Effective Action for Conformal Scalars and Anti-Evaporation of Black Holes

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abstract

We study the one-loop effective action for $N$ 4D conformally invariant scalars on the spherically symmetric background. The main part of effective action is given by integration of 4D conformal anomaly. This effective action (in large $N$ approximation and partial curvature expansion) is applied to investigate the quantum evolution of Schwarzschild-de Sitter (SdS) black holes of maximal mass. We find that the effect (recently discovered by Bousso and Hawking for $N$ minimal scalars and another approximate effective action) of anti-evaporation of nearly maximal SdS (Nariai) black holes takes also place in the model under consideration. Careful treatment of quantum corrections and perturbations modes of Nariai black hole is given being quite complicated. It is shown that exists also perturbation where black hole radius shrinks, i.e. black hole evaporates. We point out that our result holds for wide class of models including conformal scalars, spinors and vectors. Hence, anti-evaporation of SdS black holes is rather general effect which should be taken into account in quantum gravity considerations.

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1 Introduction.

The gravitational black holes radiation [1] may be considered as the one of the most brilliant manifestations of quantum gravity. Moreover, one can expect that other quantum gravity strong effects should be also searched in black holes. Recently, one such new effect (anti-evaporation of black holes) has been found in ref.[2]. The authors of above work used large $N$ and $s$-waves approximations in their calculation of effective action for 4D minimal scalar. This effective action has been used to study the quantum stability of the Schwarzschild-de Sitter black holes of maximal mass (Nariai solution). It has been shown that there is the perturbation mode where the black hole size increases and the black hole grows back to the maximal radius (anti-evaporation).

The interesting question is if that effect is quite common or it results from the specific model and (or) approximation used. To answer this question we study another model (of 4D conformal scalars) and find the effective action in another approximation (large $N$ and expansion on the curvature). The analysis of quantum corrected equations of motion is much more complicated than in Bousso-Hawking model. Nevertheless, the final result is qualitatively the same- we confirm the possibility of black holes anti-evaporation.

The contents of the paper are following. In the next section 4D anomaly induced effective action for conformal scalar is evaluated. We work in large $N$ approximation and calculate the piece of the effective action which is not anomaly induced as expansion on the curvature near spherically symmetric background. As a result we obtain effectively 2D gravitational system. Equations of motion are derived in the section 3. Section 4 is devoted to the analysis of quantum perturbations for Nariai black hole. We demonstrate the possibility of anti-evaporation of such objects. In the last section we give some comments and explain that our result is of general relevancy for a class of quantum conformal matter models.

2 Effective action for conformal scalars

In the present section we derive the effective action for conformally invariant scalars (for a general review of effective action in curved space, see[3]). Let
us start from Einstein gravity with $N$ conformal scalars $\chi_i$:

\[
S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g_{(4)}} \left\{ R_{(4)} - 2\Lambda \right\} + \frac{1}{2} \sum_{i=1}^{N} \int d^4x \sqrt{-g_{(4)}} \left( g^{(4)}_{\alpha\beta} \partial_\alpha \chi_i \partial_\beta \chi_i + \frac{1}{6} R^{(4)} \chi_i^2 \right). \tag{1}
\]

The convenient choice for the spherically symmetric spacetime is the following:

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu + f(\phi) d\Omega \tag{2}
\]

where $\mu, \nu = 0, 1$, $g_{\mu\nu}$ and $f(\phi)$ depend only from $x^0, x^1$.

Reducing 4D action (1) in accordance with the metric (2) and working in $s$-wave sector (i.e., $\chi_i = \chi_i(x^0, x^1)$), we get

\[
\frac{S}{4\pi} = -\frac{1}{16\pi G} \int d^2x \sqrt{-g} \left[ f(\phi)(R - 2\Lambda) + R_{\Omega} + 2(\nabla^\mu f^{\frac{1}{2}})(\nabla_\mu f^{\frac{1}{2}}) \right] + \frac{1}{2} \sum_{i=1}^{N} \int d^2x \sqrt{-g} \left[ f(\phi) \nabla^\alpha \chi_i \nabla_\alpha \chi_i + \frac{1}{6} \chi_i^2 (Rf(\phi) + R_{\Omega}) \right]. \tag{3}
\]

where factor $4\pi$ appears as volume of $S^2$ space and $R_{\Omega} = 2$ (or $R_{\Omega} = \frac{2}{\rho^2}$ where $\rho^2 = 1$). If we take $R_{\Omega} = \frac{2}{\rho^2}$, then $V_{S^2} = 4\pi \rho^2$.

Let us start the calculation of effective action due to scalars on the background (2). 4D scalar in Eq.(1) is conformally invariant. Let us rewrite the metric (2) as following:

\[
ds^2 = f(\phi) \left[ f^{-1}(\phi) g_{\mu\nu} dx^\mu dx^\nu + d\Omega \right]. \tag{4}
\]

In the calculation of effective action, we present effective action as following:

\[
\Gamma = \Gamma_{\text{ind}} + \Gamma[1, g^{(4)}_{\mu\nu}] \tag{5}
\]

where $\Gamma_{\text{ind}} = \Gamma[f, g^{(4)}_{\mu\nu}] - \Gamma[1, g^{(4)}_{\mu\nu}]$ is conformal anomaly induced action which is quite well-known [4], $g^{(4)}_{\mu\nu}$ is metric (4) without multiplier in front of it, i.e., $g^{(4)}_{\mu\nu}$ corresponds to

\[
ds^2 = \left[ f^{-1}(\phi) g_{\mu\nu} dx^\mu dx^\nu + d\Omega \right]. \tag{6}
\]
The conformal anomaly for $N$ 4D scalars is well-known one:

$$T = b \left( F + \frac{2}{3} \Box R \right) + b' G + b'' \Box R$$  \hspace{1cm} (7)$$

where $b = \frac{N}{120(4\pi)^2}$, $b' = -\frac{N}{360(4\pi)^2}$, $b'' = 0$ but in principle, $b''$ may be changed by the finite renormalization of local counterterm, $F$ is the square of Weyl tensor, $G$ is Gauss-Bonnet invariant.

Conformal anomaly induced effective action $\Gamma_{ind}$ may be written as follows [4]:

$$W = b \int d^4x \sqrt{-\tilde{g}} F \sigma + b' \int d^4x \sqrt{-\tilde{g}} \left\{ \sigma \left[ 2\Box^2 + 4 \mathcal{R}^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{4}{3} \Box^2 \Box - \frac{2}{3} (\nabla^\mu R) \nabla_\mu \right] \sigma \right\} + \left( G - \frac{2}{3} \Box R \right) \sigma \right\}$$

$$- \frac{1}{12} \left( b'' + \frac{2}{3} (b + b') \right) \int d^4x \sqrt{-\tilde{g}} \left[ R - 6 \Box^2 \sigma - 6 (\nabla \sigma)(\nabla \sigma) \right]$$  \hspace{1cm} (8)$$

where $\sigma = \frac{1}{2} \ln f(\phi)$, and $\sigma$-independent terms are dropped. All 4-dimensional quantities (curvatures, covariant derivatives) in Eq.(8) should be calculated on the metric (6). (We did not write subscript (4) for them.) Note that after calculation of (8) on the metric (6), we will get effectively two-dimensional gravitational theory.

The next step is to calculate second term in the right hand side of Eq.(5), i.e., conformally invariant part of EA. We work in $s$-wave approximation where scalars depend only from first two coordinates. So we have to make reduction for scalar action on the metric (6). We present now the metric (6) as follows:

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu + d\Omega$$  \hspace{1cm} (9)$$

where $\tilde{g}_{\mu\nu} = f^{-1}(\phi)g_{\mu\nu}$.

On the space with the background (9), having in mind $s$-wave approximation and reduction, we get the following two dimensional classical action for scalars:

$$\frac{S}{4\pi} = \frac{1}{2} \sum_{i=1}^{N} \int d^2x \sqrt{-\tilde{g}} \left[ \nabla^\alpha \chi_i \nabla_\alpha \chi_i + \frac{1}{6} \chi_i^2 \left( \tilde{R} + R_\Omega \right) \right].$$  \hspace{1cm} (10)$$
where all quantities in (9) (i.e., \( \tilde{R}, \Box \) and \( \nabla_\alpha \)) should be calculated on two-dimensional metric \( \tilde{g}_{\mu\nu} \). As one can easily see second term in (10) breaks the conformal invariance. Denote \( \frac{\tilde{R}+R_\Omega}{6} \equiv V \).

It is clear that it is impossible to calculate the finite part of effective action corresponding to quantum system (10) in closed form. So, we will develop expansion on the “potential term” \( V \). Zero term of this expansion (i.e., when we put \( V = 0 \)) may be found in closed form as it is standard Polyakov anomaly induced action [6]:

\[
\Gamma_0 = \frac{N}{96\pi} \int d^2x \sqrt{-\tilde{g}} \tilde{R} \frac{1}{\Box} \tilde{R}.
\] (11)

The first order term on the potential \( V \) may be found using zeta-regularization technique (see, [5] for a review). Actually, this term represents Coleman-Weinberg type term:

\[
\Gamma_V = \frac{N}{8\pi} \int d^2x \sqrt{-\tilde{g}} \left( V - V \ln \frac{V}{\mu^2} \right)
\] (12)

where \( \mu^2 \) is an arbitrary mass parameter. Note that one can calculate the next terms of the expansion on \( V \) using different techniques. If we suppose that curvature \( \tilde{R} \) is quickly changing than next leading term of expansion is of the form (11) with another (significantly smaller) coefficient than in (11).

Finally effective action in our model will take the following form:

\[
\Gamma = \Gamma_{ind} - \Gamma_0 - \Gamma_V
\] (13)

where \( \Gamma_{ind} \) is given by Eq.(8) and all quantities in Eq.(8) should be calculated on metric (6), \( \Gamma_0 \) is given by two-dimensional expression (11) and \( \Gamma_V \) is given by Eq.(12). The quantum effective action (13) should be added to classical action (3). That is combined action \( S + \Gamma \) which will define the quantum dynamics of the system under discussion.

We now consider to solve the equations of motion obtained from the above effective Lagrangians \( S + \Gamma \). In the following, we use \( \tilde{g}_{\mu\nu} \) and \( \sigma \) as a set of independent variables and we write \( \tilde{g}_{\mu\nu} \) as \( g_{\mu\nu} \) if there is not confusion. First we should note that the definitions of the Gauss-Bonnet invariant \( G \) and the Weyl tensor \( C_{\mu\nu\alpha\beta} \) are given by

\[
G \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2
\] (14)
\[ C_{\mu\nu\alpha\beta} = \frac{1}{2} (g_{\mu\beta} R_{\nu\alpha} - g_{\mu\alpha} R_{\nu\beta} + g_{\nu\alpha} R_{\mu\beta} - g_{\nu\beta} R_{\mu\alpha}) + \frac{1}{6} R (g_{\nu\alpha} g_{\mu\beta} - g_{\mu\beta} g_{\nu\alpha}) . \]  

Then the square of the Weyl tensor has the following form:

\[ C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2 R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{3} R^2 \]  

Then \( \Gamma_{\text{ind}} \) (W in Eq.(8)) is rewritten after the reduction to 2 dimensions as follows:

\[
\frac{\Gamma_{\text{ind}}}{4\pi} = b \int d^2 x \sqrt{-g} \left( R_{\mu\nu\alpha\beta} R^{(2)\mu\nu\alpha\beta} - 2 R_{(2)\alpha\beta} R^{(2)\alpha\beta} + \frac{1}{3} (R^{(2)})^2 \right.
\]
\[
+ \frac{2}{3} R \Omega R^{(2'} + \frac{1}{3} R^{(2')}_\Omega \bigg) \sigma 
\]
\[
+ b' \int d^2 x \sqrt{-g} \left\{ \sigma \left( 2 \Box^2 + 4 R^{(2)\mu\nu} \nabla_\mu \nabla_\nu - \frac{4}{3} (R^{(2)} + R \Omega) \right) \right. 
\]
\[
+ \frac{2}{3} (\nabla^\mu R^{(2)}) \nabla_\mu \bigg) \sigma 
\]
\[
+ \left( R_{\mu\nu\alpha\beta} R^{(2)\mu\nu\alpha\beta} - 4 R_{(2)\alpha\beta} R^{(2)\alpha\beta} + (R^{(2)})^2 \right. 
\]
\[
- 2 R \Omega R^{(2')} - \frac{2}{3} \Box R^{(2')} \bigg) \bigg\} 
\]
\[
- \frac{1}{12} \left\{ b'' + \frac{2}{3} (b + b') \right\} \int d^2 x \sqrt{-g} 
\]
\[
\times \left\{ (R^{(2)} + R \Omega - 6 \Box \sigma - 6 \nabla^\mu \sigma \nabla_\mu \sigma)^2 - (R^{(2)} + R \Omega)^2 \right\} . \]  

Here \( R \Omega = 2 \) is scalar curvature of \( S^2 \) with the unit radius. The suffix “(2)” expresses the quantity in 2 dimensions but we abbreviate it if there is no any confusion. We also note that in two dimensions the Riemann tensor \( R_{\mu\nu\sigma\rho} \) and \( R_{\mu\nu} \) are expressed via the scalar curvature \( R \) and the metric tensor \( g_{\mu\nu} \) as follows

\[ R_{\mu\nu\sigma\rho} = \frac{1}{2} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}) R , \quad R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R . \]
3 Equations of motion

Let us derive the equations of motion with account of quantum corrections from above effective action. In the following, we work in the conformal gauge

\[ g_{±±} = -\frac{1}{2} e^{2\rho}, \quad g_{±±} = 0 \]  

after considering the variation of the effective action \( \Gamma + S \) with respect to \( g_{\mu\nu} \) and \( \sigma \). Note that the tensor \( g_{\mu\nu} \) under consideration is the product of the original metric tensor in (9) and the \( \sigma \)-function \( e^{-2\sigma} \), the equations given by the variations of \( g_{\mu\nu} \) are the combinations of the equations given by the variation of the original metric and \( \sigma \)-equation.

Since \( g^{++} \) and \( g^{--} \) vanish in the conformal gauge (19), we can drop second or higher order terms in \( g^{++} \) and \( g^{--} \) and the scalar curvature \( R \) and de Alembertian \( \Box \) have the following forms:

\[
R = 8e^{-2\rho}\partial_+\partial_\rho + 4Q,
\]

\[
Q \equiv g^{--} \left( -\partial_\rho^2 - 2(\partial_\rho \partial_\rho)^2 \right) - \frac{1}{4} \partial_\rho^2 g^{--} - \frac{3}{2} \partial_\rho g^{--} \partial_\rho \]

\[
+ g^{++} \left( -\partial_\rho^2 - 2(\partial_\rho \partial_\rho)^2 \right) - \frac{1}{4} \partial_\rho^2 g^{++} - \frac{3}{2} \partial_\rho g^{++} \partial_\rho ,
\]

\[
\Box = -4e^{-2\rho} \partial_\rho \partial_\rho + e^{-2\rho} \partial_+ (g^{++} e^{2\rho} \partial_+) + e^{-2\rho} \partial_-(g^{--} e^{2\rho} \partial_-) . \]

In the conformal gauge (19), the action \( \Gamma_{\text{ind}} \) (17) is reduced into

\[
\frac{\Gamma_{\text{ind}}}{4\pi} = b \int d^2xe^{2\rho} \left( \frac{32}{3} e^{-4\rho}(\partial_+ \partial_\rho)^2 + \frac{16}{3} e^{-2\rho} \partial_+ \partial_\rho + \frac{2}{3} \right) \sigma
\]

\[
+ b' \int d^2xe^{2\rho} \left\{ \sigma \left[ 16e^{-2\rho} \partial_+ \partial_- (e^{-2\rho} \partial_+ \partial_\sigma) 
\right.
\]

\[
+ e^{-2\rho} \left( \frac{64}{3} e^{-2\rho} \partial_+ \partial_- + \frac{16}{3} \right) \partial_+ \partial_\sigma
\]

\[
+ \frac{8}{3} e^{-2\rho} \left( \partial_+ (e^{-2\rho} \partial_+ \partial_- \partial_\rho) \partial_\sigma + \partial_- (e^{-2\rho} \partial_+ \partial_- \partial_\rho) \partial_+ \sigma \right)
\]

\[
+ \left( 16e^{-2\rho} \partial_+ \partial_- + \frac{32}{3} e^{-2\rho} \partial_+ \partial_- (e^{-2\rho} \partial_+ \partial_-) \right) \sigma
\]

\[
- \left\{ b'' + \frac{2}{3} (b + b') \right\} \int d^2xe^{2\rho}
\]

\[
\times \left\{ 2(\partial_+ \partial_\sigma + \partial_+ \sigma \partial_\sigma) e^{-2\rho} (8e^{-2\rho} \partial_+ \partial_\rho + 2) + 24e^{-4\rho} (\partial_+ \partial_\sigma + \partial_+ \sigma \partial_\sigma)^2 \right\} . \]
and the variations of $\Gamma_{\text{ind}}$ with respect to $g^{\pm\pm}$ are given by

$$\frac{\delta}{\delta g^{\pm\pm}} \left( \frac{\Gamma_{\text{ind}}}{4\pi} \right) = \delta \left[ 8e^{2\rho} \partial_+ \sigma \partial_+ \left( e^{-2\rho} \partial_+ \partial_- \sigma \right) - 8\sigma \partial_+^2 \sigma \partial_+ \partial_- \rho \right]$$

$$+ \frac{2}{3} e^{2\rho} \partial_\pm \sigma \partial_\pm \left\{ (8e^{-2\rho} \partial_+ \partial_- \rho + 2)\sigma \right\}$$

$$+ \frac{8}{3} e^{2\rho} \sigma \partial_\pm (8e^{-2\rho} \partial_+ \partial_- \rho + 2)\partial_\pm \sigma$$

$$+ \frac{8}{3} e^{2\rho} \partial_\pm (8e^{-2\rho} \partial_+ \partial_- \rho + 2)\partial_\pm \sigma$$

$$- \left\{ b'' + \frac{2}{3}(b + b') \right\} \left[ 4e^{2\rho} \partial_\pm \sigma \partial_\pm (8e^{-2\rho} \partial_+ \partial_- \rho) \right]$$

$$- 4(\partial_\pm \sigma)^2 \partial_+ \partial_- \rho + 12e^{2\rho} \partial_\pm \sigma \partial_\pm \left\{ e^{-2\rho} (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\}$$

$$- 12(\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) (\partial_\pm \sigma)^2$$

$$+ \left\{ (-\partial_\pm^2 \rho - 2(\partial_\pm \rho)^2) - \frac{1}{4} \partial_\pm^2 + \frac{3}{2} \partial_\pm^2 \rho + \frac{3}{2} \partial_\pm \rho \partial_\pm \right\}$$

$$\times \left[ b \left( \frac{32}{3} \partial_\pm \partial_- \rho + \frac{8}{3} e^{2\rho} \right) \sigma + \frac{16}{3} b' (\sigma \partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right]$$

$$- \left\{ b'' + \frac{2}{3}(b + b') \right\} (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\} \right]$$

(22)

Usually the equation given by $g^{++}$ or $g^{--}$ can be regarded as the constraint equation with respect to the initial or boundary values. The equations obtained here are, however, combinations of the constraint and $\sigma$-equation of the motion since the tensor $g_{\mu\nu}$ under consideration is the product of the original metric tensor in (9) and the $\sigma$-function $e^{-2\sigma}$.

Polyakov’s anomaly induced action (11) has the following form in the conformal gauge (19) as usual

$$\Gamma^{(0)} = -\frac{N}{12\pi} \int d^2 x \rho \partial_+ \partial_- \rho$$

(23)

and the variations with respect to $g^{\pm\pm}$ are given by

$$\frac{\delta \Gamma^{(0)}}{\delta g^{\pm\pm}} = \frac{N}{48\pi} e^{2\rho} \left( \partial_\pm^2 \rho - (\partial_\pm \rho)^2 \right) + t^\pm (x^\pm) e^{2\rho}$$

(24)

Here $t^\pm (x^\pm)$ is a function determined by the boundary condition. The “potential” term $\Gamma^{(V)}$ in (12) has the following form in the conformal gauge
\[ \Gamma^{(V)} = \frac{N}{96\pi} \int d^2xe^{2\rho} \left( 8e^{-2\rho}\partial_+ \partial_- \rho + 2 \right) \left( 1 - \ln \frac{8e^{-2\rho}\partial_+ \partial_- \rho + 2}{6\mu^2} \right) \] (25)

and the variations with respect to \( g^{\pm\pm} \) are given by

\[ \frac{\delta \Gamma^{(V)}}{\delta g^{\pm\pm}} = -\frac{N}{24\pi} \left\{ \left( -\partial_+^2 \rho - 2(\partial_\pm \rho)^2 \right) - \frac{1}{2} \partial_\pm^2 + \frac{3}{2} \partial_\pm^2 \rho + \frac{3}{2} \partial_\pm \partial_\pm \right\} \]

\[ \times e^{2\rho} \ln \frac{8e^{-2\rho}\partial_+ \partial_- \rho + 2}{6\mu^2}. \] (26)

The variations with respect to \( \rho \) are given by

\[ \frac{\delta}{\delta \rho} \left( \Gamma_{\text{ind}} \right) = b \left\{ -\frac{64}{3} e^{-2\rho}(\partial_+ \partial_- \rho)^2 \sigma + \frac{64}{3} \partial_+ \partial_-(\sigma e^{-2\rho} \partial_+ \partial_- \rho) \right. \]

\[ + \frac{16}{3} \partial_+ \partial_- \sigma + \frac{4}{3} e^{2\rho} \sigma \right\} \]

\[ + b' \left\{ -32(\partial_+ \partial_- \sigma) e^{-2\rho} - \frac{128}{3} e^{-2\rho} \partial_+ \partial_- \rho(\sigma \partial_+ \partial_- \sigma) \right. \]

\[ + \frac{64}{3} \partial_+ \partial_- \left( e^{-2\rho} \sigma \partial_+ \partial_- \sigma \right) \]

\[ - \frac{16}{3} \left\{ -2\partial_+ \partial_- \sigma \sigma e^{-2\rho} \partial_+ \partial_- \rho + \partial_+ \partial_-(\partial_+ \sigma \partial_- \sigma e^{-2\rho}) \right\} \]

\[ + 16 \partial_+ \partial_- \sigma \]

\[ + \frac{32}{3} \left\{ -2\partial_+ \partial_- \sigma e^{-2\rho} \partial_+ \partial_- \rho + \partial_+ \partial_-(\partial_+ \partial_- \sigma e^{-2\rho}) \right\} \]

\[ - \left\{ b'' + \frac{2}{3}(b + b') \right\} \left[ 16 \partial_+ \partial_-(e^{-2\rho}(\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma)) \right. \]

\[ - 48e^{-2\rho}(\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma)^2 \] (27)

\[ \frac{\delta \Gamma^{(0)}}{\delta \rho} = -\frac{N}{6\pi} \partial_+ \partial_- \rho \] (28)

\[ \frac{\delta \Gamma^{(V)}}{\delta \rho} = +\frac{N}{48\pi} e^{2\rho} \left( 8e^{-2\rho} \partial_+ \partial_- \rho + 2 \right) \left( 1 - \ln \frac{8e^{-2\rho} \partial_+ \partial_- \rho + 2}{6\mu^2} \right) \]

\[ - \frac{N}{12\pi} \partial_+ \partial_- \left( \ln \frac{8e^{-2\rho} \partial_+ \partial_- \rho + 2}{6\mu^2} \right) \]

\[ + \frac{N}{6\pi} e^{2\rho} \partial_+ \partial_- \rho \ln \frac{8e^{-2\rho} \partial_+ \partial_- \rho + 2}{6\mu^2}. \] (29)
The variations with respect to \( \sigma \) may be found as

\[
\frac{\delta}{\delta \sigma} \left( \frac{\Gamma_{\text{ind}}}{4\pi} \right) = b e^{2\rho} \left( \frac{32}{3} e^{-4\rho} (\partial_+ \partial_- \rho)^2 + \frac{32}{3} e^{-2\rho} \partial_+ \partial_- \rho + \frac{2}{3} \right) + b' \left[ 32 \partial_+ \partial_- (e^{-2\rho} \partial_+ \partial_- \sigma) + \frac{64}{3} \left( e^{-2\rho} \partial_+ \partial_- \rho \partial_+ \partial_- \sigma + \partial_+ \partial_- (\sigma e^{-2\rho} \partial_+ \partial_- \rho) \right) + \frac{32}{3} \partial_+ \partial_- \sigma \right] + \frac{16}{3} \left( \partial_+ (e^{-2\rho} \partial_+ \partial_- \rho) \partial_- \sigma + \partial_- (e^{-2\rho} \partial_+ \partial_- \rho) \partial_+ \sigma \right) + \left( 16 \partial_+ \partial_- \rho + \frac{32}{3} \partial_+ \partial_- (e^{-2\rho} \partial_+ \partial_- \rho) \right) \] 

\[
- \left\{ b'' + \frac{2}{3} (b + b') \right\} \left[ 2 \partial_+ \partial_- (8e^{-2\rho} \partial_+ \partial_- \rho + 2) - 2 \left( \partial_+ (8e^{-2\rho} \partial_+ \partial_- \rho + 2) \partial_- \sigma + \partial_- (8e^{-2\rho} \partial_+ \partial_- \rho + 2) \partial_+ \sigma \right) + 48 \partial_+ \partial_- \left\{ e^{-2\rho} (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\} - 48 \partial_+ \left\{ e^{-2\rho} \partial_- \sigma (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\} - 48 \partial_- \left\{ e^{-2\rho} \partial_+ \sigma (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\} \right] \] 

\[
\frac{\delta \Gamma^{(0)}}{\delta \sigma} = \frac{\delta \Gamma^{(V)}}{\delta \sigma} = 0 \quad (31)
\]

It often happens that we can drop the terms linear to \( \sigma \) in (17). In particular, one can redefine the corresponding source term as it is in the case of IR sector of 4D QG [7]. In that case, Eqs. (22), (27) and (30) are reduced into

\[
\frac{\delta}{\delta g_{\pm \pm}} \left( \frac{\Gamma_{\text{ind}}}{4\pi} \right) = b' \left[ 8e^{2\rho} \partial_\pm \sigma \partial_\pm \left( e^{-2\rho} \partial_+ \partial_- \sigma \right) - 8\sigma \partial_\pm^2 \partial_\pm \partial_- \rho \right] + \frac{2}{3} \partial_\pm e^{2\rho} \partial_\pm \partial_\pm \left( (8e^{-2\rho} \partial_+ \partial_- \rho + 2) \sigma \right) + \frac{8}{3} \partial_\pm e^{2\rho} \sigma \partial_\pm (8e^{-2\rho} \partial_+ \partial_- \rho + 2) \partial_\pm \sigma \right] 

- \left\{ b'' + \frac{2}{3} (b + b') \right\} \left[ 4e^{2\rho} \partial_\pm \sigma \partial_\pm (8e^{-2\rho} \partial_+ \partial_- \rho) - 4(\partial_\pm \sigma)^2 \partial_\pm \partial_- \rho + 12e^{2\rho} \partial_\pm \sigma \partial_\pm \left\{ e^{-2\rho} (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\} \right] 
\]
\[
-12(\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma)(\partial_\pm \sigma)^2 \\
+ \left\{ (-\partial_\pm^2 \rho - 2(\partial_\pm \rho)^2) - \frac{1}{4} \partial_\pm^2 + \frac{3}{2} \partial_\pm^2 \rho + \frac{3}{2} \partial_\pm \rho \partial_\pm \right\} \\
\times \frac{16}{3} \left[ b' (\sigma \partial_+ \partial_- + \partial_+ \sigma \partial_- \sigma) \\
- \left\{ b'' + \frac{2}{3} (b + b') \right\} (\partial_\pm \partial_- \sigma + \partial_\pm \sigma \partial_- \sigma) \right],
\]

\[
\frac{\delta}{\delta \rho} \left( \frac{\Gamma_{ind}}{4\pi} \right) = b' \left[ -32(\partial_+ \partial_- \sigma)^2 e^{-2\rho} - \frac{128}{3} e^{-2\rho} \partial_+ \partial_- (\sigma \partial_\pm \partial_- \sigma) \\
+ \frac{64}{3} \partial_+ \partial_- \left( e^{-2\rho} \sigma \partial_+ \partial_- \sigma \right) \\
- \frac{16}{3} \left\{ -2 \partial_+ \sigma \partial_- \sigma e^{-2\rho} \partial_+ \partial_- \rho + \partial_+ \partial_- \left( \partial_+ \sigma \partial_- \sigma e^{-2\rho} \right) \right\} \\
- \left\{ b'' + \frac{2}{3} (b + b') \right\} \left[ 16 \partial_+ \partial_- \left\{ e^{-2\rho} (\partial_+ \partial_- \sigma + \partial_\pm \sigma \partial_- \sigma) \right\} \\
- 48 e^{-2\rho} (\partial_+ \partial_- \sigma + \partial_\pm \sigma \partial_- \sigma)^2 \right] \tag{32}
\]

\[
\frac{\delta}{\delta \sigma} \left( \frac{\Gamma_{ind}}{4\pi} \right) = b' \left[ 32 \partial_+ \partial_- (e^{-2\rho} \partial_+ \partial_- \sigma) \\
+ \frac{64}{3} \left( e^{-2\rho} \partial_+ \partial_- \partial_+ \partial_- \sigma + \partial_+ \partial_- (\sigma e^{-2\rho} \partial_+ \partial_- \rho) \right) \\
+ \frac{32}{3} \partial_+ \partial_- \sigma \\
+ \frac{16}{3} \left\{ \partial_+ \left( e^{-2\rho} \partial_+ \partial_- \rho \right) \partial_- \sigma + \partial_- \left( e^{-2\rho} \partial_+ \partial_- \rho \right) \partial_+ \sigma \right\} \right] \\
- \left\{ b'' + \frac{2}{3} (b + b') \right\} \left[ 2 \partial_+ \partial_- \left( 8 e^{-2\rho} \partial_+ \partial_- \rho + 2 \right) \right] \\
- 2 \left\{ \partial_+ \left( 8 e^{-2\rho} \partial_+ \partial_- \rho + 2 \right) \partial_- \sigma + \partial_- \left( 8 e^{-2\rho} \partial_+ \partial_- \rho + 2 \right) \partial_+ \sigma \right\} \\
+ 48 \partial_+ \partial_- \left\{ e^{-2\rho} (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\} \\
- 48 \partial_+ \left\{ e^{-2\rho} \partial_- \sigma (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\} \\
- 48 \partial_- \left\{ e^{-2\rho} \partial_+ \sigma (\partial_+ \partial_- \sigma + \partial_+ \sigma \partial_- \sigma) \right\} \right] \tag{33}
\]

The classical Einstein action

\[
S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ R^{(4)} - 2\Lambda \right\} \tag{35}
\]
is reduced into 2 dimensional one in the conformal gauge (19) as follows;
\[
\frac{S}{4\pi} = -\frac{1}{16\pi G} \int d^2x \left[ 4e^{2\sigma} \partial_+ \partial_- (\rho + \sigma) + 2e^{2\rho+2\sigma} - e^{2\rho+4\sigma} \Lambda - 4e^{2\sigma} \partial_+ \sigma \partial_- \sigma \right],
\]
(36)
and we obtain
\[
\frac{\delta}{\delta g_{\pm \pm}} \left( \frac{S}{4\pi} \right) = -\frac{1}{16\pi G} e^{2\rho+2\sigma} \left[ (\partial_{\pm} \sigma)^2 - \partial_{\pm}^2 \sigma + 2\partial_{\pm} \sigma \partial_{\pm} \rho \right] \quad (37)
\]
\[
\frac{\delta}{\delta \sigma} \left( \frac{S}{4\pi} \right) = -\frac{1}{16\pi G} \left[ 8e^{2\sigma} \left\{ 3\partial_+ \partial_- \sigma + 3\partial_+ \sigma \partial_- \sigma + \partial_+ \partial_- \rho \right\} -4e^{2\rho+4\sigma} \Lambda + 4e^{2\rho+2\sigma} \right] \quad (38)
\]
\[
\frac{\delta}{\delta \rho} \left( \frac{S}{4\pi} \right) = -\frac{1}{16\pi G} \left[ 4\partial_+ \partial_- e^{2\sigma} - 2e^{2\rho+4\sigma} \Lambda + 4e^{2\rho+2\sigma} \right]. \quad (39)
\]
Above equations give the complete system of equations of motion for the system under discussion.

4 Anti-evaporation of Nariai black holes

Following the work by Bousso-Hawking [2] we consider the Schwarzchild-de Sitter family of black holes, or more exactly its nearly degenerated case, so-called Nariai solution [8]. We are now interested in the instability of the Nariai solution. The Schwarzchild type black hole solution in the de Sitter space has two horizons, one is the usual event horizon and another is the cosmological horizon, which is proper one in the de Sitter space. The Nariai solution is given by a limit of the Schwarzchild-de Sitter black hole where two horizons coincides with each other. In the limit, the two horizons have the same temperature and they are in the thermal equilibrium. Near the limit, however, the temperature of the event horizon is higher than that of the cosmological one and we can expect that there would be a thermal flow from the event horizon to the cosmological one. Therefore the system would become instable and the black hole would evaporate. We also have to note that above cosmological black holes may naturally appear only through quantum pair creation [9] which may occur in the inflationary universe [10].
In the Nariai limit, the space-time has the topology of $S^1 \times S^2$ and the metric is given by

$$ds^2 = \frac{1}{\Lambda} \left( \sin^2 \chi d\psi^2 - d\chi^2 - d\Omega \right). \quad (40)$$

Here the coordinate $\chi$ has a period $\pi$. If we change the coordinates variables by

$$r = \ln \tan \frac{\chi}{2},$$
$$t = \frac{\psi}{4}, \quad (41)$$

we obtain

$$ds^2 = \frac{1}{\Lambda \cosh^2 x} \left( -dt^2 + dr^2 \right) + \frac{1}{\Lambda} d\Omega. \quad (42)$$

This form would correspond to the conformal gauge in two dimensions. Note that the transformation (41) has one to one correspondence between $(\psi, \chi)$ and $(t, r)$ if we restrict $\chi$ by $0 \leq \chi < \pi$ ($r$ runs from $-\infty$ to $+\infty$).

Now we solve the equations of motion. Since the Nariai solution is characterized by the constant $\phi$ (or $\sigma$), we now assume that $\sigma$ is a constant even when including quantum correction;

$$\sigma = \sigma_0 \quad (constant). \quad (43)$$

We also first consider static solutions and replace $\partial_\pm$ by $\pm \frac{1}{2} \partial_r$. Then the total constraint equation obtained by (24), (26), (32) and (37) becomes

$$0 = -\frac{N}{192\pi} e^{2\rho} \left( \partial_r^2 \rho - (\partial_\rho \rho)^2 \right) + t_0 e^{2\rho}$$
$$+ \frac{N}{96\pi} \left\{ \partial_r^2 \rho + 2(\partial_\rho \rho)^2 + \frac{1}{4} \partial_r^2 - \frac{3}{2} \partial_\rho \rho - \frac{3}{2} \partial_r \rho \partial_\rho \right\}$$
$$\times e^{2\rho} \ln \frac{-2e^{-2\rho} \partial_r^2 \rho + 2}{6\mu^2}. \quad (44)$$

The equations of motions given by (28), (29), (33) and (39) (using the variation of $\rho$) (31), (34) and (38) (using the variation of $\sigma$) have the following
forms:

\[ 0 = -\frac{1}{4\pi G} \left( -2\Lambda e^{2\rho+4\sigma_0} + 4e^{2\rho+2\sigma_0} \right) \]
\[ + \frac{16\pi}{3} b\sigma_0 \left\{ -e^{-2\rho}(\partial^2_t \rho)^2 + \partial^2_t (e^{-2\rho}\partial^2_t \rho) + e^{2\rho} \right\} \]
\[ - \frac{N}{24\pi} e^{2\rho} \left( 1 - \ln \frac{-2e^{-2\rho}\partial^2_t \rho + 2}{6\mu^2} \right) \]
\[ - \frac{N}{48\pi} \partial^2_t \left( \ln \frac{-2e^{-2\rho}\partial^2_t \rho + 2}{6\mu^2} \right) \]  

(45)

\[ 0 = -\frac{1}{4\pi G} \left( -2\Lambda e^{2\sigma_0} \partial^2_t \rho - 4\Lambda e^{2\rho+4\sigma_0} + 4e^{2\rho+2\sigma_0} \right) \]
\[ + 16\pi \left\{ b'' + \frac{2}{3}(b + b') \right\} \partial^2_t \left( e^{-2\rho}\partial^2_t \rho \right) . \]  

(46)

Assuming a solution is given by a constant 2d scalar curvature

\[ R = -2e^{-2\rho}\partial^2_t \rho = R_0 \quad \text{(constant)} \]  

(47)

and substituting (47) into (45) and (46), we find that Eqs. (45) and (46) can be satisfied if \( \sigma_0 \) and \( R_0 \) satisfy the following two algebraic equations

\[ 0 = R_0 - 4\Lambda e^{4\sigma_0} + 4 \]  

(48)

\[ 0 = -\frac{1}{2\pi G} \left( -\Lambda e^{4\sigma_0} + 2e^{2\sigma_0} \right) + \frac{4\pi b}{3} \left( -R_0^2 + 4 \right) \sigma_0 \]
\[ - \frac{N}{24\pi} R_0 \left( 1 - \ln \frac{R_0 + 2}{6\mu^2} \right) . \]  

(49)

The above equation can be solved with respect to \( \sigma_0 \) and \( R_0 \) in general although it is difficult to get the explicit expression. A special case is given by dropping the term corresponding to the terms linear to \( \sigma \) in (17) (i.e., by dropping the second term proportional to \( b \) in (49)) and by choosing the renormalization scale \( \mu \) to be

\[ \mu^2 = \frac{R_0 + 2}{6} . \]  

(50)

In this case, the solution of (48) and (49) is given by

\[ R_0 = \pm 4\sqrt{1 + \frac{NG\Lambda}{12}} \]
\[ e^{2\sigma} = \frac{1}{\Lambda} \left( 1 \pm \sqrt{1 + \frac{NGA}{12}} \right). \] (51)

If we choose + sign in front of the root in the above equations, the solution reduces to the classical solution in the classical limit of \( N \to 0 \).

Equation (47) can be integrated to be

\[ (\partial_t \rho)^2 = -\frac{R_0}{2} e^{2\rho} + C. \] (52)

Here \( C \) is a constant of integration. Substituting (47) and (52) into (44), we find Eq. (44) is satisfied if and only if

\[ t_0 = -\frac{N}{192\pi} C. \] (53)

Eq. (52) can be integrated to become

\[ e^{2\rho} = \frac{2C}{R_0} \cdot \frac{1}{\cosh^2 \left( r\sqrt{C} \right)}. \] (54)

We now consider the perturbation around the Nariai type solution (43) and (54);

\[ \rho = \rho_0 + \epsilon R(t, r) \]
\[ \sigma = \sigma_0 + \epsilon S(t, r) \]
\[ t_0 = -\frac{N}{192\pi} C + \epsilon T^\pm. \] (55)

Here \( \epsilon \) is a infinitesimal small parameter and \( \rho_0 \) is a solution given by (54):

\[ e^{2\rho_0} = \frac{2C}{R_0} \cdot \frac{1}{\cosh^2 \left( r\sqrt{C} \right)}. \] (56)

Then we obtain the following equations

\[ 0 = -8\pi \frac{b'\sigma_0 C}{\cosh^2 \left( r\sqrt{C} \right)} \partial^2_\pm S \]
\[
\begin{align*}
+4\pi \left( -\frac{16}{3} b' \sigma_0 - \frac{2}{3} (b + b') \right) & \left( \frac{3}{8} \frac{1}{\cosh^2 (r \sqrt{C})} - \frac{1}{2} \right) C \partial_+ \partial_- S \\
+4\pi \left( -\frac{16}{3} b' \sigma_0 - \frac{2}{3} (b + b') \right) & \left( -\frac{1}{4} \partial^2_\pm \partial_+ \partial_- S \mp \frac{3}{4} \sqrt{C} \tanh (r \sqrt{C}) \partial_\pm \partial_\mp S \right) \\
\frac{-N}{24\pi R_0} & \left( -\frac{C}{2} R + \partial^2_\pm R \pm \sqrt{C} \tanh (r \sqrt{C}) \partial_\pm R \right) \\
\frac{2C}{R_0 \cosh^2 (r \sqrt{C})} & \left( \frac{NC}{96\pi} + T^\pm \right) \\
+ \frac{N}{24\pi} & \left[ \frac{C}{2R_0} \partial^2_\pm R \frac{1}{\cosh^2 (r \sqrt{C})} \ln \frac{R_0 + 2}{6\mu^2} \\
- \frac{4C}{R_0 \cosh^2 (r \sqrt{C})} & \left( -\frac{1}{4} \partial^2_\pm R \mp \frac{\sqrt{C}}{2} \tanh (r \sqrt{C}) \partial_\pm R \right) \\
+ \frac{8C}{R_0} & \left\{ \left( \frac{3}{8} \frac{1}{\cosh^2 (r \sqrt{C})} - \frac{1}{2} \right) \partial_+ \partial_- R \\
- \frac{1}{4} \partial^2_\pm \partial_+ \partial_- R \mp \frac{3\sqrt{C}}{4} & \tanh (r \sqrt{C}) \partial_\pm \partial_\pm R \right\} \right] \\
- \frac{1}{16\pi G R_0 \cosh^2 (r \sqrt{C})} & e^{2\sigma_0} \left( -\partial^2_\pm S \mp \sqrt{C} \tanh (r \sqrt{C}) \partial_\pm S \right) \\
0 & = 4\pi b' \left\{ -\frac{16}{3} R_0 \sigma_0 \partial_+ \partial_- S + \frac{32}{3} R_0 \sigma_0 \partial_+ \partial_- \left( \cosh^2 (r \sqrt{C}) \partial_+ \partial_- S \right) \right\} \\
-4\pi (b + b') & \frac{16}{3} \frac{R_0}{C} \partial_+ \partial_- \left( \cosh^2 (r \sqrt{C}) \partial_+ \partial_- S \right) \\
\frac{N}{12\pi} & \frac{1}{R_0} \left\{ \frac{2R_0}{R_0 + 2} - \frac{1}{R_0 + 2} \left( -2R_0 R + \frac{4R_0}{C} \cosh^2 (r \sqrt{C}) \partial_+ \partial_- R \right) \right\} \\
+ \frac{N}{12\pi} & \frac{1}{R_0 + 2} \left\{ -2R_0 \partial_+ \partial_- R + \frac{4R_0}{C} \partial_+ \partial_- \left( \cosh^2 (r \sqrt{C}) \partial_+ \partial_- R \right) \right\} \\
- \frac{1}{16\pi G} & e^{2\sigma_0} \left[ 8\partial_+ \partial_- S - \frac{8AC}{R_0 \cosh^2 (r \sqrt{C})} e^{2\sigma_0} (R + 2S) \right]
\end{align*}
\]
When deriving the above equations, we have dropped the terms linear to $\sigma$ in (17), for simplicity.

The equations (58) and (59) can be solved by assuming $R$ and $S$ are given by

$$R(t,r) = P e^{\pm t \sqrt{C}} \cosh^\alpha r \sqrt{C}$$
$$S(t,r) = Q e^{\pm t \sqrt{C}} \cosh^\alpha r \sqrt{C}.$$  

(60)

Here $P$, $Q$ and $\alpha$ are constants. We should note that $\alpha < 0$ if we require $R$ and $S$ are finite in the limit of $r \to \pm \infty$. Then we obtain the following equations

$$\cosh^2 \left( r \sqrt{C} \right) \partial_+ \partial_- R = AR$$
$$\cosh^2 \left( r \sqrt{C} \right) \partial_+ \partial_- S = AS$$

$$A \equiv \frac{\alpha(\alpha - 1)C}{4}.$$  

(61)

Note that there is one to one correspondence between $A$ and $\alpha$ if we restrict $A > 0$ and $\alpha < 0$. Using (61), we can rewrite (58) and (59) (using (48)) as follows:

$$0 = \left\{ \frac{64\pi b' \sigma_0 R_0}{3} \left( -A + 2 \frac{A^2}{C} \right) - \frac{64\pi (b + b') R_0 A^2}{3} \frac{1}{C} - \frac{1}{4\pi G} e^{2\sigma_0} (2A - C) \right\} Q$$

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In order that the above two algebraic equations have a non trivial solution for \( P \) and \( Q \), \( A \) should satisfy the following equation

\[
0 = \left\{ \frac{64\pi b'R_0}{3}\left(-A + \frac{2}{C}A^2\right) - \frac{64\pi (b+b') R_0}{C} A^2 - \frac{e^{2\sigma_0}}{4\pi G} \left(6A - C - \frac{4C}{R_0}\right) \right\} Q
\]

\[
+ \left\{ \frac{64\pi b'\sigma_0 R_0}{3}\left(-A + \frac{2}{C}A^2\right) - \frac{e^{2\sigma_0}}{4\pi G} \left(6A - C - \frac{4C}{R_0}\right) \right\} P .
\]

(62)

In order that the above two algebraic equations have a non trivial solution for \( P \) and \( Q \), \( A \) should satisfy the following equation

\[
0 = \left\{ \frac{64\pi b'\sigma_0 R_0}{3}\left(-A + \frac{2}{C}A^2\right) - \frac{64\pi (b+b') R_0}{C} A^2 - \frac{1}{4\pi G} e^{2\sigma_0} (2A - C) \right\}
\]

\[
\times \left\{ \frac{64\pi b'\sigma_0 R_0}{3}\left(-A + \frac{2}{C}A^2\right) - \frac{e^{2\sigma_0}}{4\pi G} (2A - C) \right\}
\]

\[
- \left\{ \frac{N C}{6\pi R_0} \left(1 + \ln \frac{R_0 + 2}{6\mu^2}\right) + \frac{N C}{6\pi R_0 + 2} \left(-1 + \frac{2}{C}A\right) \right\}
\]

\[
+ \frac{N R_0}{6\pi R_0 + 2} \left(-A + \frac{2}{C}A^2\right) - \frac{C}{8\pi G} e^{2\sigma_0} \left(-1 + \frac{4}{R_0}\right) \right\}
\]

\[
\times \left\{ \frac{64\pi b'R_0}{3}\left(-A + \frac{2}{C}A^2\right) - \frac{64\pi (b+b') R_0}{C} A^2
\]

\[
- \frac{e^{2\sigma_0}}{4\pi G} \left(6A - C - \frac{4C}{R_0}\right) \right\}
\]

\[
\equiv F(A) .
\]

(63)

If the equation (63) has a positive \((A > 0)\) solution for \( A \), i.e., a real negative solution for \( \alpha \), there is a solution where \( S \) increases exponentially in time and the system would be unstable. It is generally difficult to find the parameter region where \( A \) has a positive solution but it might not be so difficult to show the existence of such a parameter region.

In order to show the existence, we now consider the solution of (51) and
\( R_0 \to 4 \) limit \((NG\Lambda \to 0)\). Then \( F(0) \) is given by

\[
F(0) \to \frac{C^2}{4\pi^2G^2\Lambda^2} > 0 . \tag{64}
\]

On the other hand, when \( A = \frac{C}{2} \)

\[
F \left( A = \frac{C}{2} \right) \to -\frac{N}{24\pi} \left( \frac{91\pi b}{2^3 \cdot 3^2 \cdot 5\pi} + \frac{1}{2\pi G} \right) C^2 < 0 . \tag{65}
\]

Eqs.(64) and (65) tell that there is a solution of the equation \( F(A) = 0 \) when \( 0 < A < \frac{C}{2} \), i.e.,

\[
0 > \alpha > -1 . \tag{66}
\]

In order to show that the existence of the solution (66) implies the instability of the system, we consider a linear combination of (60):

\[
S(t,r) = Q \cosh t\alpha\sqrt{C}\cosh^\alpha r\sqrt{C} . \tag{67}
\]

It should be noted that any linear combination of two solutions is always a solution since the perturbative equations of motion are linear differential equations. The horizon is given by the condition

\[
\nabla \sigma \cdot \nabla \sigma = 0 . \tag{68}
\]

Substituting (67) into (68), we find the horizon is given by

\[
r = \alpha t . \tag{69}
\]

Therefore on the horizon, we obtain

\[
S(t,r(t)) = Q \cosh^{1+\alpha} t\alpha\sqrt{C} . \tag{70}
\]

It should be noted \( 1 + \alpha > 0 \) in the solution of (66), which implies the system is unstable. Since the radius of the horizon \( r_h \) is given by

\[
r_h = e^{\sigma} = e^{\sigma_0 + \epsilon S(t,r(t))} , \tag{71}
\]

the radius increases monotonically in time if the initial perturbation at \( t = 0 \) is positive \( Q > 0 \), on the other hand, if the initial perturbation is negative \( Q < 0 \), the radius shrinks, i.e., the black hole evaporates.
The solution in $0 > \alpha > -1$ shows that the system is unstable. The perturbation, however, becomes stable if there is a solution where $\alpha < -1$, i.e., $A > C^2$. In order to show the existence of the solution where $A > C^2$, we consider $\tilde{F}(A)$ in the limit of $A \to +\infty$. As in the previous case, we consider the limit near the classical solution $R_0 \sim 4$. Then we find

$$F(A \to +\infty) \to \left(\frac{2^{18}}{3^4} \left(\frac{b \ln \frac{A}{2}}{2}\right)^2 + \frac{2^9 \pi N b}{3^2}\right) \frac{A^4}{C^2} > 0$$

Combining (65) and (72), we find that there is a solution where $A > C^2$, i.e., $\alpha < -1$. This implies that the perturbation is stable since the perturbation becomes exponentially small in time on the horizon (70)

$$S(t, r(t)) \to Q e^{(1+\alpha)t|\alpha|\sqrt{C}}.$$  

Therefore if the initial perturbation is negative $Q < 0$ and the perturbed radius of horizon is smaller than that of the Nariai limit, which is the thermal equilibrium, the radius increases in time and approaches to the Nariai limit asymptotically, i.e., the black hole anti-evaporates as observed by Bousso and Hawking [2]. It might be interesting that there are both of stable and unstable perturbations in the model discussed here although the treatment of the quantum corrections is rather different from that of Bousso and Hawking.

In the model by Bousso and Hawking, it depends on the initial conditions if the perturbation is stable or unstable, i.e., the perturbation is stable and the black hole anti-evaporates if its initial value does not vanish but its time derivative vanishes, on the other hand, the perturbation is unstable if its initial value vanishes but its time derivative does not vanish. In the model discussed in this paper, however, the initial conditions for the value and the time derivative of the stable and unstable perturbations are completely the same, i.e., the initial value does not vanish but the initial time derivative vanishes for both cases. The reason why these initial conditions lead to the physically different results could be because the effective Lagrangian contains higher derivatives and the initial value and the initial time derivative are not sufficient as a set of the initial conditions and we need the initial conditions for the higher derivatives to determine the dynamics.
5 Discussion

In summary, we demonstrated that in the model under discussion black holes may evaporate or anti-evaporate. We have to note that our consideration has quite common character. We limited to the case of conformal scalars but we could start from the theory of free conformal scalars, spinors and vectors. (As is known [3] many asymptotically free GUTs in the early Universe are also asymptotically conformally invariant and they may be represented as collection of free conformal fields). Then, the anomaly induced effective action $W$ would have the same form as in section 2 where only coefficients $b$ and $b'$ would be changed (but their signs are the same). And in the Eqs.(11) and (12) again only numerical coefficients would be changed due to spinor and vector corrections (in Eq.(12) also signs of some coefficients could be changed). Hence, our result actually corresponds to quantum conformal scalar-spinor-vector system.

There are different possibilities to extend our results. From one side, one can study the similar, anti-evaporation effect in black holes realised in gravity with torsion. The corresponding anomaly induced effective action is given in last ref. of [4]. That could help to find finally some phenomenon where torsion effects may be significaly strong to prove (or disprove) the existence of torsion in the Nature.

From another side, using the effective action of section 2 one can investigate the quantum cosmology of Kantowski-Sacks form [12]. Recently, such cosmology has been studied in large $N$ and $s$-waves approximation [11]. In the above works it was found that realization of non-singular Universe for the models under discussion is problematic. The results of the present work (if exchange time and radius) indicate to the possibility of construction of non-singular Kantowski-Sacks early Universe.

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