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CURRENT ALGEBRA AND GAUGE VARIANCE

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ABSTRACT

It is urged that the lesson of gauge invariance in quantum electrodynamics implies the irrelevance of "Schwinger term" difficulties in current algebra. The divergence equations of Veltman form the basis of a gauge-variation formalism in which these questions are avoided.

Experience with the vacuum polarization tensor in quantum electrodynamics suggests that the anomalous commutators of Gato and Imamura ¹⁾ [and later Schwinger, and others ¹⁾] are irrelevances to be avoided in one of at least two ways. The first way is that of regularization ²⁾. This has been illustrated already in some detail for a simple model ³⁾. It was suggested that in relations between propagators at fixed energy the canonical commutators are correct, and only for sum rules involving infinite energy do the anomalous commutators become significant - in a less trivial model they would then be related to the obscure question of subtractions in dispersion relations. The second way is that of avoiding as much as possible the creaking machinery of field theory and simply imposing gauge invariance on the matrix elements of interest. Veltman ⁴⁾, in a fundamental paper, has taken the essential step in generalizing this approach to cover much of the algebra of the quark model. The purpose of this note is to spell out the relation of his equations to gauge variance of S matrix elements, and the translation into this language of a class of applications of current algebra.

The approach will be illustrated as usual with the quark model. However, instead of taking the algebraic properties of currents as the essential feature, we take certain gauge properties of S matrix elements. Although the starting point of the two approaches is the same, the quark model, it should be noted that by abstracting different aspects we end up by covering different classes of generalized theory.

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2. QUARK MODEL ALGEBRA

First recall the situation in the quark model. We define 18 vector and axial currents

$$J_{\mu}^n = \bar{\Psi} i \gamma_{\mu} \lambda^n \Psi \quad (1)$$

where Ψ is the quark field (with suppressed Dirac and unitary spin indices), $\lambda^0 - \lambda^8$ are the 3×3 unitary spin matrices of Gell-Mann, and

$$\lambda^{9+n} = \lambda^n \gamma_5, \quad n = 0 - 8$$

In the absence of interactions we have the Dirac equation

$$\gamma_{\mu} \partial_{\mu} \Psi + M \Psi = 0$$

where M , a 3×3 matrix acting on unitary spin indices, is

$$M = \begin{pmatrix} M_p & 0 & 0 \\ 0 & M_n & 0 \\ 0 & 0 & M_{\lambda} \end{pmatrix}$$

From this

$$\partial_{\mu} J_{\mu}^n = -i \Psi^{\dagger} [\lambda^n, M \beta] \Psi$$

We suppose that the quark-quark interactions are such that this remains true. Suppose now that we add an interaction with hypothetical external C number fields a_{μ}^n , i.e., modify the basic Lagrangian

$$L \rightarrow L + a_\mu^n J_\mu^n$$

The modified Lagrange equations of motion then yield

$$\partial_\mu J_\mu^n = -i\psi^\dagger[\lambda^n, M\beta]\psi + ia_\mu^m \psi^\dagger[\lambda^n, i\beta\gamma_\mu\lambda^m]\psi \quad (2)$$

The term involving the external fields is a linear combination of the original currents.

The relation of this to current algebra is most immediately exhibited by the canonical equation of motion

$$\partial_\mu J_\mu^n = i [J_\mu^n, P_\mu]$$

where P_μ is the four vector of total momentum and energy. This yields

$$\begin{aligned} \partial_\mu J_\mu^n &= -i \left[\psi^\dagger \lambda^n \psi, \int d\vec{x} \psi^\dagger M \beta \psi \right] \\ &\quad + i \left[\psi^\dagger \lambda^n \psi, \int d\vec{x} \psi^\dagger i\beta\gamma_\mu a_\mu^m \lambda^m \psi \right] \end{aligned}$$

which reduces to the above on using the formula of current algebra

$$\left[\psi^\dagger \alpha \psi, \int d\vec{x} \psi^\dagger(\vec{x}) \beta \psi(\vec{x}) \right] = \psi^\dagger [\alpha, \beta] \psi$$

which follows from simple-minded use of canonical commutation relations. However, there are notorious ambiguities associated with such commutators of currents ²⁾. It is an attraction of the new approach that such questions are side-tracked. It avoids commutators of currents, working directly with "currents of commutators".

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3. GAUGE FORMALISM

If the mass term is neglected in the above model, the equations of motion are invariant under the infinitesimal gauge transformation of the second kind

$$\psi(x) \rightarrow \psi(x) + i \epsilon^n(x) \lambda^n \psi(x)$$

$$a_\mu^\ell(x) \rightarrow a_\mu^\ell(x) + \epsilon^m C^{mn\ell} a_\mu^n - \partial_\mu \epsilon^\ell(x)$$

where

$$[\lambda^\ell, i \lambda^m] = C^{\ell m n} \lambda^n$$

If the gauge parameters $\epsilon(x)$ vanish at infinity, the asymptotic fields ψ are unaltered. It follows that the S matrix in the presence of external fields a_μ is unaltered by the indicated change

$$\frac{\delta}{\delta \epsilon_\mu(x)} (\beta | S | \alpha) = 0$$

Note also that by the action principle the effect of a small change of the external fields is given by

$$\frac{\delta}{\delta a_\mu^\ell(x)} (\beta | S | \alpha) = \langle \beta | i J_\mu^\ell(x) | \alpha \rangle_{\text{out}}^{\text{in}}$$

where on the right-hand side the states are the incoming and outgoing states of the Heisenberg representation. These properties of the S matrix can be made the basis of a generalized formulation.

In what follows we will be concerned with matrix elements of the S matrix between fixed arbitrary initial and final states α and β , which will not however be indicated explicitly, for simplicity of notation. The S matrix is supposed to depend on external fields a_μ^n , and we define

$$i J_\mu^n(x) = \frac{\delta S}{\delta a_\mu^n(x)}$$

Applying the gauge transformation

$$a_\mu^l(x) \rightarrow a_\mu^l(x) + \epsilon^m(x) a_\mu^n(x) C^{mnl} - \partial_\mu \epsilon^l(x)$$

to the external fields, we have

$$\begin{aligned} \delta S &= \int dx \frac{\delta S}{\delta a_\mu^l(x)} (\epsilon^m(x) a_\mu^n(x) C^{mnl} - \partial_\mu \epsilon^l(x)) \\ &= \int dx i \epsilon^m(x) (C^{mnl} a_\mu^n(x) J_\mu^l(x) + \frac{\partial J_\mu^m(x)}{\partial x_\mu}) \end{aligned}$$

Thus

$$\frac{1}{i} \frac{\delta S}{\delta \epsilon^m(x)} = \partial_\mu J_\mu^m(x) + C^{mnl} a_\mu^n(x) J_\mu^l(x) \quad (3)$$

This is the generalized Veltman equation. It is true by definition. Physics enters through the assumption that the currents so defined are appropriate in weak and electromagnetic interactions, and in hypotheses

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about $\delta S / \delta \epsilon^m$. For the parameters ϵ_n corresponding to the isospin current we would impose, CVC, $\delta S / \delta \epsilon = 0$, giving the conservation of the current in the absence of external fields. For those parameters corresponding to the isovector axial current, $\delta S / \delta \epsilon$ would be a suitable object for the hypothesis of pion pole dominance. In any case one sees here that the equation is a considerably generalized version of (2). That his equation could be so generalized beyond the quark model was already known to Veltman ⁵⁾ for conserved vector current in a theory where elementary pseudoscalar mesons as well as quarks are postulated. Also for the conserved vector currents, Kroll ⁶⁾ gave a general derivation of the equation in classical Lagrangian field theory.

4. APPLICATIONS

In applications we are concerned with objects that may be represented diagrammatically by

in which the wiggly lines are external fields. These external fields are to be replaced in specific applications by photon wave functions, or lepton pair wave functions, or perhaps two of them have to be contracted together with an intermediate boson ⁷⁾ or photon ⁸⁾ propagator. In the quark model these quantities are just matrix elements of time ordered products of currents

$$T(J_\mu^m(x) \dots J_\nu^n(y) \dots)$$

From the canonical current commutators one obtains

$$\begin{aligned} \partial_\mu T(J_\mu^m(x) \dots J_\nu^n(y) \dots) \\ = T(\partial_\mu J_\mu^m(x) \dots J_\nu^n(y) \dots) \\ - i \delta(x-y) T(\dots C^{mn\ell} J_\nu^\ell(y) \dots) \\ \dots \end{aligned} \quad (4)$$

where we indicate by dots that more than two currents may be present, and that then there are additional δ function terms arising from the other equal time commutators.

More generally the time ordered product is not the required covariant amplitude. What we need is the manifestly covariant multiple functional derivative of the S matrix

$$\frac{\delta \dots \delta \dots S}{\delta a_\mu^m(x) \dots \delta a_\nu^n(y) \dots} \Big|_{a=0}$$

in which the external fields are after the differentiation set equal to zero. For simplicity consider only the two point function. By functional differentiation of (3)

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$$\begin{aligned}
 & \partial_\mu \frac{\delta^2 S}{\delta a_\mu^m(x) \delta a_\nu^n(y)} \Big|_{a=0} \\
 &= \frac{\delta}{\delta a_\mu^n(y)} \frac{\delta S}{\delta E^m(x)} \Big|_{a=0} \\
 & \quad - i \delta(x-y) C^{mnl} J_\mu^l(x) \Big|_{a=0}
 \end{aligned} \tag{5}$$

in which the δ function term now arises from

$$\frac{\delta}{\delta a_\mu^n(x)} a_\nu^m(y) = \delta_{nm} \delta_{\mu\nu} \delta(x-y)$$

Generalizing to the many-point function, by adding rows of dots, we have the analogue of (4)

Note that the gauge formalism gives directly only results of the "zero-energy theorem" type, or more generally relations involving finite energies only. To obtain sum rules involving summation to infinite energy then requires assumptions about subtractions in dispersion relations. It has been suggested already elsewhere ³⁾ that this same situation holds in the algebraic approach when the ambiguities in commutation relations are resolved by regularization.

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