($W$–boson, $t$–quark and charged Higgs) are integrated out simultaneously at the scale $\mu_W$. In the context of a NLO calculation this should be a reasonable approximation provided $m_H$ is of the same order of magnitude as $M_W$ or $m_t$.

Before giving the results of the three steps listed above, we should briefly mention that instead of the original Wilson coefficients $C_i(\mu)$ it is convenient to use certain linear combinations of them, the so–called “effective Wilson” coefficients $C_i^{\text{eff}}(\mu)$ introduced in [13,21]:

$$
C_i^{\text{eff}}(\mu) = C_i(\mu), \quad (i = 1, \ldots, 6),
$$

$$
C_7^{\text{eff}}(\mu) = C_7(\mu) + \sum_{i=1}^{6} y_i C_i(\mu), \quad C_8^{\text{eff}}(\mu) = C_8(\mu) + \sum_{i=1}^{6} z_i C_i(\mu),
$$

(14)

where $y_i$ and $z_i$ are defined in such a way that the leading order matrix elements $\langle s\gamma|O_i|b\rangle$ and $\langle sg|O_i|b\rangle$ ($i = 1, 6$) are absorbed in the leading orders terms in $C_7^{\text{eff}}(\mu)$ and $C_8^{\text{eff}}(\mu)$.

The explicit values of $\{y_i\}$ and $\{z_i\}$, $y = (0, 0, -\frac{1}{3}, -\frac{4}{9}, -\frac{20}{3}, -\frac{80}{9})$, $z = (0, 0, 1, -\frac{1}{6}, 20, -\frac{10}{3})$ were obtained in ref. [21] in the $\overline{\text{MS}}$ scheme with fully anticommuting $\gamma_5$ (also used in the present paper).

3.1. NLO Wilson coefficients at the matching scale $\mu_W$: $C_i^{\text{eff}}(\mu_W)$

To give the results for the effective Wilson coefficients $C_i^{\text{eff}}$ at the matching scale $\mu_W$ in a compact form, we write:

$$
C_i^{\text{eff}}(\mu_W) = C_i^{0,\text{eff}}(\mu_W) + \frac{\alpha_s(\mu_W)}{4\pi} C_i^{1,\text{eff}}(\mu_W).
$$

(15)

The LO Wilson coefficients at this scale are well known [42,28]. We decompose them in such a way to render explicit their dependence on the couplings $X$ and $Y$:

$$
C_2^{0,\text{eff}}(\mu_W) = 1
$$
\[
C_{i}^{0, \text{eff}}(\mu_w) = 0 \quad (i = 1, 3, 4, 5, 6)
\]
\[
C_{7}^{0, \text{eff}}(\mu_w) = C_{7, \text{SM}}^{0} + \{Y\}^{2} C_{7, \text{YY}}^{0} + (XY^{*}) C_{7, \text{XY}}^{0}
\]
\[
C_{8}^{0, \text{eff}}(\mu_w) = C_{8, \text{SM}}^{0} + \{Y\}^{2} C_{8, \text{YY}}^{0} + (XY^{*}) C_{8, \text{XY}}^{0}.
\]  

(16)

The coefficients \(C_{i}^{0, \text{SM}}(\mu_w)\) and \(C_{i}^{0, \text{SM}}(\mu_w)\) are functions of \(x = m_{t}^{2}/M_{w}^{2}\), while \(C_{7, j}^{0}(\mu_w)\) and \(C_{8, j}(\mu_w)\) \((j = \text{YY, XY})\) are functions of \(y = m_{t}^{2}/m_{H}^{2}\); their explicit forms are given in Appendix A. Note that there is no explicit dependence of the matching scale \(\mu_w\) in these functions. Whether there is an implicit \(\mu_w\)-dependence via the \(t\)-quark mass depends on the precise definition of this mass which has to be specified when going beyond leading logarithms. If one chooses for example to work with \(\overline{m}_{t}(\mu_w)\), then there is such an implicit \(\mu_w\)-dependence of the lowest order Wilson coefficient; in contrast, when working with the pole mass \(m_{t}\) there is no such dependence. We choose to express our (NLO) results in terms of the pole mass \(m_{t}\).

The NLO pieces \(C_{i}^{1, \text{eff}}(\mu_w)\) of the Wilson coefficients have an explicit dependence on the matching scale \(\mu_w\) and for \(i = 7, 8\) they also explicitly depend on the actual definition of the \(t\)-quark mass. Initially, when the heavy particles are integrated out, it is convenient to work out the matching conditions \(C_{i}^{1, \text{eff}}(\mu_w)\) for \(i = 7, 8\) in terms of \(\overline{m}_{t}(\mu_w)\). Using eq. (13), it is then straightforward to get the corresponding result expressed in terms of the pole mass \(m_{t}\). As in the LO case we give them in a form where the dependence of the couplings \(X\) and \(Y\) is explicit:

\[
C_{1}^{1, \text{eff}}(\mu_w) = 15 + 6 \ln\frac{\mu_{w}^{2}}{M_{w}^{2}}
\]
\[
C_{4}^{1, \text{eff}}(\mu_w) = E_{0} + \frac{2}{3} \ln\frac{\mu_{w}^{2}}{M_{w}^{2}} + \{Y\}^{2} E_{H}
\]
\[
C_{i}^{1, \text{eff}}(\mu_w) = 0 \quad (i = 2, 3, 5, 6)
\]
\[
C_{7}^{1, \text{eff}}(\mu_w) = C_{7, \text{SM}}^{1, \text{eff}}(\mu_w) + \{Y\}^{2} C_{7, \text{YY}}^{1, \text{eff}}(\mu_w) + (XY^{*}) C_{7, \text{XY}}^{1, \text{eff}}(\mu_w)
\]
\[
C_{8}^{1, \text{eff}}(\mu_w) = C_{8, \text{SM}}^{1, \text{eff}}(\mu_w) + \{Y\}^{2} C_{8, \text{YY}}^{1, \text{eff}}(\mu_w) + (XY^{*}) C_{8, \text{XY}}^{1, \text{eff}}(\mu_w),
\]  

(17)

where for \(i = 7, 8\) the three terms on the right hand side can be written in the form

\[
C_{i, \text{SM}}^{1, \text{eff}}(\mu_w) = W_{i, \text{SM}} + M_{i, \text{SM}} \ln\frac{\mu_{w}^{2}}{M_{w}^{2}} + T_{i, \text{SM}} \left(\ln\frac{m_{t}^{2}}{\mu_{w}^{2}} - \frac{4}{3}\right)
\]
\[
C_{i, \text{YY}}^{1, \text{eff}}(\mu_w) = W_{i, \text{YY}} + M_{i, \text{YY}} \ln\frac{\mu_{w}^{2}}{m_{H}^{2}} + T_{i, \text{YY}} \left(\ln\frac{m_{t}^{2}}{\mu_{w}^{2}} - \frac{4}{3}\right)
\]
\[
C_{i, \text{XY}}^{1, \text{eff}}(\mu_w) = W_{i, \text{XY}} + M_{i, \text{XY}} \ln\frac{\mu_{w}^{2}}{m_{H}^{2}} + T_{i, \text{XY}} \left(\ln\frac{m_{t}^{2}}{\mu_{w}^{2}} - \frac{4}{3}\right).
\]  

(18)

Note that in eq. (18) the \(W_{i,j}\) and the \(M_{i,j}\) \((j = \text{SM, XY, YY})\) terms would be the full result when working in terms of \(\overline{m}_{t}(\mu_w)\). The \(T_{i,j}\) terms result when expressing \(\overline{m}_{t}(\mu_w)\) in terms of the pole mass \(m_{t}\) in the corresponding lowest order coefficients. Thus, for
Notice that if one worked with the running $t$-quark mass $\bar{m}_t(\mu_t)$ normalized at the scale $\mu_t$ instead of the pole mass $m_t$, the third terms on the right hand sides of eqs. (18) would have to be replaced by $T_{i,j}\ln(\mu_t^2/\mu_w^2)$ ($j = \text{SM, YY, XY}$). The functions $W_{i,j}$, $M_{i,j}$ and $T_{i,j}$ ($j = \text{SM, YY, XY}$), together with $E_0$ and $E_H$, are listed in Appendix A. Our results for $C_{i,\text{eff}}^1(\mu_w)$ and $C_{i,\text{eff}}^3(\mu_w)$ agree with those in refs. [33] and [17], when taking into account that the latter results are expressed in terms of $\bar{m}_t(\mu_w)$. We disagree, however, with the form of the coefficients $C_{i,\text{eff}}^1(\mu_w)$ and $C_{i,\text{eff}}^3(\mu_w)$, whose matching scale dependence is forgotten in [27,17]. As it will be seen later on, this neglect has consequences in the determination of the matching scale which yields the lowest estimate of $\text{BR}(\bar{B} \to X_s\gamma)$.

3.2. NLO Wilson coefficients at the low–scale $\mu_b$: $C_i^{\text{eff}}(\mu_b)$

The evolution from the matching scale $\mu_w$ down to the low–energy scale $\mu_b$ is described by the RGE:

$$\frac{\mu}{d\mu}C_i^{\text{eff}}(\mu) = C_j^{\text{eff}}(\mu) \gamma_{ij}^{\text{eff}}(\mu).$$  \hspace{1cm} (20)

The initial conditions $C_i^{\text{eff}}(\mu_w)$ for this equation are given in Section 3.1, and the anomalous dimension matrix $\gamma_{ij}^{\text{eff}}$ is given in Appendix B up to order $\alpha_s^2$, which is the precision needed for a NLO calculation. The solution of eq. (20), obtained through the procedure described in [43], yields for the coefficient

$$C_i^{\text{eff}}(\mu_b) = C_i^{0,\text{eff}}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_i^{1,\text{eff}}(\mu_b),$$  \hspace{1cm} (21)

the LO term:

$$C_i^{0,\text{eff}}(\mu_b) = \eta_i^{14} C_i^{0,\text{eff}}(\mu_w) + \frac{8}{3} \left( \eta_i^{14} - \eta_i^{25} \right) C_s^{0,\text{eff}}(\mu_w) + \sum_{i=1}^8 h_i \eta_i^{23} C_2^{0,\text{eff}}(\mu_w),$$  \hspace{1cm} (22)

and the NLO one:

$$C_i^{1,\text{eff}}(\mu_b) = \eta_i^{23} C_i^{1,\text{eff}}(\mu_w) + \frac{8}{3} \left( \eta_i^{23} - \eta_i^{25} \right) C_s^{1,\text{eff}}(\mu_w)$$

$$+ \frac{297664}{14283} \left( \eta_i^{16} - \frac{716446}{357075} \eta_i^{14} + \frac{256868}{14283} \eta_i^{25} - \frac{6698884}{357075} \eta_i^{23} \right) C_s^{0,\text{eff}}(\mu_w)$$

$$+ \frac{37208}{4761} \left( \eta_i^{23} - \eta_i^{32} \right) C_7^{0,\text{eff}}(\mu_w)$$

$$+ \sum_{i=1}^8 (e_i \eta_i^{14} C_4^{1,\text{eff}}(\mu_w) + (f_i + k_i \eta) C_2^{0,\text{eff}}(\mu_w) + l_i \eta C_4^{1,\text{eff}}(\mu_w)) \eta_i^{a_i},$$  \hspace{1cm} (23)
The symbol $\eta$ is defined as $\eta = \alpha_s(\mu_w)/\alpha_s(\mu_b)$; the vectors $a_i$, $h_i$, $e_i$, $f_i$, $k_i$ and $l_i$ are listed in Appendix C. Notice that eq. (23) can be used in this form for all models in which the same set of coefficients $C_i(\mu_w)$ are non vanishing. It is more general than the corresponding eq. (21) given in [21], which can be used only when the matching scale $\mu$ is fixed at the value $M_w$. The use made of that equation in [27], where the dependence of $\text{BR}(\bar{B} \to X_s\gamma)$ on the matching scale is studied, is inappropriate.

As far as the other Wilson coefficients are concerned, they are only needed to LO precision in the complete NLO analysis of $\text{BR}(\bar{B} \to X_s\gamma)$. In this precision, the Wilson coefficients of the four–Fermi operators $(i = 1, \ldots, 6)$ are the same as in the SM. As the coefficients $C_3^{\text{eff}}(\mu_b), \ldots, C_6^{\text{eff}}(\mu_b)$ are numerically much smaller than $C_2^{\text{eff}}(\mu_b)$, we neglect contributions proportional to these small Wilson coefficients in the amplitude for $\text{BR}(\bar{B} \to X_s\gamma)$ and list here only the LO expressions for $C_1^{\text{eff}}(\mu_b)$, $C_2^{\text{eff}}(\mu_b)$, and $C_8^{\text{eff}}(\mu_b)$:

$$
C_1^{0,\text{eff}}(\mu_b) = (\eta^6 - \eta^2) C_2^{0,\text{eff}}(\mu_w) \\
C_2^{0,\text{eff}}(\mu_b) = \left(\frac{2}{3} \eta^\frac{6}{23} + \frac{1}{3} \eta^\frac{12}{23}\right) C_2^{0,\text{eff}}(\mu_w) \\
C_8^{0,\text{eff}}(\mu_b) = \eta^\frac{12}{23} C_8^{0,\text{eff}}(\mu_w) + \sum_{i=1}^{5} h'_i \eta^\epsilon_i C_2^{0,\text{eff}}(\mu_w),
$$

When discussing the contributions due to the charged Higgs boson, it is convenient to split the Wilson coefficients at the scale $\mu_b$ into the contributions $C_i^{\text{eff}}(\mu_b)$, $C_{i,\text{SM}}(\mu_b)$, and $C_{i,\text{eff}}(\mu_b)$:

$$
C_i^{\text{eff}}(\mu_b) = C_i^{\text{eff}}(\mu_b) + |Y|^2 C_i^{\text{eff}}(\mu_b) + (X Y^*) C_i^{\text{eff}}(\mu_b).
$$

As the solution (23) of the renormalization group equation (20) is linear in the initial conditions, this splitting is given in an obvious way in terms of the corresponding splitting at the matching scale $\mu_w$, presented in Section 3.1.

When calculating NLO results in the numerical analyses in Section 4, we use the NLO expression for the strong coupling constant:

$$
\alpha_s(\mu) = \frac{\alpha_s(M_Z)}{v(\mu)} \left[ 1 - \frac{\beta_1}{\beta_0} \frac{\alpha_s(M_Z)}{4\pi} \ln \left( \frac{\mu}{v(\mu)} \right) \right],
$$

with

$$
v(\mu) = 1 - \frac{\beta_0}{2\beta_1} \frac{\alpha_s(M_Z)}{\beta_0} \ln \left( \frac{M_Z}{\mu} \right),
$$

where $\beta_0 = \frac{23}{3}$ and $\beta_1 = \frac{116}{3}$. However, for LO results we always use the LO expression for $\alpha_s(\mu)$, i.e., $\beta_1$ is put to zero in eq. (26).

### 3.3. Branching ratio for $\text{BR}(\bar{B} \to X_s\gamma)$

We first give the formulas for the quark decay $b \to X_s\gamma$ and discuss the meson decay $\bar{B} \to X_s\gamma$ later. In a NLO calculation, $b \to X_s\gamma$ involves the subprocesses $b \to s\gamma$
(including virtual corrections) and \( b \to s \gamma g \), i.e., the gluon bremsstrahlung process. The amplitude for the first can be written as

\[
A(b \to s \gamma) = -\frac{4G_F}{\sqrt{2}} V^*_{ts} V_{tb} \mathcal{D}(s \gamma|O_7|b)_{\text{tree}},
\]

where the reduced amplitude \( \mathcal{D} \) is

\[
\mathcal{D} = C_7^{0, \text{eff}}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} \left( C_7^{0, \text{eff}}(\mu_b) + V(\mu_b) \right).
\]

The symbol \( V(\mu_b) \) is defined as:

\[
V(\mu_b) = \sum_{i=1}^{8} C_i^{0, \text{eff}}(\mu_b) \left[ r_i + \frac{1}{2} \gamma_i^{0, \text{eff}} \ln \frac{m_b^2}{\mu_b^2} - \frac{16}{3} C_i^{0, \text{eff}}(\mu_b) \right].
\]

In writing eq. (28) we directly converted the running mass factor \( m_b(\mu_b) \), which appears in the definition of the operator \( O_7 \) in eq. (12), into the pole mass \( m_b \) by making use of eq. (13). This conversion is absorbed into the function \( V(\mu_b) \) and consequently the symbol \( (s \gamma|O_7|b)_{\text{tree}} \) is the tree-level matrix element of the operator \( O_7 \), where the running mass factor \( m_b(\mu_b) \) is understood to be replaced by the pole mass \( m_b \). In the previous literature, this procedure was done in two step. First, \( m_b(\mu_b) \) was expressed in terms of \( m_b(m_b) \) and then in turn \( m_b(m_b) \) was converted into the pole mass \( m_b \). This last step brought into the game the quantity \( F \) (see e.g. ref. [18]). The elements \( \gamma_i^{0, \text{eff}} \) of the anomalous dimension matrix and the virtual correction functions \( r_i \) in (30) are given in Appendix B and D, respectively. Note, that some parts of the bremsstrahlung contributions associated with \( O_7 \) are effectively transferred to \( r_7 \) as detailed in ref. [18].

A splitting analogous to that in (25) holds also for the reduced amplitude \( \mathcal{D} \):

\[
\mathcal{D} = \mathcal{D}_{SM} + |Y|^2 \mathcal{D}_{YY} + (X Y^*) \mathcal{D}_{XY}.
\]

From \( A(b \to s \gamma) \) in eq. (28) the decay width \( \Gamma(b \to s \gamma) \) is easily obtained to be

\[
\Gamma(b \to s \gamma) = \frac{G_F^2}{32\pi^4} |V^*_{ts} V_{tb}|^2 \alpha_{em} m_b^5 |\mathcal{D}|^2.
\]

In the numerics we discard those terms in \( |\mathcal{D}|^2 \) which are explicitly of order \( \alpha_s^2 \). Note, however, that there are implicit higher order terms which are retained. Such terms arise for example because we evaluate the quantity \( \eta = (\alpha_s(\mu_W)/\alpha_s(\mu_b)) \) using the NLO expression for \( \alpha_s \) also in those Wilson coefficients which are needed only in LO precision.

To obtain the inclusive rate for \( b \to X_s \gamma \) consistently at the NLO level, we have to take into account the bremsstrahlung contributions [10,44]. The corresponding decay width \( \Gamma(b \to s \gamma g) \) is of the form

\[
\Gamma(b \to s \gamma g) = \frac{G_F^2}{32\pi^4} |V^*_{ts} V_{tb}|^2 \alpha_{em} m_b^5 A,
\]

(33)
where $A$ is

$$A = \frac{\alpha_s(m_b)}{\pi} \sum_{i,j=1; i\neq j}^{8} \text{Re} \left\{ C_{i}^{0, \text{eff}}(m_b) \left[ C_{j}^{0, \text{eff}}(m_b) \right]^* f_{ij} \right\}. \quad (34)$$

As in the virtual contributions we put $C_i^{0, \text{eff}} = 0$ for $i = 3, \ldots, 6$. In contrast to ref. [21], we do not introduce a cut–off when the photon gets soft. In order to cancel the infrared singularity, which appears in this case, we include as in ref. [18] the virtual photonic correction to the process $b \to s\gamma$, which we absorb into the quantity $f_{88}$ (see eq. (B.9) in ref. [18]). Note also, that in our approach the term $f_{77}$ is already absorbed into the function $r_7$. The non–vanishing $f_{ij}$ terms, which have to be taken into account explicitly in eq. (34), are listed in Appendix E.

In order to get the decay width for the meson decay $\bar{B} \to X_s\gamma$ we take into account the non–perturbative corrections which scale like $1/m_b^2$ [23] and those which scale like $1/m_c^2$ [26]. The decay width $\Gamma(\bar{B} \to X_s\gamma)$ then reads:

$$\Gamma(\bar{B} \to X_s\gamma) = \frac{G_F^2}{32\pi^4} |V_{ts}|^2 \alpha_{em} m_b^5$$

$$\left\{ |\bar{B}|^2 + A + \frac{\delta_{\gamma}^{\text{NP}}}{m_b^2} |C_1^{0, \text{eff}}(m_b)|^2 + \frac{\delta_{\epsilon}^{\text{NP}}}{m_c^2} \text{Re} \left\{ \left[ C_2^{0, \text{eff}}(m_b) \right]^* \left( C_2^{0, \text{eff}}(m_b) - \frac{1}{6} C_1^{0, \text{eff}}(m_b) \right) \right\} \right\}. \quad (35)$$

Since we work in the new operator basis, a contribution to the term in $1/m_b^2$ comes not only from $O_3$, as it was incorrectly assumed in [17], but also from $O_1$. The non–perturbative quantities $\delta_{\gamma}^{\text{NP}}$ [23] and $\delta_{\epsilon}^{\text{NP}}$ [26] in (35) are:

$$\delta_{\gamma}^{\text{NP}} = \frac{\lambda_1}{2} - \frac{9\lambda_2}{2}, \quad \delta_{\epsilon}^{\text{NP}} = -\frac{\lambda_2}{9}, \quad (36)$$

where $\lambda_1$ and $\lambda_2$ parametrize the kinetic energy of the $b$–quark and the chromomagnetic interactions, respectively. Their values are: $\lambda_1 = -0.5 \text{ GeV}^2$ and $\lambda_2 = -0.12 \text{ GeV}^2$. Of these two parameters, the first has larger uncertainties than the second one. As it will appear from the final formula for $\text{BR}(\bar{B} \to X_s\gamma)$, the overall $\lambda_1$ dependence cancels to a large extent.

The branching ratio is then obtained as

$$\text{BR}(\bar{B} \to X_s\gamma) = \frac{\Gamma(\bar{B} \to X_s\gamma)}{\Gamma_{sL}} \text{BR}_{sL}, \quad (37)$$

where $\text{BR}_{sL}$ is the measured semileptonic branching ratio and the semileptonic decay width $\Gamma_{sL}$ is given by:

$$\Gamma_{sL} = \frac{G_F^2}{192\pi^3} |V_{cb}|^2 m_b^5 g(z) \left( 1 - \frac{2\alpha_s(m_b)}{3\pi} f(z) + \frac{\delta_{sL}^{\text{NP}}}{m_c^2} \right); \quad z = \frac{m_c^2}{m_b^2}. \quad (38)$$

The phase space function $g(z)$ and the (approximated) QCD–radiation function $f(z)$ [45] in (38) are:

$$g(z) = 1 - 8z + 8z^3 - z^4 - 12z^2 \ln z, \quad f(z) = \left( \pi^2 - \frac{31}{4} \right) (1 - \sqrt{z})^2 + \frac{3}{2}, \quad (39)$$

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and the non–perturbative correction $\delta^{NP}_{SL}$ reads [24,25]:

$$\delta^{NP}_{SL} = \frac{\lambda_1}{2} + \frac{3\lambda_2}{2} \left[ 1 - 4 \left(\frac{1-z}{g(z)}\right)^4 \right].$$  \hfill (40)

In eq. (38) $\bar{\mu}_b$ is the renormalization scale relevant to the semileptonic process, which is a priori different from the renormalization scale $\mu_b$ in the radiative decay, as stressed in [27]. However, as pointed out in ref. [46], the identification $\bar{\mu}_b = \mu_b$ turns out to be more conservative. We therefore use $\bar{\mu}_b = \mu_b$ in the numerical analysis. Furthermore, in the evaluation of eq. (37), we do not expand $1/\Gamma_{SL}$ in powers of $\alpha_s$.

4. Results

We discuss in this section results and theoretical uncertainties of the NLO calculation of $\text{BR}(B \to X_s\gamma)$ in 2HDMs. As it will be shown, these uncertainties can be very large, in contrast with what is found in the SM. For reference, we give our SM result:

$$\text{BR}(B \to X_s\gamma) = \left(3.57 \pm 0.01_{0.12} (\mu_b) \pm 0.00_{0.08} (\mu_w) \pm 0.29_{0.27} \text{ (param)}\right) \times 10^{-4}. \hfill (41)$$

The central value $3.57 \times 10^{-4}$ is obtained for $\mu_b = 4.8$ GeV, $\mu_w = M_W$ and the central values of the input parameters listed in Table 1 of Appendix F. The low–scale variation in the interval $[2.4, 9.6]$ GeV gives the maximum value for $\text{BR}(B \to X_s\gamma)$ at $\mu_b = 4.2$ GeV and the minimum at $\mu_b = 9.6$ GeV. The matching scale dependence of the branching ratio is monotonically decreasing for increasing $\mu_w$: from the central value in (41), $\text{BR}(B \to X_s\gamma)$ is reduced by 2% at $\mu_w = m_t$. The value $3.57 \times 10^{-4}$ in eq. (41) reduces to $3.46 \times 10^{-4}$ when the factor $1/\Gamma_{SL}$ in eq. (37) is expanded in $\alpha_s$.

Our results for the branching ratio $\text{BR}(B \to X_s\gamma)$ in the class of 2HDMs considered, is parametrized in terms of $\{X, Y, m_H\}$. We limit the range of this three–dimensional parameter space as follows. We fix $Y$ to small real values of $O(1)$ and scan $X$ in the complex plane without ever violating the perturbativity requirement on $X$ and $Y$ discussed in [32]. As explained in [47], the value of $m_H$ in 2HDMs can be as low as the LEP I lower bound of 45 GeV. We do not strictly apply cuts on $\{X, Y, m_H\}$ due to the measurement of $R_b$ or of other processes like $B\bar{B}$ mixing with virtual exchange of charged Higgs bosons, which could remove some corners of the parameter space considered. Our interest is more to show some theoretical features of the NLO calculation of $\text{BR}(B \to X_s\gamma)$ in regions as wide as possible and to discuss constraints due to the decay $B \to X_s\gamma$ itself. For the numerical evaluations, unless otherwise specified, we use the central value of the input parameters listed in Table 1 in Appendix F. The values of the matching and low–scale, chosen respectively in the intervals $[M_W, \text{Max}(m_t, m_H)]$ and $[2.4, 9.6]$ GeV, are explicitly given for each result presented.

In order to investigate the reliability of the NLO prediction for the branching ratio, we study the term $|\mathcal{D}|^2$, which dominates the expression within curly brackets in (35).
Notice that $|\bar{D}|^2$ encapsulates all the matching scale dependence and the bulk of the dependence on the low–scale. At the NLO level, the procedure of squaring $\bar{D}$ dictated by perturbation theory, in which terms of $O(\alpha_s^2)$ are omitted, gives:

$$|\bar{D}|^2 = |C_7^{0,\text{eff}}(\mu_b)|^2 \left\{ 1 + 2 \text{Re} (\Delta \bar{D}) \right\}, \quad (42)$$

where $\Delta \bar{D}$ is defined as (see eq. (29)):

$$\Delta \bar{D} \equiv \frac{\bar{D} - C_7^{0,\text{eff}}(\mu_b)}{C_7^{0,\text{eff}}(\mu_b)} = \left( \frac{\alpha_s(\mu_b)}{4\pi} \right) \frac{C_7^{1,\text{eff}}(\mu_b) + V(\mu_b)}{C_7^{0,\text{eff}}(\mu_b)}. \quad (43)$$

If $|\Delta \bar{D}|$ is not small, the formally Next to Next to Leading Order (NNLO) term $|\Delta \bar{D}|^2$, dropped in eq. (42), is numerically relevant. Its omission can lead to branching ratios with a large dependence on $\mu_b$, or in extreme cases, even to negative values for the branching ratio. In these situations, the truncation of the perturbative expansion of $\bar{D}$ at the NLO level is certainly not justified. As we shall see, the size of $|\Delta \bar{D}|$ depends crucially on the values of $X$, $Y$ and $m_H$. We split our analysis as follows. In Sect. 4.1, we disentangle the effect of the couplings $X$ and $Y$ by studying the NLO contributions to the Higgs components $\bar{D}_{XY}$ and $\bar{D}_{YY}$ of the reduced amplitude $\bar{D}$. In Sect. 4.2 we illustrate predictions for the branching ratio for real $X$ and $Y$ couplings, giving particular emphasis to cases in which the results are highly unstable or altogether unacceptable. In Sect. 4.3 we discuss in detail Type II models, for which the NLO corrections are under control, and Type I models for which the reliability of the theoretical prediction depends strongly on the point of parameter space considered. Finally, in Sect. 4.4, we study branching ratios and CP asymmetries in the presence of complex couplings, and outline regions of parameter space where these predictions are trustworthy.

4.1. Amplitude: $X$–$Y$ independent analysis

Before analyzing the size of the various corrections within 2HDMs, it is worth reviewing the situation in the SM. There, as it was first observed in [21], the NLO contribution to the matching condition tends to cancel the contribution due to the NLO evolution from the matching scale $\mu_W$ to the low–energy scale $\mu_b$ (see upper frame in Fig. 1). Indeed, $C_7^{1,\text{eff}}$, which is 12% of $C_7^{0,\text{eff}}$ at $\mu_W$, is practically negligible for $\mu_b \in [2.4,9.6]$ GeV. The dominant NLO effect is therefore due to $V_{SM}(\mu_b)$ which, in the same range of $\mu_b$, is at most 20% of the leading term $C_7^{0,\text{eff}}(\mu_b)$ and vanishes exactly at $\mu_b \sim 3.5$ GeV. These features are illustrated in Fig. 2. The size of $V_{SM}(\mu_b)$ results from cancellations among the individual contributions of the four operators $O_1$, $O_2$, $O_7$ and $O_8$ retained in eq. (30). The details of the above described cancellations and, more generally, of the distribution of the complete NLO correction among the individual terms in eq. (29), are specific to the $\overline{\text{MS}}$ scheme (with anticommuting $\gamma_5$) and could be altered by a different choice of scheme. We remind that only the complete $\bar{D}_{SM}$ term as well as the final BR($\bar{B} \to X_s\gamma$) are scheme–independent, up to higher order.
Figure 1: RGE evolution of the Wilson coefficients $C_{7,j}^{\text{eff}}$, at the LO (with NLO $\alpha_s$) (dashed lines) and at the NLO (solid lines), for $\mu_W = M_W$. The upper frame shows the SM coefficients, the lower one the coefficients $C_{7,YY}^{\text{eff}}, C_{7,XY}^{\text{eff}}$, for $m_H = 100$ GeV. The needed input parameters are fixed at their central values listed in Table 1 in Appendix F.
Figure 2: Low-energy dependence of the term \( \bar{D} \), (solid line), of the LO coefficient \( C_0^{\text{eff}}(\mu_b) \) (with NLO \( \alpha_s \)) (dashed line), and of \( C_7^{\text{eff}}(\mu_b) \) (dotted lines), for \( \mu_W = M_W \). For \( \bar{D}_{XY} \) and \( \bar{D}_{YY} \), the value \( m_H = 100 \) GeV was used. The needed input parameters are fixed at their central values listed in Table 1 in Appendix F.

One is naturally led to ask whether the cancellations among different sets of NLO corrections observed in the SM are spoiled in 2HDMs. To address this issue, we investigate the building blocks \( \bar{D}_{XY} \) and \( \bar{D}_{YY} \) defined in eq. (31), for a specific value of \( m_H \). In Fig. 2, we show the real parts of \( \bar{D}_{XY} \) and \( \bar{D}_{YY} \) for \( m_H = 100 \) GeV, together with the real part of \( \bar{D}_{SM} \). Even when considering real X and Y couplings, imaginary parts to \( \bar{D} \) come from the absorptive terms of the loop corrections in (30). As it can be seen from eq. (42), the imaginary parts of \( \bar{D}_{XY} \), \( \bar{D}_{YY} \) and \( \bar{D}_{SM} \) do not contribute to \( \text{BR}(\bar{B} \to X_s\gamma) \) at this order of perturbation theory if X and Y are real. In the remainder of this Section, therefore, any reference to these components is understood as a reference to their real parts.

Unlike in the SM, the NLO corrections coming from \( C_{7,XY}^1(\mu_b) \) and \( V_{XY}(\mu_b) \), have roughly the same size (~20%) and the same sign, for \( m_H = 100 \) GeV. The combined correction

\[
\Delta \bar{D}_{XY} \equiv \frac{\bar{D}_{XY} - C_{7,XY}^{0,\text{eff}}(\mu_b)}{C_{7,XY}^{0,\text{eff}}(\mu_b)} = \left( \frac{\alpha_s(\mu_b)}{4\pi} \right) \frac{C_{7,XY}^{1,\text{eff}}(\mu_b) + V_{XY}(\mu_b)}{C_{7,XY}^{0,\text{eff}}(\mu_b)}
\]

amounts to the considerable values of \(-43\%\) to \(-36\%\) when varying \( \mu_b \in [2.4, 9.6] \) GeV. (We warn here that all components of the LO Wilson coefficient \( C_7^{0,\text{eff}}(\mu_b) \) discussed in
Figure 3: Low-scale dependence of the term $\mathcal{D}$, (solid lines), of the LO coefficient $C_{7}^{0,\text{eff}}(\mu_{b})$ (with NLO $\alpha_{s}$) (dashed lines), and of $C_{7}^{\text{eff}}(\mu_{b})$ (dotted lines), for $m_{H} = 500$ GeV. The matching scale $\mu_{W} = M_{W}$ is used in the upper frame and to $\mu_{W} = m_{H}$ in the lower one. The needed input parameters are fixed at their central values listed in Table 1 in Appendix F.
this Section and plotted in Figs. 1 and 2, are evaluated with NLO $\alpha_s$.) Differently than in the SM, there is no scale $\mu_b$ in the range considered, at which the LO and NLO prediction for $\mathcal{D}_{XY}$ coincide. Similar results hold for $\mathcal{D}_{YY}$.

The change in matching scale from $\mu_W = M_W$ (value used for Fig. 2) to, say, $\mu_W = m_H$, is practically inconsequential for $m_H = 100$ GeV. It becomes very relevant for large values of $m_H$, since it crucially affects the size of the NLO corrections. For $m_H = 500$ GeV, as shown in Fig. 3, the correction $\Delta\mathcal{D}_{XY}$ reduces to $-37\%$, $-31\%$ when varying $\mu_b$ in the usual interval $[2.4, 9.6]$ GeV, whereas values of $-51\%$, $-45\%$ are obtained for $\mu_W = M_W$. Notice that, for $m_H = 500$ GeV, $|\mathcal{D}_{XY}|$ is roughly ten times smaller than $|\mathcal{D}_{SM}|$ and $\mathcal{D}_{YY}$ completely negligible. When using the same matching scale $\mu_W = 500$ GeV for the SM contribution, the cancellation between the NLO correction to the matching condition and the NLO evolution of the Wilson coefficient $C_{7,SM}^{\text{eff}}$ does not occur anymore. Large cancellations are instead observed between $V_{SM}(\mu_b)$ and $C_{7,SM}^{1,\text{eff}}(\mu_b)$. The complete correction $|\Delta\mathcal{D}_{SM}|$ ranges still between $19\%$ and $2\%$ for $\mu_b \in [2.4, 9.6]$ GeV, but the point where NLO and LO predictions for $\mathcal{D}_{SM}$ coincide is pushed to the higher end of $\mu_b$. These results imply that for heavy $H^+$ and large enough X and Y couplings (to lift at least $\mathcal{D}_{XY}$ to be of $\mathcal{O}(\mathcal{D}_{SM})$, a choice of $\mu_W$ of $\mathcal{O}(m_H)$, instead than $\mathcal{O}(M_W)$, minimizes the size of the NLO corrections.

We draw attention to the fact that the sensitivity of the reduced amplitude $\mathcal{D}$ to $m_H$ is weaker than that of the Wilson coefficient $C_{7,SM}^{\text{eff}}(\mu_W)$ at the matching scale. It is interesting to see that the coefficients $C_{7,SM}^{\text{eff}}$ and $C_{7,XY}^{\text{eff}}$ are almost identical at the matching scale $\mu_W$ for $m_H = 100$ GeV. This feature is clearly visible in Fig. 1, where the matching scale $\mu_W = M_W$ is used. (It remains true for all $\mu_W \in [M_W, m_t]$.) We observe in Fig. 1 that the RGE flow from $\mu_W$ to $\mu_b$ introduces a large gap between $C_{7,SM}^{\text{eff}}(\mu_b)$ and $C_{7,XY}^{\text{eff}}(\mu_b)$, which is then somewhat widened at the level of the reduced amplitude $\mathcal{D}$ by the inclusion of the $V(\mu_b)$–contribution. At $\mu_b \in [2.4, 9.6]$ GeV, $\mathcal{D}_{XY}$ is about a factor of three smaller than $\mathcal{D}_{SM}$.

When going to physical observables such as BR($\mathcal{B}$ → $X_\gamma$), the individual building blocks of $\mathcal{D}$, discussed above in a coupling–independent way, are weighted according to the values of the couplings X and Y.

4.2. Branching Ratio: Real Couplings

The almost perfect flatness of $\mathcal{D}_{YY}$ and $\mathcal{D}_{XY}$ shown in Figs. 2 and 3, should not lead to the conclusion that the NLO prediction for BR($\mathcal{B}$ → $X_\gamma$) is well–behaved. It was explicitly shown in [27], by expanding the Wilson coefficients around $\mu_b = m_b$, that the dominant scale dependence of the form $\alpha_s(m_b)\ln(\mu_b/m_b)$ is cancelled in the complete NLO expression for $\mathcal{D}$ in eq. (29) and consequently also in $|\mathcal{D}|^2$. If the NLO corrections are large enough to reduce substantially the magnitude of the LO term, $|\mathcal{D}|^2$ becomes sensitive to higher order dependence on $\mu_b$ of the form $\alpha_s^2(m_b)\ln^p(\mu_b/m_b)$ ($p = 1, 2$). The
Figure 4: Branching Ratio as a function of the coupling $X$, for $Y = 1$ and $m_H = 100$ GeV. The upper (lower) frame shows the LO (NLO) result for $\mu_b = 4.8$ GeV (solid lines), $\mu_b = 2.4$ GeV (dashed lines) and $\mu_b = 9.6$ GeV (dash–dotted lines), and matching scale $\mu_W = m_H = 100$ GeV. The needed input parameters are fixed at their central values listed in Table 1 in Appendix F. Superimposed is the range of values allowed by the CLEO measurement.
flatness of $\overline{D}$ seems to indicate that the omitted NNLO term $|\Delta D|^2$ would cancel to a large extent this remaining $\mu_b$ dependence. Nevertheless, whether the omitted $|\Delta D|^2$ in eq. (42) is the bulk of the NNLO corrections can only be decided when a complete NNLO calculation is at hand.

It is clear that the reliability of the NLO prediction for the branching ratio, which is linked to the size of $\Delta \overline{D}$, depends on the values of X and Y. For a given $m_H$, it is possible to choose these couplings in such a way that the reduced amplitude $\overline{D}$ is dominated, for example, by the Higgs contribution $D_{XY}$. For such points in the parameter space $\{X, Y, m_H\}$, the size of the NLO corrections to the branching ratio is then roughly $2\text{Re}(\Delta \overline{D}_{XY})$, i.e. about $-80\%$ for $m_H = 100$ GeV. In this case, the NLO corrections substantially reduce the leading order prediction. As expected, the resulting scale dependence of the branching ratio $\text{BR}(\overline{B} \to X_s \gamma)$ is large, viz. about $40\%$. For $Y = 1$ and the same value of $m_H$, $D_{XY}$ completely dominates the branching ratio for $|X| \geq 20$, outside the range shown in Figs. 4. The prediction for $\text{BR}(\overline{B} \to X_s \gamma)$ is then, however, far above the band of values allowed by the CLEO measurement.

Moreover, it is possible to choose X and Y in such a way that $\text{Re}(\Delta \overline{D}) < -50\%$. As eq. (42) shows, this choice leads to a negative NLO prediction for $\text{BR}(\overline{B} \to X_s \gamma)$. Needless to say, in such a situation higher order corrections to the NLO calculation are mandatory to obtain sensible results. In our representative case of $m_H = 100$ GeV and $Y = 1$ illustrated in Fig. 4, the range of X corresponding to a negative branching ratio is roughly $-10 < X < -2$. A comparison with the upper frame of this Figure shows that, given the very large scale dependence of the LO calculation, one could have guessed the pathological situation which is encountered at the NLO level.

Reliable predictions for the branching ratio with a mild scale dependence are obtained only for $X > -1$ (see lower frame of Fig. 4). For these values, the SM contribution dominates, but the Higgs contribution proportional to $XY^*$ is still large enough to produce a sharp raise of $\text{BR}(\overline{B} \to X_s \gamma)$ when X increases from $-1$ to 2. Notice that for $Y = 1$, the values $X = -1$ and $X = 1$ correspond respectively to the ordinary 2HDM of Type I and Type II with $\tan\beta = 1$. Type I models, however, are not always so stable and well-behaved as in the case described above. For $Y = 1$ and $X = -1$ (corresponding to $\tan\beta = 1$), it is enough to lower $m_H$ to 45 GeV to find a low-scale dependence of $+40\%$ (see Fig. 7). Keeping $m_H$ at 100 GeV and increasing Y to 2, we find that for $X = -2$ (corresponding to a Type I model with $\tan\beta = 0.5$), the branching ratio is negative for all values of $\mu_b \in [2.4, 9.6]$ GeV.

Going to heavier Higgs masses, e.g. $m_H = 500$ GeV, we find for $Y = 1$ the same qualitative features as in Fig. 4, but shifted to larger ranges of $|X|$. We should warn, however, that for large enough $|X|$, when the Higgs contribution dominates over the SM one, the stability of the branching ratio becomes worse. The NLO corrections have a more dramatic dependence on the particular matching scale chosen, than in the case $m_H = 100$ GeV. Indeed, for $\mu_W = m_H = 500$ GeV, the relevant correction for the branching ratio $2\text{Re}(\Delta \overline{D}_{XY})$ is $\sim -70\%$, but exceeds $-100\%$ for $\mu_W = M_W$. 
Figure 5: Branching Ratio as a function of $m_H$ in a 2HDM of Type II, for three different choices of the low–scale: $\mu_b = 2.4, 4.8, 9.6$ GeV and $\mu_W = m_H$ (upper frame), and for three different choices of the matching scale: $\mu_W = M_W, m_t, m_H$ and $\mu_b = 4.8$ GeV. The needed input parameters are fixed at their central values listed in Table 1 in Appendix F.
Figure 6: Contour plot in \((\tan \beta, m_H)\) obtained by using the NLO expression for the branching ratio \(\text{BR}(B \rightarrow X_s\gamma)\) and possible experimental upper bounds (see text). The allowed region is below the corresponding curves.

4.3. Type I and Type II Models

Theoretical predictions for the branching ratio in Type II models stand, in general, on a rather solid ground. In Fig. 5, we show these predictions in a Type II model with \(\tan \beta = 2\), for \(100 \leq m_H \leq 600\) GeV. This value of \(\tan \beta\) is particularly interesting since already for \(\tan \beta \geq 2\), the branching ratio becomes insensitive to the actual value of this variable: the contribution \(D_{YY}\) is, in fact, suppressed by the coupling \(|Y|^2 \leq 1/4\), whereas the contribution \(D_{XY}\) is multiplied by the coupling \(X Y^* = 1\), for any value of \(\tan \beta\). For \(\tan \beta \leq 2\), the branching ratio grows very rapidly when \(\tan \beta\) decreases and can be made compatible with existing measurements only for large values of \(m_H\).

The upper frame of Fig. 5 shows the low–scale dependence of \(\text{BR}(B \rightarrow X_s\gamma)\) for matching scale \(\mu_W = m_H\), for \(m_H > 100\) GeV. It is less than 10% for any value of \(m_H\) above the LEP lower bound of 45 GeV. Such a small scale uncertainty is a generic feature of Type II models and remains true for values of \(\tan \beta\) as small as 0.5. The lower frame in Fig. 5 shows the (very weak) matching scale dependence as obtained by varying \(\mu_W\) in the interval \([M_W, \text{Max}(m_t, m_H)]\). We should point out that the lowest estimate of \(\text{BR}(B \rightarrow X_s\gamma)\) comes from the largest value of \(\mu_W\) in this interval. We disagree on this point with ref. [17] where the lowest estimate is obtained with the smallest value of \(\mu_W\). The result of [17] is presumably due to the neglect of the term proportional to \(\ln(\mu_W^2/M_W^2)\).
Figure 7: Branching Ratio as a function of $m_H$ in a 2HDM of Type I, for three different choices of the low scale: $\mu_b = 2.4, 4.8, 9.6$ GeV and for $\mu_W = m_H$. The needed input parameters are fixed at their central values listed in Table 1 in Appendix F.

for the coefficients $C_{1}^{1,\text{eff}}(\mu_W)$ and $C_{4}^{1,\text{eff}}(\mu_W)$ in eq. (17).

In Type II models, the theoretical estimate of $\text{BR}(\bar{B} \to X_s\gamma)$ can be well above the range $(1 - 4.2) \times 10^{-4}$ indicated by the CLEO Collaboration as the band of experimentally allowed values. It is therefore interesting to establish with some accuracy which values of \{tan$\beta$, $m_H$\} are excluded by possible measurements of the branching ratio $\text{BR}(\bar{B} \to X_s\gamma)$. Our results are given in Fig. 6, where we show the contour of the region excluded by the upper bound obtained by CLEO, $4.2 \times 10^{-4}$, as well as for other two hypothetical values, $3.4 \times 10^{-4}$ and $5.0 \times 10^{-4}$; the latter one is not far from the upper bound obtained by the ALEPH Collaboration. These contours are obtained by finding the minimum of $\text{BR}(\bar{B} \to X_s\gamma)$, when varying simultaneously the input parameters within their errors (see Table 1 in Appendix F) and the two scales $\mu_b$ and $\mu_W$ in the ranges $2.4 \leq \mu_b \leq 9.6$ GeV and $M_W \leq \mu_W \leq \text{Max}(m_t, m_H)$. For tan$\beta = 0.5, 1, 5$, we exclude respectively $m_H \leq 375, 289, 255$ GeV. Notice that the flatness of the curves shown in Fig. 5 towards the higher end of $m_H$, causes a high sensitivity of these bounds on all details of the calculation. Different treatments of the infrared singularities when the photon gets soft (e.g. with a cut in the photon energy [21]) and the possible expansion of $1/T_{SL}$ in powers of $\alpha_s$ could alter the branching ratio at the 1% level, i.e. well within the estimated theoretical uncertainty. These details, however, could still produce shifts of several tens of GeV, in either direction, in the lower bounds quoted above.
Branching ratios in Type I models can be reliably predicted for $m_H > 100$ GeV and $\tan\beta > 1$. Theoretical results for this range of parameters fall within the CLEO band $(1 - 4.2) \times 10^{-4}$, as it can be seen in Fig. 7 for $\tan\beta = 1$ and $100 \leq m_H \leq 600$ GeV. Larger values of $\tan\beta$ decrease the Higgs contribution to the branching ratio, giving therefore values closer to the SM prediction. Lower values of $m_H$ (and/or $\tan\beta < 1$) can produce results outside the CLEO range. For these parameters, however, the theoretical predictions are unstable under scale variation and, at times, ill-defined. As shown in Fig. 7, a scale dependence of $\pm 40\% - 60\%$ is obtained for $m_H = 45$ GeV and $\tan\beta = 1$. For $\tan\beta < 1$, negative values of the branching ratio are found already for $m_H = 100$ GeV.

4.4. Branching Ratio: Complex Couplings and Rate Asymmetries

In this section we study several aspects of the decay $\bar{B} \to X_s \gamma$ in the presence of complex X and Y couplings. Particularly motivating is the observation that the measurement of $\text{BR} (\bar{B} \to X_s \gamma)$ yields strong constraints on $\text{Im} (X Y^*)$ [31,32]. In turn, these constraints limit the possibility of having large indirect CP asymmetries in neutral $B$ decays such as $B \to \psi K_s$ [32]. This observation is based on a LO analysis whose reliability and stability under scale variation was not enquired. We plan to investigate this aspect and to check how the bound in [32] may be modified by the inclusion of NLO corrections.

The bound of ref. [32] on $\text{Im} (X Y^*)$ is obtained as follows. Since the LO branching ratio is proportional to $|C^0_{7,S,M}(\mu_b)|^2$, the upper bound $\text{BR} (\bar{B} \to X_s \gamma) \leq 4.2 \times 10^{-4}$ from the CLEO measurement implies an upper bound on $|C^0_{7,YY}(\mu_b)|^2$, and therefore on $\text{Im} (X Y^*)$ when:

$$C^0_{7,YY}(\mu_b) = \left\{ C^0_{7,S,M}(\mu_b) + |Y|^2 C^0_{7,YY}(\mu_b) + \text{Re} (X Y^*) C^0_{7,XY}(\mu_b) \right\} + i \text{Im} (X Y^*) C^0_{7,XY}(\mu_b)$$

i.e., when the real parts of the charged Higgs contributions cancel the SM coefficient $C^0_{7,S,M}(\mu_b)$. An inspection of the upper frame of Fig. 4 shows that, for $m_H = 100$ GeV and $Y = 1$, a vanishing branching ratio is induced by the the real coupling $X = -2$. The choice of complex couplings X and Y with $Y = 1$ and $\text{Re} X = -2$, therefore, fulfills to a good approximation the condition (45), for all values of $\text{Im} X$. The corresponding branching ratio, obtained with the central value of the input parameters, $\mu_w = m_H$, and $\mu_b = m_b$, and shown in Fig. 8 as a function of $\text{Im} X$, equals $4.2 \times 10^{-4}$ at $|\text{Im} X| \sim 2$. For the chosen Higgs mass, $m_H = 100$ GeV, therefore, the upper bound on $|\text{Im} (X Y^*)|$ is $\sim 2$. Notice that this procedure yields only a first estimate for the actual LO bound, since the errors of the input parameters have not been considered.

We observe that a variation of the low-scale $\mu_b$ in the usual range leads to large uncertainties for the branching ratio, throughout the whole range of $\text{Im} X$. In particular, for $|\text{Im} X| \sim 2$, this uncertainty amounts to $\pm 25\%$. When including NLO corrections,
Figure 8: Branching Ratio as a function of Im X, with fixed Re X = −2 and Y = 1, for $\mu_W = m_H$ and different low–scales: $\mu_b = 4.8$ GeV (solid lines), $\mu_b = 2.4$ GeV (dashed lines) and $\mu_b = 9.6$ GeV (dash–dotted lines). The needed input parameters are fixed at their central values listed in Table 1 in Appendix F. Superimposed is the range of values allowed by the CLEO measurement. The upper (lower) frame shows the LO (NLO) result.
the situation does not improve, as the lower frame of Fig. 8 shows. It is interesting to
see that the value of $BR(\bar{B} \to X_s \gamma)$ for $\mu_b = m_b$, at $|\text{Im} X| \sim 2$ drops from $4.2 \times 10^{-4}$ to roughly $1 \times 10^{-4}$. The intersection of the NLO curve for $\mu_b = m_b$ with the CLEO upper bound is at $\text{Im} X \sim 4$. This procedure is essentially the construction of the NLO bound for $|\text{Im}(XY^*)|$ in the sense specified before. (In the actual construction, one should have cancelled also small real parts of $C_7^{\text{eff}}$ with NLO $\alpha_s$ and small terms coming from absorptive parts of virtual corrections.)

The inclusion of NLO corrections verifies explicitly the instability of the LO upper bound on $|\text{Im}(XY^*)|$. Given the fact that the NLO predictions for $BR(\bar{B} \to X_s \gamma)$ are plagued by even larger scale uncertainties, it is hard to believe that the NLO candidate qualifies as a reliable bound.

Not all complex $X$ and $Y$ couplings yield NLO predictions as problematic as those shown in Fig. 8. A typical case in which the perturbative expansion of $BR(\bar{B} \to X_s \gamma)$ can be safely truncated at the NLO level is identified by: $Y = 1/2$, $X = 2 \exp(i\phi)$, and $m_H = 100$ GeV. Indeed, for these parameters, the real and imaginary part of $|Y|^2 D_{YY}$ and $(XY^*)D_{XY}$ are dominated by $D_{SM}$. Therefore, the low–scale dependence of the branching ratio is not much larger than the very mild one obtained for the SM estimate, as shown in Fig. 9. This case is particularly interesting since it gives rise to a theoretical prediction of $BR(\bar{B} \to X_s \gamma)$ consistent with the CLEO measurement, even for a relatively small value of $m_H$. Such a light charged Higgs can contribute to the decays of the $t$-quark, through the mode $t \to H^+ b$.

We now investigate CP asymmetries induced by complex couplings. It is well known that in the SM a non–vanishing direct CP rate asymmetry

\begin{equation}
  a_{CP} = \frac{BR(\bar{B} \to X_s \gamma) - BR(\bar{B} \to \overline{X_s} \gamma)}{BR(\bar{B} \to X_s \gamma) + BR(\bar{B} \to \overline{X_s} \gamma)}
\end{equation}

is due to non–trivial relative weak– and relative strong phases in the decay amplitude for $b \to s \gamma$ (as well as the one for $b \to s \gamma g$). We write the amplitude of $b \to s \gamma$ and of the CP conjugated process as:

\begin{align}
  A(b \to s \gamma) &= (V_{ub}V_{us}^*) A_u + (V_{cb}V_{cs}^*) A_c + (V_{tb}V_{ts}^*) A_t, \\
  A(\bar{b} \to \overline{s} \gamma) &= (V_{ub}V_{us}^*)^* A_u + (V_{cb}V_{cs}^*)^* A_c + (V_{tb}V_{ts}^*)^* A_t,
\end{align}

where the dependence on the CKM matrix is manifest. Working in the LO approximation, the quantities $A_u$, $A_c$, and $A_t$ are real, hence $|A(\bar{b} \to \overline{s} \gamma)|^2 = |A(b \to s \gamma)|^2$, and consequently $a_{CP} = 0$. Relative strong phases among $A_u$, $A_c$, and $A_t$ only appear at the NLO level, due to absorptive terms in the loop diagrams. Thus, a non–vanishing CP asymmetry results if $V_{ub}V_{us}^*$, $V_{cb}V_{cs}^*$, and $V_{tb}V_{ts}^*$ have also relative phases. In the approximation $V_{ub}V_{us}^* = 0$, which we used so far, the CP asymmetry vanishes, since unitarity implies that $V_{cb}V_{cs}^* = -V_{tb}V_{ts}^*$. Undoing this approximation, the SM CP rate asymmetry turns out to be below 1\% [48].
Figure 9: Branching Ratio as a function of $\phi$, where $\phi$ parametrizes $X$: $X = 2 \exp(i\phi)$, for $Y = 1/2$, $m_H = 100$ GeV, and $\mu_W = m_H$. Solid, dashed and dash–dotted lines correspond respectively to $\mu_b = 4.8, 2.4$ and 9.6 GeV. The needed input parameters are fixed at their central values listed in Table 1 in Appendix F. Superimposed is the range of values allowed by the CLEO measurement. The upper (lower) frame shows the LO (NLO) result.
Figure 10: CP rate asymmetry $a_{cp}$ as a function of $\phi$, where $\phi$ parametrizes $X$: $X = 2 \exp(i\phi)$, for $Y = 1/2$, $m_H = 100$ GeV, and $\mu_W = m_H$. Solid, dashed and dash–dotted lines correspond respectively to $\mu_b = 4.8, 2.4$ and $9.6$ GeV. The needed input parameters are fixed at their central values listed in Table 1 in Appendix F. In the upper (lower) frame the LO (NLO) expression for the denominator in eq. (46) is used. See text for more details.
Since the CP rate asymmetry is so tiny in the SM, it would be exciting if the imaginary parts of the X and Y couplings would induce –together with the absorptive parts of the NLO loop-functions– measurable CP rate asymmetries. In order to investigate this question, we switch off the SM asymmetry by working in the limit $V_{ub}V_{us}^* = 0$. As an illustrative example we calculate the CP asymmetries for the same values of parameters as in Fig. 9, i.e., for $X = 2 \exp(i\phi)$, $Y = 1/2$ and for $m_H = 100$ GeV, where the branching ratio appears to be well–behaved. As mentioned above, the NLO prediction for the branching ratio in the numerator of eq. (46) is required to obtain the first non–vanishing term for the rate asymmetry. For the denominator, one can either use the LO or the NLO estimate for the two branching ratios. The difference amounts to higher order terms which are not systematically calculated. The respective results are shown in the upper and lower frame in Fig. 10 as a function of $\phi$. For this specific choice of X and Y the asymmetries are rather modest, at the 1% level. The scale dependence shown by the lower frame is certainly smaller than that in the upper frame. However, since both procedure are in principle viable, one cannot conclude that the result for the CP asymmetries is as reliable as indicated in the lower frame. Although we have not systematically scanned the parameter space $\{X, Y, m_H\}$, this is our generic result: choices of couplings X and Y which render the branching ratio stable, i.e. not plagued by a large dependence on the scale $\mu_b$, induce small values of $a_{CP}$.

5. Conclusions

We have presented NLO predictions for the decay $B \to X_s \gamma$ in generic 2HDMs with flavour–conserving tree–level neutral couplings. We include in this definition Multi–Higgs Doublet models where all charged Higgs bosons, except one, are heavy and completely decoupled at the electroweak scale. Their existence leaves an imprint only in the Yukawa potential where the couplings X and Y multiplying respectively the down– and up–term are not necessarily correlated, and in general complex. This generalization allows a simultaneous study of $B \to X_s \gamma$ for different types of 2HDMs by continuously varying the couplings X and Y. Results for the well–known Type I and Type II models are then obtained for specific choices of real couplings: only one combination of X and Y, usually denoted as $\tan \beta$, occurs. Since supersymmetric models have an enlarged Higgs sector with two Higgs doublets of Type II, the results presented here are also a first step towards a complete evaluation of the rate of $B \to X_s \gamma$ in these models. They constitute already a good approximation for those supersymmetric scenarios, such as gauge–mediated supersymmetric models with large $\tan \beta$, where the Higgs contribution dominates by far over the genuinely supersymmetric contributions [49].

Our calculation is carried out using the effective Hamiltonian formalism with on–shell operators. The NLO matching condition for the decay $B \to X_s \gamma$ was already completely calculated in [17] and partially in [33]. For the Wilson coefficients $C^e_{7}(\mu_w)$ and $C^e_{8}(\mu_w)$, we find agreement with the existing results, but we disagree with ref. [27,17] for the
matching scale dependence of the coefficients $C_{\text{eff}}^1(\mu_W)$ and $C_{\text{eff}}^4(\mu_W)$. We generalize the solution of the RGE needed to obtain the coefficient $C_{\text{eff}}^7(\mu_b)$ at the low–scale $\mu_b$ for values of $\mu_W$ different from $M_W$. We also correct the dependence on the Wilson coefficients of one of the non–perturbative contribution to $\Gamma(\bar{B} \to X_s\gamma)$, which is erroneously reported in [17].

We have given predictions for $\text{BR}(\bar{B} \to X_s\gamma)$ in 2HDMs as functions of the parameters $\{X,Y,m_H\}$. We found that the theoretical uncertainties of these NLO calculations are in general larger than those obtained in the SM. The quality of our predictions, therefore, depends strongly on the values of the parameters considered. As a first step, we summarize the theoretical features of our results, ignoring a comparison with the existing experimental data. We distinguish several cases: i) When these parameters are such that the SM contribution is much larger than the Higgs contribution to $\text{BR}(\bar{B} \to X_s\gamma)$, the reliability of our predictions does not differ much from that in the SM. This, however, does not prevent the $\text{BR}(\bar{B} \to X_s\gamma)$ from being rather sensitive to $m_H$. ii) For values of parameters which bring the Higgs contributions to the same level as the SM one, with constructive interference between the two, we find in general larger scale dependence than in the SM. iii) When the Higgs contributions are still of the same order as the SM contribution, but interfere destructively with it, we find very large scale dependences and, in specific cases, we obtain negative values of $\text{BR}(\bar{B} \to X_s\gamma)$. iv) When $X$ and $Y$ make the Higgs contributions largely dominate over the SM one, we find a scale dependence in $\text{BR}(\bar{B} \to X_s\gamma)$ of $\sim 40\%$ for $m_H$ of $\mathcal{O}(M_W)$. For much larger values of $m_H$, the quality of the theoretical prediction depends strongly on the matching scale considered.

Type II models are typical of case i) and barely become of case ii) for $m_H$ at the LEP lower bound of 45 GeV [47] and for $\tan\beta \sim 0.5$. The theoretical uncertainty for such models is in general below 10%. Similar uncertainties are obtained for Type I models when $m_H$ and/or $\tan\beta$ are large enough. Rather unstable results are, however, obtained for $m_H$ at the lower end of allowed values and $\tan\beta \lesssim \mathcal{O}(1)$.

When these problematic regions are avoided, then all predictions obtained for Type I models are consistent with the CLEO measurement of $\text{BR}(\bar{B} \to X_s\gamma)$. The same is not true for Type II models where the theoretical predictions are always above the SM result. Measurements become then highly constraining. Taking into account only the CLEO result, without combining it with the still preliminary one from ALEPH, we exclude $m_H \lesssim 375$ GeV for $\tan\beta = 0.5$ and $m_H \lesssim 255$ GeV for $\tan\beta = 5$. These bounds are very sensitive to details in the definition of the branching ratio (e.g. whether it requires or not an expansion in $\alpha_s$ of $1/\Gamma_{\text{sl}}$), which give rise to uncertainties usually not included in the theoretical error for this observable. For the generic 2HDMs which we consider, we find wide regions of parameter space where the theoretical predictions for $\text{BR}(\bar{B} \to X_s\gamma)$ are reliable and within the band of values $(1.0-4.2) \times 10^{-4}$ allowed by the CLEO measurement. In particular, we find that within these models, charged Higgs bosons can still be light enough to be produced through a decay of the $t$–quark.

Since $X$ and $Y$ are in general complex, new CP violating effects are induced. In partic-
ular, the combination $\text{Im}(XY^*)$ is important as it may affect the indirect CP asymmetries in $B \to \psi K_s$ already at the LO. It is known that the measurement of $\text{BR}(B \to X_s\gamma)$ constrains $\text{Im}(XY^*)$ [31] and an upper bound for this quantity has been obtained in [32] using the LO calculation. We find that the LO estimate of the branching ratio at the values of $X$ and $Y$ which determine the upper bound extracted in [32] is unstable under scale variation. The addition of NLO corrections tend to shift the LO bound to a higher value. Nevertheless, the scale dependence of $\text{BR}(B \to X_s\gamma)$ for the new combination of $X$ and $Y$ needed, is still too large to conclude that the bound obtained at the NLO level is stable against higher order corrections. The large instabilities encountered even at this order are due to the fact that the construction of this upper bound requires the almost complete cancellation of the SM contribution.

At the NLO, complex values of $X$ and $Y$ induce also direct CP rate asymmetries, in $\mathcal{B} \to X_s\gamma$, which is essentially vanishing in the SM. If sizable, a measurement of this observable, could provide a handle to detect some of these models. Unfortunately, we find that combinations of couplings where $\text{BR}(\mathcal{B} \to X_s\gamma)$ is reliably predicted, lead to rate asymmetries only at the 1% level.

We find that within 2HDMs, the truncation of the perturbative series at the NLO level is often inappropriate. This is in sharp contrast with the SM case, where the large LO theoretical uncertainties are drastically reduced in the NLO calculation, and the overall size of the NLO corrections is, in comparison, rather modest. It is somehow disturbing to find the problematic features described in this paper in models which structurally do not differ very much from the SM and it is conceivable that other extensions may suffer from the same diseases. One may wonder about the deeper reasons for the cancellations observed in the SM. Their understanding may help in clarifying also the reasons for the features encountered in 2HDMs.
Appendix A. Wilson Coefficients at $\mu_W$

We list here the functions introduced in the text, which define the Wilson coefficients at
the matching scale (see eqs. (16)–(18)).

SM case

The LO functions are ($x = m_t^2/M_W^2$):

$$C^7_{0, SM} = \frac{x}{24} \left[ -8x^3 + 3x^2 + 12x - 7 + (18x^2 - 12x) \ln x \right] \frac{1}{(x - 1)^4}$$

$$C^8_{0, SM} = \frac{x}{8} \left[ -x^3 + 6x^2 - 3x - 2 - 6x \ln x \right] \frac{1}{(x - 1)^4}.$$  \hspace{1cm} (A1)

In the $\overline{\text{MS}}$ scheme, we have at the NLO:

$$E_0 = \frac{x(2x + 11x - 18)}{12(x - 1)^3} + \frac{x^2(4x^2 - 16x + 15)}{6(x - 1)^4} \ln x - \frac{2}{3} \ln x - \frac{2}{3} \frac{1}{(x - 1)^4}$$

$$W^7_{0, SM} = -\frac{16x^4 - 122x^3 + 80x^2 - 8x}{9(x - 1)^4} \text{Li}_2 \left( \frac{1}{x} - \frac{1}{x} \right) + \frac{6x^4 + 46x^3 - 28x^2}{3(x - 1)^5} \ln^2 x$$

$$+ \frac{-102x^5 - 588x^4 - 2262x^3 + 3244x^2 - 1364x + 208}{81(x - 1)^5} \ln x$$

$$+ \frac{1646x^4 + 12205x^3 - 10740x^2 + 2509x - 436}{486(x - 1)^4}$$

$$W^8_{0, SM} = -\frac{4x^4 + 40x^3 + 41x^2 + x}{6(x - 1)^4} \text{Li}_2 \left( \frac{1}{x} - \frac{1}{x} \right) + \frac{-17x^4 - 31x^2}{2(x - 1)^5} \ln^2 x$$

$$+ \frac{-210x^5 + 1086x^4 + 4893x^3 + 2857x^2 - 1994x + 280}{216(x - 1)^5} \ln x$$

$$+ \frac{737x^4 - 14102x^3 - 28209x^2 + 610x - 508}{1296(x - 1)^4}$$

$$M^7_{0, SM} = \frac{82x^5 + 301x^4 + 703x^3 - 2197x^2 + 1319x - 208 - (162x^4 + 1242x^3 - 756x^2) \ln x}{81(x - 1)^5}$$

$$M^8_{0, SM} = \frac{77x^5 - 475x^4 - 1111x^3 + 607x^2 + 1042x - 140 + (918x^3 + 1674x^2) \ln x}{108(x - 1)^5}$$

$$T^7_{0, SM} = \frac{x}{3} \left[ 47x^3 - 63x^2 + 9x + 7 - (18x^3 + 30x^2 - 24x) \ln x \right] \frac{1}{(x - 1)^5}.$$
\[ T_{8,SM} = 2x \left[ \frac{-x^3 - 9x^2 + 9x + 1 + (6x^2 + 6x) \ln x}{(x - 1)^5} \right]. \]  

(A2)

### 2HDM case, coupling $|Y|^2$

The functions relative to charged Higgs contributions, with coupling $|Y|^2$, are at the LO ($y = m_t^2/m_H^2$):

\[
C_{7,YY}^0 = \frac{1}{3} C_{7,SM}^0 (x \to y) \\
C_{8,YY}^0 = \frac{1}{3} C_{8,SM}^0 (x \to y); \quad (A3)
\]

at the NLO:

\[
E_H = \frac{1}{36} y \left[ \frac{7y^3 - 36y^2 + 45y - 16 + (18y - 12) \ln y}{(y - 1)^4} \right] \\
W_{7,YY} = \frac{2}{9} y \left[ \frac{8y^3 - 37y^2 + 18y}{(y - 1)^4} \text{Li}_2 \left( 1 - \frac{1}{y} \right) + \frac{3y^3 + 23y^2 - 14y}{(y - 1)^5} \ln^2 y \right. \\
+ \left. \frac{21y^4 - 192y^3 - 174y^2 + 251y - 50}{9(y - 1)^5} \ln y \right] \left\{ - \frac{4}{9} E_H \right\} \\
W_{8,YY} = \frac{1}{6} y \left[ \frac{13y^3 - 17y^2 + 30y}{(y - 1)^4} \text{Li}_2 \left( 1 - \frac{1}{y} \right) - \frac{17y^2 + 31y}{(y - 1)^5} \ln^2 y \right. \\
+ \left. \frac{42y^4 + 318y^3 + 1353y^2 + 817y - 226}{36(y - 1)^5} \ln y \right] \left\{ - \frac{1}{6} E_H \right\} \\
M_{7,YY} = \frac{1}{36} y \left[ \frac{-14y^4 + 149y^3 - 153y^2 - 13y + 31 - (18y^3 + 138y^2 - 84y) \ln y}{(y - 1)^5} \right] \\
M_{8,YY} = \frac{1}{36} y \left[ \frac{-7y^4 + 25y^3 - 279y^2 + 223y + 38 + (102y^2 + 186y) \ln y}{(y - 1)^5} \right] \\
T_{7,YY} = \frac{1}{3} T_{7,SM} (x \to y) \\
T_{8,YY} = \frac{1}{3} T_{8,SM} (x \to y). \quad (A4)
2HDM case, coupling \((XY^*)\)

Similarly, the functions relative to the charged Higgs contributions proportional to \((X Y^*)\) are at the LO:

\[
C_{7,XY}^0 = \frac{1}{12} y \left[ \frac{-5y^2 + 8y - 3 + (6y - 4) \ln y}{(y - 1)^3} \right]
\]

\[
C_{8,XY}^0 = \frac{1}{4} y \left[ \frac{-y^2 + 4y - 3 - 2 \ln y}{(y - 1)^3} \right],
\]

and at the NLO:

\[
W_{7,XY} = \frac{4}{3} y \left[ \frac{8y^2 - 28y + 12}{3(y - 1)^3} \operatorname{Li}_2 \left( 1 - \frac{1}{y} \right) + \frac{3y^2 + 14y - 8}{3(y - 1)^4} \ln^2 y 
+ \frac{4y^3 - 24y^2 + 2y + 6}{3(y - 1)^4} \ln y + \frac{-2y^2 + 13y - 7}{(y - 1)^3} \right]
\]

\[
W_{8,XY} = \frac{1}{3} y \left[ \frac{17y^2 - 25y + 36}{2(y - 1)^3} \operatorname{Li}_2 \left( 1 - \frac{1}{y} \right) - \frac{17y + 19}{(y - 1)^4} \ln^2 y 
+ \frac{14y^3 - 12y^2 + 187y + 3}{4(y - 1)^4} \ln y - \frac{3(29y^2 - 44y + 143)}{8(y - 1)^3} \right]
\]

\[
M_{7,XY} = \frac{2}{9} y \left[ \frac{-8y^3 + 55y^2 - 68y + 21 - (6y^2 + 28y - 16) \ln y}{(y - 1)^4} \right]
\]

\[
M_{8,XY} = \frac{1}{6} y \left[ \frac{-7y^3 + 23y^2 - 97y + 81 + (34y + 38) \ln y}{(y - 1)^4} \right]
\]

\[
T_{7,XY} = \frac{2}{3} y \left[ \frac{13y^2 - 20y + 7 - (6y^2 + 4y - 4) \ln y}{(y - 1)^4} \right]
\]

\[
T_{8,XY} = 2y \left[ \frac{-y^2 - 4y + 5 + (4y + 2) \ln y}{(y - 1)^4} \right].
\]

\[(A5)\]

\[(A6)\]

Appendix B. Anomalous Dimension Matrix

For completeness, we give the anomalous dimension matrix which govern the evolution of the Wilson Coefficients from \(\mu_W\) to \(\mu_b\). It can be expanded perturbatively as:

\[
\gamma_{eff} j_i (\mu) = \frac{\alpha_s(\mu)}{4\pi} \gamma_{0,eff} j_i + \frac{\alpha_s^2(\mu)}{(4\pi)^2} \gamma_{1,eff} j_i + \mathcal{O}(\alpha_s^3).
\]

\[(B1)\]
The matrix \( \gamma^{0,\text{eff}}_{ji} \) is given by:

\[
\begin{pmatrix}
-4 & \frac{8}{3} & 0 & -\frac{2}{9} & 0 & 0 & -\frac{208}{243} & \frac{173}{162} \\
12 & 0 & 0 & \frac{4}{3} & 0 & 0 & \frac{416}{81} & \frac{70}{27} \\
0 & 0 & 0 & -\frac{52}{3} & 0 & 2 & -\frac{176}{81} & \frac{14}{27} \\
0 & 0 & -\frac{40}{9} & -100 & \frac{4}{9} & 5 & -\frac{152}{243} & -\frac{587}{162} \\
0 & 0 & 0 & -\frac{256}{3} & 0 & 20 & -\frac{6272}{81} & \frac{6596}{27} \\
0 & 0 & -\frac{256}{9} & \frac{56}{9} & \frac{40}{9} & -\frac{2}{3} & \frac{4624}{243} & \frac{4772}{81} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{32}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{32}{3} & \frac{28}{3} \\
\end{pmatrix}, \quad \text{(B2)}
\]

and in the \(\overline{\text{MS}}\) scheme with fully anticommuting \(\gamma_5\), \(\gamma_{ji}^{1,\text{eff}}\) is [21]:

\[
\begin{pmatrix}
-\frac{355}{9} & -\frac{502}{27} & -\frac{1412}{243} & -\frac{1369}{243} & \frac{134}{81} & -\frac{35}{243} & -\frac{818}{243} & \frac{3779}{324} \\
-\frac{35}{3} & -\frac{28}{3} & -\frac{416}{243} & -\frac{1280}{81} & \frac{56}{27} & \frac{35}{27} & \frac{508}{81} & \frac{1641}{108} \\
0 & 0 & -\frac{4468}{81} & -\frac{31469}{81} & \frac{400}{81} & \frac{3373}{108} & \frac{22348}{243} & \frac{10178}{81} \\
0 & 0 & -\frac{8158}{243} & -\frac{59399}{243} & \frac{468}{81} & \frac{648}{81} & \frac{243}{243} & \frac{648}{81} \\
0 & 0 & -\frac{251680}{81} & -\frac{128648}{81} & \frac{23836}{243} & \frac{6106}{27} & \frac{1183696}{243} & \frac{2901296}{243} \\
0 & 0 & -\frac{58640}{243} & -\frac{26334}{243} & -\frac{14324}{243} & -\frac{2551}{14} & \frac{2480344}{2187} & \frac{3296257}{27} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{4668}{27} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{2192}{81} & \frac{4963}{27} \\
\end{pmatrix}, \quad \text{(B3)}
\]

Appendix C. “Running” numbers

The vectors \(\{a_i\}, \{h_i\}, \{a'_i\}, \{h'_i\}\), needed for the evaluation of the low–scale Wilson coefficients \(C_7^{0,\text{eff}}(\mu_b)\) and \(C_8^{0,\text{eff}}(\mu_b)\) are:

\[
\{a_i\} = \left\{ \frac{14}{23}, \frac{16}{23}, \frac{3}{23}, -\frac{12}{23}, -\frac{0.4086}{23}, -\frac{0.4230}{23}, -\frac{0.8994}{23}, 0.1456 \right\},
\]

\[
\{h_i\} = \left\{ \frac{626126}{272277}, \frac{-56281}{51730}, \frac{-3}{7}, -\frac{1}{14}, -\frac{0.6494}{7}, -\frac{0.0380}{7}, -\frac{0.0186}{7}, -\frac{0.0057}{7} \right\}, \quad \text{(C1)}
\]

\[
\{a'_i\} = \left\{ \frac{14}{23}, 0.4086, -0.4230, -0.8994, 0.1456 \right\}
\]

\[
\{h'_i\} = \left\{ \frac{313063}{363063}, -0.9135, 0.0873, -0.0571, 0.0209 \right\}; \quad \text{(C2)}
\]

those needed for \(C_7^{1,\text{eff}}(\mu_b)\):
\{e_i\} = \left\{ \frac{4661194}{816831}, \frac{8516}{2217}, 0, 0, -1.9043, -0.1008, 0.1216, 0.0183 \right\}
\{f_i\} = \{-17.3023, 8.5027, 4.5508, 0.7519, 2.0040, 0.7476, -0.5385, 0.0914\}
\{k_i\} = \{9.9372, -7.4878, 1.2688, -0.2925, -2.2923, -0.1461, 0.1239, 0.0812\}
\{l_i\} = \{0.5784, -0.3921, -0.1429, 0.0476, -0.1275, 0.0317, 0.0078, -0.0031\} . \ (C3)

\textbf{Appendix D. Virtual correction functions } r_i

The renormalization scale independent parts of the virtual corrections, encoded in the functions \( r_i \) in eq. (30), read:

\begin{align*}
r_1 &= -\frac{1}{6} r_2 \\
r_2 &= \frac{2}{243} \left\{ -833 + 144\pi^2 z^{3/2} \\
           &+ \left[ 1728 - 180\pi^2 - 1296 \zeta(3) + (1296 - 324\pi^2)L + 108L^2 + 36L^3 \right] z \\
           &+ \left[ 648 + 72\pi^2 + (432 - 216\pi^2)L + 36L^3 \right] z^2 \\
           &+ \left[ -54 - 84\pi^2 + 1092L - 756L^2 \right] z^3 \right\} \\
           &+ \frac{16\pi i}{81} \left\{ -5 + \left[ 45 - 3\pi^2 + 9L + 9L^2 \right] z + \left[ -3\pi^2 + 9L^2 \right] z^2 + \left[ 28 - 12L \right] z^3 \right\} \\
           &+ \mathcal{O}(z^4)
\end{align*}

\begin{align*}
r_7 &= \frac{32}{9} - \frac{8}{9}\pi^2 \\
r_8 &= -\frac{4}{27} \left( -33 + 2\pi^2 - 6i\pi \right), \quad \text{(D1)}
\end{align*}

where \( z \) is defined as \( z = m_c^2/m_b^2 \) and the symbol \( L \) denotes \( L = \ln(z) \). Notice that \( r_3, r_4, r_5, \) and \( r_6 \) are not used in the approximation \( C_i^{\text{th, eff}}(\mu_b) = 0 \ (i = 3, 4, 5, 6) \) used for the matrix elements.

\textbf{Appendix E. Bremsstrahlung } f_{ij} \text{ terms}

The expressions for the bremsstrahlung functions \( f_{ij} \) we use in the present paper are obtained after integrating one variable in the expressions given in Appendix B in ref. [18]. The explicit expressions read (converted to the operator basis 12):

\[ f_{11} = \frac{1}{36} f_{22} \]
\[ f_{12} = -\frac{1}{3} f_{22} \]
\[ f_{17} = -\frac{1}{6} f_{27} \]
\[ f_{18} = -\frac{1}{6} f_{28} \]
\[ f_{22} = \frac{16z}{27} \int_0^{1/z} dt (1 - zt)^2 \left| \frac{G(t)}{t} + \frac{1}{2} \right|^2 \]
\[ f_{27} = -\frac{8z^2}{9} \int_0^{1/z} dt (1 - zt) \left( G(t) + \frac{t}{2} \right) \]
\[ f_{28} = -\frac{1}{3} f_{27} \]
\[ f_{78} = \frac{8}{9} \left( \frac{25}{12} - \frac{\pi^2}{6} \right) \]
\[ f_{88} = \frac{1}{27} \left( \frac{16}{3} - \frac{4\pi^2}{3} + 4 \ln \frac{m_b}{\mu_b} \right), \quad (E1) \]

where \( z = m_z^2/m_b^2 \) and \( \mu_b \) is the renormalization scale. The function \( G(t) \) appearing in eq. (E1) reads
\[
G(t) = \begin{cases} 
-2 \arctan^2 \sqrt{\frac{t}{4-t}} & \text{for } t < 4 \\
-\pi^2/2 + 2 \ln^2 \left( \frac{\sqrt{t} + \sqrt{4-t}}{2} \right) - 2i\pi \ln \left( \frac{\sqrt{t} + \sqrt{4-t}}{2} \right) & \text{for } t \geq 4.
\end{cases} \quad (E2)
\]

**Appendix F. Input Parameters**

We list in table 1 the values of input parameters used in our calculation. The masses

| \( \alpha_s(M_Z) \) (GeV) | \( m_t \) (GeV) | \( m_c/m_b \) | \( m_b - m_c \) (GeV) | \( \alpha_{em}^{-1} \) | \( |V_{ts}V_{tb}/V_{cb}|^2 \) | BR_{SL} |
|-------------------------|----------------|-------------|-----------------------------|----------------|-------------------------------|--------|
| 0.119 | 175 | 0.29 | 3.39 | 130.3 | 0.95 | 0.1049 |
| ±0.004 | ±5.0 | ±0.02 | ±0.04 | ±2.3 | ±0.03 | ±0.0046 |

Table 1: Central value of our input parameters (first line) and their uncertainties (second line).

\( m_t, m_b, \) and \( m_c \) are understood to be the pole masses of the top, bottom and charm quark. The value of the \( m_t \) is obtained by combining the results given in [50,51]; those of the two combinations of \( m_c \) and \( m_b \) are taken from [24]. For \( \alpha_{em} \) and \( |V_{ts}V_{tb}/V_{cb}|^2 \) we refer to [52] and [53], respectively. We take \( \alpha_s(M_Z) = 0.119 \pm 0.004 \), as an average between a pessimistic and optimistic estimate of the error [54], as suggested in [55]. The
value of the semileptonic branching ratio $BR_{sL}$ has been recently obtained by the CLEO Collaboration [56]. The other constants used in the calculation are $M_w = 80.33$ GeV, $\lambda_1 = -0.5$ GeV$^2$, and $\lambda_2 = 0.12$ GeV$^2$. 
References


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